CHAPTER 1

INTRODUCTION

1.1 Perspective

Communication systems are becoming progressively complex as technological advances permit faster transmission over links of greater capacity. Similarly, computer architectures are undergoing a wide range of development due to the world’s insatiable appetite for high performance computing. Performance models of computer and communication systems have been studied for several decades with a view to assisting optimization and guiding the design of new generation systems as they play a key role in the design, development, configuration and tuning of these systems. With the wide increase in complexity associated with the systems of the future, formal models of performance are necessary for efficient and reliable design and/or optimization (Bertsekas and Gallager (1992), Haverkort (1998)). Queueing models have been efficiently used to find the performance measures of computer and communication systems (Dshalalow (1997), Tijms (2003), Parthasarathy and Lenin (2004)).

The study of queueing theory has been the subject of considerable research interest for many decades since the appearance of the base paper by A.K. Erlang on congestion in telephone traffic. From computer and communications networks to epidemiology, queueing theory has proved to be very useful in various fields of science and engineering.

The transient analysis of state-dependent queues is more difficult than their corresponding steady-state analysis. Due to this difficulty, there are few explicit expressions
available even for simple models, for example, the transient solution of an M/M/1 queue involves the modified Bessel function of the first kind. In real world problems the underlying arrival and service rates are state-dependent or state- and time-dependent. The difficulty is compounded in the transient analysis of such models. Hence, it is pertinent to develop effective techniques to solve such systems so as to gain an insight into the behaviour of the various system characteristics. In this thesis, generating functions, continued fractions and transforms are used to find the values of time-dependent probabilities, mean and variance of the system size for several birth-death queueing models with state-dependent rates.

In the analysis of queueing systems, the emphasis is often laid on the steady state analysis while the transient or time-dependent analysis has received less attention. The assumptions required to obtain the steady state solutions to queueing systems are seldom satisfied in the design and analysis of real systems. Further, steady state measures of system performance simply do not make sense for systems that do not approach equilibrium. These situations frequently occur in communication networks where the load on the network usually depends on time. Moreover, the steady state solutions are inappropriate in situations wherein the time horizon of operations is finite. Hence, in many applications, the practitioner needs a knowledge of the transient behaviour of the system rather than easily obtainable steady state results. For example, while analysing algorithms, it is important to study the time-dependent behaviour of the average queue length. The purpose is to identify the performance bottlenecks in the system/algorithm and improve the performance by rectifying the situation. Also, transient solutions are available for a wider class of problems and contribute to a more finely tuned analysis of the costs and benefits of the systems. For example, when buffers are allocated in real time by a central processor, the equilibrium distribution of buffer content may be used to determine the
required number of buffers, but the fluctuations will determine the load on the central processor for buffer allocation (Whitt (1983)).

Abate and Whitt (1988) have studied the transient analysis of the single server queue, that is, the BDP with fixed birth and death rates using the scaling factor via Laplace transforms and have expressed the same in terms of first passage time distributions while Baccelli and Massey (1989) have obtained the transient distribution for the queue length using the sample path approach. Parthasarathy (1987) has obtained the time-dependent solution in a simple and direct approach for this process. Leguesdron et al. (1993) have given a new analytical expression of the transient probabilities using the generating function of the transient probabilities of the uniformized Markov chain associated with the M/M/1 queue. Krinik (1992) has given an explicit Taylor series solution for empty system size probability $P_0(t)$, and describes an iterative procedure for deriving $P_n(t)$. Tarabia (2002) gives a new formula to derive the transient behaviour of a non-empty M/M/1/$\infty$ queue.

Saaty (1960) has considered the Multiserver queue and derived the Laplace transform of the distribution of the number in the system at time $t$. For this multiserver queue, Parthasarathy and Sharafali (1989) have derived the transient probability values of the number present in the system at any time in a simple and elegant way, by defining the generating function in an unusual manner. Recently Parthasarathy and Sudhesh (2007) have obtained the exact transient solution for the system size probabilities of an M/M/c queue with N-policy.

There has been an increasing attention in the queueing literature towards the study of systems in which the parameters of arrival and service depend on the instantaneous state of the system (Gross and Harris (1998), Hanbali and Boxma (2010)). Many actual service facilities possess defence mechanisms against long waiting lines. The servers
may increase the rate at which they provide service under the pressure of a large backlog of work.

During the last three decades, great attention has been paid to the description of biological, physical and engineering systems subject to various types of catastrophes, a catastrophe being defined as a random event resulting in the extinction of all individuals or customers in the system. There has been an increasing interest in queueing systems and networks with negative customers due to their applications in telecommunication and computer networks. When a negative customer arrives at the queue, it has the effect of a signal which induces ordinary (positive) customers to leave immediately from the system. Since their introduction by Gelenbe (1989), the name G-queues and G-networks have been adopted for this kind of queues. A review of the main results on this topic can be found in Gelenbe (2000). It should be pointed out that queues with catastrophes could be considered as the basic models of computer systems in the presence of a virus or a reset order. In computer systems or networks, if a job (or a station) is infected, this job may transmit a virus when it is transferred to other processors (CPU, I/O, diskettes, etc). In this case, catastrophes have a role of a clearing operation of all stored messages present in the system.

Many types of applications in computer and communication systems require time-dependent parameters; such situations frequently occur in communication networks where the load on the network depends on time (Knessl and Yang (2002)). Clarke (1956) has considered the time-varying rates for a single server queue, and has shown that the difference-differential equations can be reduced to a partial differential equation, “the Telegrapher’s equation”, and the boundary value problem can then be formulated in terms of the solution of a Volterra-type integral equation, which can be solved numerically. Zhang and Coyle (1991) have given analytical arguments for this boundary probability.
Margolius (2005) has derived an integral equation for the transient probabilities and expected number in the queue for the multiserver queue with time-varying arrival and departure rates, and a time-varying number of servers by using the generating functions.

Since the mid of 20th century, due to the rapid advance of computer technology, flexible manufacturing systems, telecommunication networks, and supply chain management have been becoming more and more popular in many organizations. To evaluate and eventually increase the performance and efficiency, queueing models were developed to analyze the operations of these hi-tech systems. However, due to the increasing complexity of these stochastic models, classical queueing theory, which was once quite successful in modeling telephone systems, became inadequate. Vacation queueing models was developed in the 1970’s as an extension of the classical queueing theory. In a queueing system with vacations, other than serving randomly arriving customers, the server is allowed to take vacations. The vacations may represent server’s working on some supplementary jobs, performing server maintenance inspection Many studies on vacation models were summarized in the survey paper by Doshi (1986).

In the classical single server queues with server vacations, the server stops working during vacation periods. However, that a system can be staffed with a substitute server during the times the main server is taking vacations. The service rate of the substitute server is different from (and probably lower than) that of the main server. This is the notion of working vacations recently introduced by Servi and Finn (2002). The motivation of their study is that the M/M/1/WV queue can be used to approximate a multi-queue system whose service rate is one of two service speeds such that the fast speed mode cyclically moves from queue to queue with exhaustive service. They try to apply the M/M/1/WV queue to model a wavelength division multiplexing (WDM) optical access network using multiple wavelengths which can be reconfigured. This work is motivated
and the transient analysis are discussed in this thesis.

In this thesis, we mainly concentrate a time-dependent solution of state-dependent birth and death queueing models with in the presence and in the absence of catastrophe, state-dependent birth-death queues with finite and infinite capacity and queues with working vacations by employing Laplace transforms, generating functions and continued fractions.

1.2 Birth-Death Processes

A Birth and Death Process (BDP) is a continuous-time Markov chain $\mathcal{X} = \{X(t), t \geq 0\}$ with state space $\mathcal{S} = \{0, 1, 2, \ldots \}$ for which transitions from state $i$ can only go to either state $i - 1$ or state $i + 1$. The state of the process is usually thought of as representing the size of some population and when the state increases by 1 we say that a birth occurs, and when it decreases by 1, we say that a death occurs. If there is no birth and no death then the state remains in the state $i$. Define

$$P_{mn}(t) := P(X(t) = n | X(0) = m), \quad m, n \in \mathcal{S},$$

the conditional (transition) probabilities that the population size is $n$ at time $t > 0$ given it was $m$ at time $t = 0$. It is assumed that in an interval $(t, t + \delta t)$ each individual in the population has a probability $\lambda_n \delta t + o((\delta t)^2)$ of giving birth to a new individual and a probability $\mu_n \delta t + o((\delta t)^2)$ of dying. That is, as $t \to 0^+$ the transition probabilities obey

$$P_{mn}(t) = \begin{cases} 
\lambda_mt + o(t) & \text{if } n = m + 1 \\
\mu_mt + o(t) & \text{if } n = m - 1 \\
1 - \lambda_mt - \mu_mt + o(t) & \text{if } n = m \\
o(t) & \text{if } |m - n| > 1.
\end{cases}$$
The parameters $\lambda_n > 0$ and $\mu_n > 0$ are called the birth and death rates respectively, when the population size is $n$.

The transition probability matrix is

$$P(t) = (P_{mn}(t)), \ m, n \in SS.$$ 

Since the process is time homogeneous, $P_{mn}(t)$ does not depend on how the system reached state $m$ but depends only on $m, n$ and the time $t$ lapsed in moving from state $m$ to state $n$. This is equivalent to

$$P(s + t) = P(s)P(t).$$

By considering $P_{mn}(t + \delta t)$ in terms of $P_{m,n-1}(t)$, $P_{mn}(t)$ and $P_{m,n+1}(t)$, we have the following system of differential-difference equations known as forward Kolmogorov equations:

$$P'_{m0}(t) = -\lambda_0 P_{m0}(t) + \mu_1 P_{m1}(t)$$

$$P'_{mn}(t) = \lambda_{n-1} P_{m,n-1}(t) - (\lambda_n + \mu_n) P_{mn}(t) + \mu_{n+1} P_{m,n+1}(t), \ n \in S \setminus \{0\}$$

whence $0 \leq P_{mn}(t) \leq 1$ and $\sum_{n=0}^{\infty} P_{mn}(t) = 1$, subject to the initial condition $P_{mn}(0) = \delta_{nm}$ (Kronecker delta) for some $m \in S$.

In some applications, we shall allow $\lambda_n = 0, \ n \geq N$, with $N$ being a given positive integer. In this case, the state space of $X(t)$ is truncated to $\{0, 1, 2, \ldots, N\}$.

We now introduce the quantities

$$A = \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n},$$

$$B = \sum_{n=0}^{\infty} \pi_n,$$

$$C = \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=0}^{n} \pi_i,$$

$$D = \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=n+1}^{\infty} \pi_i,$$ (1.2)
where the potential coefficients $\pi_n$ are given by

$$\pi_0 = 1 \quad \text{and} \quad \pi_n = \pi_{n-1} \frac{\lambda_{n-1}}{\mu_n} = \frac{\lambda_0 \ldots \lambda_{n-1}}{\mu_1 \ldots \mu_n}, \quad n = 1, 2, \ldots \quad (1.3)$$

An interpretation for the values of $A, B, C$ and $D$ is given below.

The BDP $\{X(t)\}$ is recurrent (transient) if $A = \infty \ (A < \infty)$ and when $\{X(t)\}$ is recurrent, it is positive recurrent (null recurrent) if in addition $B < \infty \ (B = \infty)$ (Karlin and McGregor (1957)). In order to establish the uniqueness of the solution of the Kolmogorov equations, we need to introduce a classification of boundaries at infinity of BDPs. We denote the boundary by $\infty$. The boundary at infinity of a BDP is said to be regular if $C, D < \infty$, exit if $C < \infty$ and $D = \infty$, entrance if $C = \infty$ and $D < \infty$, and natural if $C = D = \infty$.

The transition probability functions are uniquely determined by the birth and death rates if and only if $A + B = \infty$, i.e., at least one of $C$ and $D$ diverges, since $AB = C + D$. If the series $C$ diverges, then the BDP $\{X(t)\}$ is non-explosive, which means that the process makes at most finitely many jumps in a finite time with probability one. If $C < \infty$ and the series $D$ diverges ($\infty$ is an exit boundary), then the process is explosive and the boundary $\infty$ is absorbing. In the case of a regular boundary, the transition probabilities are not uniquely determined by the birth and death rates. That is, the Kolmogorov equations have infinitely many solutions. In this case, we say that the rate problem associated with $\{X(t)\}$ is indeterminate (see, for details, van Doorn (1989), Kijima (1997)).

If $P_{mn}(t) > 0$ for some $t$ and for every $m$ and $n$ in $S$, the process $\mathcal{X}$ is said to be **irreducible**. When $\mathcal{X}$ is irreducible and $p_n = \lim_{t \to \infty} P_{mn}(t)$ exists and is positive for all $n$, we say that the BDP $\mathcal{X}$ is **ergodic**. The quantities $p_n, \ n \in S$ are called the **steady state probabilities**.
If these probabilities exist, they are obtained by setting $P'_{mn}(t) = 0$ in (1.1) and denoting $P_{mn}(t)$ by $p_n$, we get

$$0 = \lambda_{n-1}p_{n-1} - (\lambda_n + \mu_n)p_n + \mu_{n+1}p_{n+1}, \quad n = 0, 1, \ldots.$$ 

or

$$\mu_{n+1}p_{n+1} - \lambda np_n = \mu np_n - \lambda_{n-1}p_{n-1}$$

$$= \vdots$$

$$= \mu_1 p_1 - \lambda_0 p_0 = 0.$$ 

Therefore,

$$p_j = \frac{\lambda_0 \lambda_1 \ldots \lambda_{j-1}}{\mu_1 \mu_2 \ldots \mu_j} p_0, \quad j = 1, 2, \ldots.$$ 

(1.4)

Since $\sum_{j=0}^{\infty} p_j = 1$, we have

$$p_0 = \frac{1}{1 + \sum_{j=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{j-1}}{\mu_1 \mu_2 \ldots \mu_j}}.$$ 

Note that the probabilities $p_n$, $n \geq 1$, depend on $p_0$, and $p_0$ in turn depends on the convergence of the series in the above expression. For ergodic processes this series converges and hence for these processes steady-state probabilities exist.

### 1.3 Continued Fractions

Continued Fractions (CFs) play a fundamental role in many investigations of the classical moment problems. Approximations employing CFs often provide a good representation for transcendental functions. They are generally much more valid than the classical representation by power series. A systematic study of the theory of CFs with stress on computations can be found in Jones and Thron (1980). Its application to the study of BDPs was initiated by Murphy and O’Donohoe (1975).
CF approximations occupy a remarkable place in mathematical literature due to their interesting convergence properties and also due to their connections with many branches of mathematics like number theory, special functions, differential equations, moment problems, Orthogonal Polynomials and so on (Lorentzen and Waadeland (1992)). On account of their algorithmic nature, they are used in numerical analysis, computer science, automata, electronic communication etc. Their importance has grown further with the advent of fast computing facilities.

A CF is denoted by

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \ldots$$

or equivalently by

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \ldots$$

(1.5)

where $a_n$ and $b_n$ are real or complex numbers. This fraction can be terminated by retaining the terms $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ and dropping all the remaining terms $a_{n+1}, b_{n+1}, \ldots$. The number obtained by this operation is called the $n^{th}$ convergent or $n^{th}$ approximant and is denoted by $A_n/B_n$. Both $A_n$ and $B_n$ satisfy the recurrence relation

$$U_n = a_n U_{n-2} + b_n U_{n-1}$$

(1.6)

with initial values $A_0 = 0$, $A_1 = a_1$ and $B_0 = 1$, $B_1 = b_1$.

The CF (1.5) is said to be convergent if atmost a finite number of its denominators $B_n$ vanish and the limit of its sequence of approximants

$$\lim_{n \to \infty} \frac{A_n}{B_n}$$

(1.7)

exists and is finite. Otherwise, the CF is said to be divergent. The value of a CF is defined to be the limit of its sequence of approximants given by (1.7). No value is assigned to a divergent CF (Wall (1948)).
For any CF the exact value of the fraction lie between any two neighboring convergents. All even numbered convergents lie to the left of the exact value, that is they give an approximation to the exact value by defect. All odd numbered convergents lie to the right of the exact value, that is they give an approximation to the exact value by excess. For a terminating CF, that is, a fraction with finite number of terms, the last convergent coincides with the exact value. If a CF is non-terminating, the sequence of convergents is infinite.

The CF representation of $\pi$ is

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \frac{1}{1 + \cdots}}}}}.$$  

Its successive convergents are $(3/1)$, $(22/7)$, $(333/106)$, $(355/113)$ etc.,. We observe that

$$\left|\pi - \frac{355}{113}\right| < 3 \times 10^{-7}.$$  

This shows how fast the CF is converging to the exact value.

The representation of a CF is not unique. The CF $\left(\frac{1.5}{1}\right)$ can be transformed by equivalence transformation by multiplying the numerator and denominator of successive fractions by numbers different from zero:

$$\frac{c_1a_1}{c_1b_1} + \frac{c_1c_2a_2}{c_1c_2b_2} + \frac{c_2c_3a_3}{c_2c_3b_3} + \cdots.$$  

Different types of CFs exist and they are $C$-fractions, regular $C$-fractions, $S$-fractions, $g$-fractions, associated CFs, $J$-fractions, $T$-fractions, general $T$-fractions and $M$-fractions. A detailed study of these CFs is carried out by Jones and Thron (1980) and Brezinski (1991).

The CF which we have used throughout the thesis is the J-fraction and it is given by

$$\frac{1}{z + b_1 - \frac{a_1}{z + b_2 - \frac{a_2}{z + b_3 - \cdots}}}.$$
The $n^{th}$ convergent of this fraction is denoted by
\[ \frac{A_n}{B_n}. \]
By using the recurrence relations given in (1.6), $n^{th}$ numerator and denominator polynomial can be represented in a determinant form. The zeros of these polynomials are the eigenvalues of a tridiagonal matrix which is obtained by using the properties of tridiagonal determinants. There is a close connection between BDPs and CFs (Flajolet and Guillemin (2000)).

There are many types of CFs exist and which are $C$-fractions, regular $C$-fractions, $S$-fractions, $g$-fractions, associated CFs, $J$-fractions, $T$-fractions, general $T$-fraction and $M$-fractions. A detailed study of these CFs has been carried out by Jones and Thron (1980).

The CF which we have used in this thesis is the $J$-fraction and it is given by
\[ f(x) = \frac{1}{z + b_0 - \frac{c_0}{z + b_1 - \frac{c_1}{z + b_2 - \cdots}}} \]  
(1.8)
If $b_n = a_{2n} + a_{2n-1}$, $c_n = a_{2n}a_{2n+1}$, $n = 0, 1, \ldots$ with $a_{-1} = 0$ and $a_n > 0$ then we write a CF (1.8) as follows:
\[ f(x) = \frac{x}{1+ \frac{a_0 x}{1+ \frac{a_1 x}{1+ \frac{a_2 x}{1+ \cdots}}}} \]  
(1.9)
where $x = 1/z$ and it is known as $S$-fraction (Stieltjes fraction).

In the study of CFs there are two main problems which arise in a rather natural way. First is the problem of obtaining a continued fraction expansion making use of the coefficients of a given power series, second being converse of the first, that is, to evaluate the power series coefficients from a known continued fraction expansion.

Let us assume that the power series expansion of a given CF be
\[ \frac{1}{1+ \frac{a_0 x}{1+ \frac{a_1 x}{1+ \frac{a_2 x}{1+ \cdots}}} = \sum_{n=0}^{\infty} A_n (-x)^n } \]  
(1.10)
with $A_0 = 1$.

The problem of converting a continued fraction into a power series has been studied by many distinguished mathematicians and number of partial results are known (Rogers (1907), Wall (1948), Zajta and Pandikow (1975), Flajolet (1980), Goulden and Jackson (1983), Berndt (1989)).

(i) Rogers (1907) has given the first few coefficients of the above power series and commented that, “the power series coefficients from CF can be determined successively, but apparently not generally.”

(ii) Ramanujan (Entry 17, Berndt (1989)) gives a recurrence relation for these coefficients.

(iii) Wall (1948) has given a power series expansion for the $J$-fraction and these coefficients are obtained from two dimensional recurrence relations which involves an infinite Stieltjes matrix equation.

(iv) The power series coefficients of a CF (1.10) given by Zajta and Pandikow (1975) are

\[
A_n = A_n(a_0, a_1, a_2, \ldots, a_{n-1}) = \sum_{a_0} a_0^{n_0} \prod_{k=1}^{n-1} \left( \frac{n_{k-1} + n_k - 1}{n_k} \right) a_k^{n_k}, \quad n = 1, 2, \ldots (1.11)
\]

where the exponents $n_k$ ($k = 0, 1, 2, \ldots, n - 1$) are non-negative integers and the summation is taken over all partitions of positive integers $n$, with parts $n_k$, $\sum_{k=0}^{n-1} n_k = n$. The coefficients $A_n$ are also satisfy a convolution formula

\[
A_{n+1}(a_0, a_1, a_2, \ldots, a_n) = a_0 \sum_{k=0}^{n} A_k(a_0, a_1, a_2, \ldots, a_{k-1}) A_{n-k}(a_1, a_2, a_3 \ldots, a_{n-k})
\]

with $A_0 = 1$ and $A_1 = a_0$

(v) Flajolet (1980), Goulden and Jackson (1983) have given a geometrical interpretation for this CF (1.10) and have obtained the power series coefficients (1.11) by a direct “non computational” approach.
Parthasarathy and Sudhesh (2006) have given a procedure to find the power series coefficients of CFs and obtained these coefficients in terms finite summation.

1.4 Connection Between BDPs and CFs

Approximate transient system size probability values for a BDP can be obtained using the CF technique as explained below.

Denote
\[ P_n^*(s) = \int_0^\infty e^{-st}P_n(t)dt, \quad n = 0, 1, 2, \ldots, \]
where, for notational convenience, we use \( P_n(t) \) for \( P_{mn}(t) \).

Taking Laplace transform of the system of equations given by (1.1) and assuming that there are zero units initially in the system, \( P_0^*(s) \) simplifies to the expression
\[ P_0^*(s) = \frac{1}{s + \lambda_0 - \mu_1 P_0^*(s)}. \] (1.12)

Similarly, from (1.1),
\[ \frac{P_n^*(s)}{P_{n-1}^*(s)} = \frac{\lambda_{n-1}}{s + \lambda_n + \mu_n - \mu_{n+1} \frac{P_{n+1}^*(s)}{P_n^*(s)}}, \quad n = 1, 2, 3, \ldots. \] (1.13)

Iterating this equation, we get, for \( n = 1, 2, 3, \ldots \),
\[ \frac{P_n^*(s)}{P_{n-1}^*(s)} = \frac{\lambda_{n-1}}{s + \lambda_n + \mu_n - \frac{\lambda_{n-1} P_{n-1}^*(s)}{s + \lambda_{n+1} + \mu_{n+1} - \cdots}}. \] (1.14)

This results in
\[ P_0^*(s) = \frac{1}{s + \lambda_0 - \frac{\lambda_0 \mu_1}{s + \lambda_1 + \mu_1 - \frac{\lambda_1 \mu_2}{s + \lambda_2 + \mu_2 - \cdots}}} \] (1.15)

Let the \( r^{th} \) approximant of (1.15) be given by \( \frac{A_r(s)}{B_r(s)} \) where
\[
\begin{align*}
A_1(s) &= 1 \\
A_2(s) &= s + \lambda_1 + \mu_1 \\
A_r(s) &= (s + \lambda_{r-1} + \mu_{r-1})A_{r-1}(s) - \lambda_{r-2}\mu_{r-1}A_{r-2}(s), \quad r = 3, 4, \ldots
\end{align*}
\] (1.16)
and

\[ B_1(s) = s + \lambda_0 \]
\[ B_2(s) = (s + \lambda_1 + \mu_1)B_1(s) - \lambda_0\mu_1 \]
\[ B_r(s) = (s + \lambda_{r-1} + \mu_{r-1})B_{r-1}(s) - \lambda_{r-2}\mu_{r-1}B_{r-2}(s), \ r = 3, 4, \ldots . \quad (1.17) \]

Then

\[ P^*_0(s) \approx \frac{A_r(s)}{B_r(s)}. \quad (1.18) \]

This expression is then inverted to the time domain by using the zeros of the denominator polynomial (Parthasarathy and Lenin (1998)). This CF methodology has been employed successfully in the numerical study of state-dependent Markovian queues, inventory systems and population models Parthasarathy et al. (1999)).

Conolly and Langaris (1992) and Parthasarathy and Lenin (1997) have applied CF methodology, which was till then used only to obtain numerical solutions, to obtain the transient solution of BDPs analytically. We also apply this technique to obtain analytically the transient buffer content distribution for the fluid models under consideration. Our methodology makes use of the connection between fluid queues, BDPs and CFs and involves the identification of the resultant CF with the well-known identities for particular forms of the birth and death rates. These are explained in subsequent Chapters.

1.5 Special Numbers

In this section we review some classical work on combinatorics.
Catalan Numbers

*Catalan numbers* \( C_n \) are given by

\[
C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}.
\]

The Catalan numbers appear in tree enumeration problem. The Catalan numbers turn up in many related types of problems. \( C_{n-1} \) gives the number of binary bracketings of \( n \) letters. The Catalan numbers also give the solution to the ballot problem, the number of rooted planar binary trees with \( n \) internal nodes, the number of non-crossing handshakes possible across a round table between \( n \) pairs of people etc.

The generating function of the Catalan numbers is given by

\[
C(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z},
\]

for \(|z| \leq 1/4\), where \( C(z) \) is a complex function.

1.6 Description of the Research Work

The thesis will have six chapters; a brief description of which follows.

Chapter 1

Chapter 1 presents a short introduction to importance of transient analysis, birth and death process with in the presence and in the absence of catastrophes, state-dependent queues and working vacation. It also introduces related concepts like usual birth and death processes, continued fractions and their relations. Also, the problem of converting a continued fraction into a power series has been studied in detail.
We use CF techniques to obtain analytically the time-dependent probabilities of number of customers in the system for state-dependent queues. These are explained in subsequent Chapters.

**Chapter 2**

The aim of the chapter 2 is to investigate the transient solutions of a general state-dependent birth and death process with catastrophe. Further we discussed a connection between birth and death process with in the presence and in the absence of catastrophes. The steady-state probabilities of BDP with in the presence of catastrophe are also discussed. In the study of BDP with catastrophe, the underlying forward Kolmogorov differential-difference equations are first transformed into a set of linear algebraic equations using Laplace transforms. The algebraic equations are converted as continued fractions and these expressed as power series. The inversion power series leads to closed form transient probabilities of state-dependent BDP with catastrophe. Several interesting examples are provided to illustrate this approach.

Let $P_n(t)$ and $\hat{P}_n(t)$ be the probability that time-dependent system size probabilities of a birth-death process with catastrophes and removing the possibility of catastrophes respectively. Then for $n = 1, 2, 3, \ldots$,

$$P_n(t) = e^{-\xi t} \hat{P}_n(t) + \xi \int_0^t e^{-\xi y} \hat{P}_n(y) \, dy,$$

then for $n = 1, 2, 3, \ldots$ the probabilities $\hat{P}_n(t)$ are given by

$$\hat{P}_n(t) = L_{n-1} \sum_{m=0}^{\infty} (-1)^m A(m, 2n) \frac{t^{m+n}}{(m+n)!},$$

where

$$A(m, n) = \sum_{i_1=0}^{n} a_{i_1} \sum_{i_2=0}^{i_1+1} a_{i_2} \sum_{i_3=0}^{i_2+1} a_{i_3} \cdots \sum_{i_m=0}^{i_{m-1}+1} a_{i_m} \quad \forall \ n \in \mathbb{N},$$
and \( L_{n-1} = a_0a_2a_4 \cdots a_{2n-2} \), \( L_{-1} = 1 \) with \( A(0, n) = 1 \).

Also, \( P_n(t) \) is given by

\[
P_n(t) = L_{n-1} \sum_{m=0}^{\infty} (-1)^m \frac{A(m, 2n)}{\xi^{m+n}} - L_{n-1} e^{-\xi t} \sum_{k=0}^{\infty} \frac{(\xi t)^k}{k!} \sum_{m=\max(k-n+1,0)}^{\infty} (-1)^m \frac{A(m, 2n)}{\xi^{m+n}}
\]

**Chapter 3**

Chapter 3 aims at presenting an alternate approach to derive the exact transient system size probabilities of Markovian queues. A connection between continued fraction and its power series coefficients are given and the power series coefficients are obtained in closed form. It has been proved by Wall (1948) that, in general, a power series in \( x \), \( \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots \), can be represented as a continued fraction of the form

\[
\frac{1}{b_1 + \frac{a_1}{b_2 + \frac{a_2}{b_3 + \ldots}}}
\]

then it can be expressed as a power series of the form

\[
P \left( \frac{1}{x} \right) = \sum_{r=0}^{\infty} (-1)^r \frac{c_r}{x^{r+1}}
\]

The coefficients of the continued fraction can be connected by the relations

\[
c_{r+q} = k_{0p}k_{oq} + a_1 k_{1,p}k_{1,q} + a_1 a_2 k_{2,p}k_{2,q} \ldots ,
\]

where

\( k_{00} = 1; \ k_{r,s} = 0 \) if \( r > s \) and where the \( k_{r,s} \) for \( s \geq r \) are given recurrently by the matrix equation

\[
\begin{pmatrix}
k_{00} & 0 & 0 & \cdots \\
k_{01} & k_{11} & 0 & \cdots \\
k_{02} & k_{12} & k_{22} & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
b_1 & 1 & 0 & 0 & \cdots \\
a_1 & b_2 & 1 & 0 & \cdots \\
0 & a_2 & b_3 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
= 
\begin{pmatrix}
k_{01} & k_{11} & 0 & 0 & 0 & \cdots \\
k_{02} & k_{12} & k_{22} & 0 & 0 & \cdots \\
k_{03} & k_{13} & k_{23} & k_{33} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]
The given matrix relation is can be written as

\[ k_{mn} = \begin{cases} 
0 & \text{if } m > n \\
1 & \text{if } m = n \\
\quad \quad \quad \quad \quad k_{m-1,n-1} + b_{m+1}k_{m,n-1} + a_{m+1}k_{m+1,n-1} & \text{if } 0 < m < n
\end{cases} \]

The power series coefficient \( k_{0n}, n = 0, 1, 2, \ldots \) is obtained in closed form. The power series coefficient is used to evaluate the time-dependent system size probabilities and busy period distribution of a classical single server queue are deduced using continued fraction and its power series. Numerical illustrations are also presented.

Further, a new approach is discussed to find the stationary probabilities of general state-dependent Markovian queues in this chapter. First we find the steady-state probabilities of finite birth and death process from the continued fractions and its associated tridiagonal determinants. The continued fraction of finite BDP is expressed as a rational function of two tridiagonal determinants. That is,

\[
f_0(s) = \frac{A_N^{(1)}(s)}{B_{N+1}(s)},
\]

\[
f_n(s) = \left[ \prod_{i=1}^{n} a_{2i-2} \right] \frac{A_{N-n}^{(2n+1)}(s)}{A_N^{(1)}(s)} f_0(s), \quad n = 1, 2, \ldots, N,
\]

where \( f_n(s) \) are Laplace transform of \( P_n(t), n = 0, 1, 2, \ldots \) and

\[
B_{N+1} = \begin{vmatrix}
    s + a_0 & a_0 \\
    a_1 & s + a_1 + a_2 & a_2 \\
    a_3 & s + a_3 + a_4 & \ddots \\
    a_{2N-1} & s + a_{2N-1}
\end{vmatrix}_{(N+1) \times (N+1)}
\]

and for \( n = 0, 1, 2, \ldots N \), \( A_{N-n}^{(2n+1)}(s) \) is obtained from \( B_{N+1}(s) \) by deleting the first \( n + 1 \) rows and first \( n + 1 \) columns with \( A_0^{(1)}(s) = 1. \)
The tridiagonal determinants are expressed as polynomials. The applications of limit theorems in Laplace transform yields the stationary probabilities of finite capacity state-dependent queues. Further, the stationary probabilities of infinite systems are also deduced.

Chapter 4

In chapter 4, we derive the time dependent system size probabilities of finite state BDP with state-dependent rates. The associated continued fractions are expressed as a rational function of tridiagonl determinants and this can be expressed as a power series. The inversion of power series yield the time-dependent system size probabilities. The empty state system size probability is given by

\[ p_0(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{k=0}^{\min(m,N)} (-1)^{k+\max(m-N, 0)} \theta_{\min(m,N)-k} \psi_{k+\max(m-N, 0)} \]

where \( \phi_n \)'s, \( \theta_n \)'s and \( \psi_n \)'s are calculated as follows:

\[
\phi_n = \sum_{i_1=1}^{2} \sum_{i_2=i_1+2}^{4} \cdots \sum_{i_{n-1}=i_{n-1}+2}^{2n} a_{i_1}a_{i_2} \cdots a_{i_n} \phi_0 = 1,
\]

\[
\theta_n = \sum_{i_1=2}^{2} \sum_{i_2=i_1+2}^{4} \cdots \sum_{i_{n-1}=i_{n-1}+2}^{2n} a_{i_1}a_{i_2} \cdots a_{i_n}, \theta_0 = 1
\]

\[
\psi_n = \begin{bmatrix}
\phi_1 & \phi_0 & 0 & 0 & 0 & \ldots & 0 \\
\phi_2 & \phi_1 & \phi_0 & 0 & 0 & \ldots & 0 \\
\phi_3 & \phi_2 & \phi_1 & \phi_0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\phi_{n-1} & \phi_{n-2} & \cdots & \cdots & \phi_0 \\
\phi_n & \phi_{n-1} & \phi_{n-2} & \cdots & \phi_1
\end{bmatrix}
\]
Chapter 5

The transient probabilities of birth and death processes with state- and time-dependent birth and death rates cannot be obtained easily. However, for an unusual choice of these parameters, these probabilities are obtained in closed form.

Chapter 6

The main objective of chapter 6 is to derive the time-dependent system size probabilities of classical M/M/1 queue with vacations, in which the server works with different service rates rather than completely stops service during the vacation period. Service times during vacation period, service times during service period and vacation times are all exponentially distributed. An exact transient system size probability distributions are derived in closed form.

Let $P_{jr}(t)$ be the probability that there are $j$ customers in the system when $r = B$ or $V$. If $r = B$, the server is not on a working vacation and $r = V$, the server is on working
vacation. Then for $j = 1, 2, 3\ldots$

$$P_jB(t) = \gamma \int_0^t e^{-(\lambda + \mu_B)(t-y)} \sum_{m=1}^{\infty} \beta_B^{j-m} P_mV(y) [I_{j-m}(\alpha_B(t-y)) - I_{j+m}(\alpha_B(t-y))] dy,$$

$$P_jV(t) = \lambda \beta_V^{j-1} \int_0^t e^{-(\lambda + \mu_V + \gamma)y} [I_{j-1}(\alpha_V y) - I_{j+1}(\alpha_V y)] P_{00}(t-y) dy$$

where

$$P_{00}(t) = \frac{2}{\alpha V} \sum_{m=0}^{\infty} \left( \frac{\beta_V}{\beta_B} \right)^m \sum_{n_1+n_2+n_3=m} \frac{m!}{n_1! n_2! n_3!} \left( \frac{\beta_B(\mu_V + \gamma)}{\lambda} \right)^{n_2} \left( \frac{\alpha V \gamma - \mu_V - \gamma}{\sqrt{\lambda \mu_V}} \right)^{n_3}$$

$$\times \left\{ \frac{\alpha V \alpha_B}{4} [I_{m+n_3}(\alpha_V t) - I_{m+n_3+2}(\alpha_V t)] e^{-(\lambda + \mu_V + \gamma)t} \ast [I_{n_1+n_3-1}(\alpha_B t) - I_{n_1+n_3+1}(\alpha_B t)] e^{-(\lambda + \mu_B)t} - \lambda \mu_B [I_{m+n_3+1}(\alpha_V t) - I_{m+n_3+3}(\alpha_V t)] e^{-(\lambda + \mu_V + \gamma)t} \ast [I_{n_1+n_3}(\alpha_B t) - I_{n_1+n_3+2}(\alpha_B t)] e^{-(\lambda + \mu_B)t} \right\}.$$

Further, we obtained the time-dependent system size probability in the absence of working vacation. A numerical illustration is provided by considering various parameters.