Chapter 4

EoS in Quantum RHO

4.1 Introduction

In this model quarks are under the action of a Lorentz scalar plus a Lorentz vector mean field potential of the form

$$V = (1 + \gamma_0) A^2 r^2$$  \hspace{1cm} (4.1)

where $A$ is the confinement mean field parameter, $\gamma$ is the Dirac matrix,

$$\gamma_a = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and $r$ is the mean distance of the particle from the center. The confined state of the quarks under this potential is given by the Dirac equation

$$[i\gamma^\mu \partial_\mu - M_q - V_r] \psi_q(r) = 0$$ \hspace{1cm} (4.2)
where $M_q$ is the quark mass.

This model had been used satisfactorily to describe light hadron spectroscopy, baryon magnetic moments as well as properties of glue balls. RHO model is more realistic in its approach compared to MIT Bag Model. Here we approach the problem from the known fact that RHO model of hadrons quite successfully explains many properties of hadrons. QGP formed in RHICs and compact stars may be viewed as a large hadron with large number of quarks and gluons, confined in a RHO confining potential. Hence, a study of statistical mechanics and thermodynamics of such a system will give EoS of QGP. As a first step, here we neglect effects like colour-coulombic interactions, hadron formations etc and see how the EoS of QGP differs from ideal EoS. Earlier, similar calculations has been done [1] violating the Stefan-Bolzmann law and hence here we reconsider the problem using the method applied in Landau’s theory of diamagnetism and de Hass Alphen effect [2] for the calculation of number of states.

4.2 EoS of Gluon Plasma in Quantum RHO

Let us now discuss EoS of QGP in RHO model. From the earlier study of hadrons in RHO model, we have the single particle energy levels [3]

$$\epsilon_n^2 = C_g^2 (2n + 1) \quad (4.3)$$

for gluons where $C_g$ is the spring constant of gluon in RHO confining potential. $n = 1, 2, 3, \ldots$ are the oscillator quantum numbers. Let us consider gluon plasma without quarks which is a Bose system. Following the stan-
standard procedure of statistical mechanics, pressure or logarithm of partition function is given by,

$$\frac{PV}{T} = -\sum_{n=1}^{\infty} g_n g_I \ln(1 - e^{-\beta \epsilon_n}), \quad (4.4)$$

where $g_n$ is the degeneracy appropriate to finite volume. Here $V$ is the volume, $\beta$ is the inverse temperature and $g_I = 16$ is the internal degrees of freedom of gluons. Note that $g_n = n(n + 1)/2$ is a degeneracy for RHO with infinite volume. Consider a particle bounded inside a volume $V$, but subjected to RHO potential. We can calculate $g_n$ by evaluating the number of microstates in the phase-space volume at zero potential. This is based on the fact that the almost continuous set of the zero-potential levels ‘coalesce together’ to discrete oscillator levels. Hence, the number of microstates bounded by the energies corresponding to $n$ and $n + 1$ may appear as the degeneracy of the $n^{th}$ level. This degeneracy can be calculated as below.

In continuum 3D phase space the number of states between $p$ and $p + dp$ is given by

$$g(p)dp = \frac{4\pi p^2 dp V}{3h^3} \quad (4.5)$$

Analogous to this in discrete phase space the number of states can be written as

$$g_n = \frac{4\pi V (p_{n+1}^3 - p_n^3)}{3h^3} \quad (4.6)$$

The corresponding $p_{n+1}$ and $p_n$ are substituted and then $g_n$ is

$$g_n = \frac{V C_g^3}{6\pi^2} \left( (2n + 3)^{3/2} - (2n + 1)^{3/2} \right) \quad (4.7)$$

in natural units. Note that the above expression is just a generalization of the degeneracy used in Landau’s theory of diamagnetism and de Hass
Alphen effect [2]. In Landau’s theory, particles are confined in 2-dimensions perpendicular to a uniform magnetic field inside a finite volume whereas here we have particles, confined in 3-dimensions by RHO potential inside a finite volume $V$. Since gluons are bosons, the fugacity $z = 1$. For fermions like quarks $z \neq 1$ and pressure is given by,

$$\frac{PV}{T} = \sum_{n=1}^{\infty} g_n g_f \ln(1 + z e^{-\beta \epsilon_n})$$  \hspace{1cm} (4.8)$$

where $g_f$ is the internal degrees of quarks. Above expressions for pressure are exact and one need to do infinite sum which may be done numerically. Substituting $g_n$ from Eq (4.7), $\epsilon_n$ from Eq (4.3), Eq (4.4) becomes

$$\frac{P}{T} = -\frac{C_g^3 g_f}{6\pi^2} \sum_{n=1}^{\infty} ((2n + 3)^{3/2} - (2n + 1)^{3/2}) \ln(1 - e^{-\frac{C_g}{T} \sqrt{2n+1}})$$  \hspace{1cm} (4.9)$$

Changing the summation from 1 to $\infty$ to 0 to $\infty$ and taking $n = 0$

$$\frac{P}{T} = -\frac{C_g^3 g_f}{6\pi^2} 3\sqrt{3} \ln \left( 1 - e^{-\frac{C_g}{T}} \right)$$

and with

$$\ln(1 - x) = \sum_{l=1}^{\infty} (-1)^l \frac{x^l}{l}$$

Eq (4.9) can be rewritten as

$$\frac{P}{T} = \frac{C_g^3 g_f}{6\pi^2} \sum_{l=1}^{\infty} \frac{1}{l} \sum_{n=0}^{\infty} ((2n + 3)^{3/2} - (2n + 1)^{3/2}) e^{-\frac{C_g}{T} \sqrt{2n+1}}$$

$$+ \frac{C_g^3 g_f}{6\pi^2} 3\sqrt{3} \ln \left( 1 - e^{-\frac{C_g}{T}} \right)$$  \hspace{1cm} (4.10)$$

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Since the levels are continuously placed we can use the Euler-Macularin summation formula

\[ \sum_{n=a}^{b} f(n) = \int_{a}^{b} f(n) \, dn + \frac{f(a) + f(b)}{2} \]

Then the Eq (4.10) becomes

\[
\begin{align*}
P \frac{T}{T} &= \frac{C^3_gI}{6\pi^2} \sum_{l=1}^{\infty} \frac{1}{l} \sum_{n=0}^{\infty} ((2n + 3)^{3/2} - (2n + 1)^{3/2}) \, e^{-\frac{C_gI}{T} \sqrt{2n+1}} \\
&\quad + \frac{C^3_gI}{12\pi^2} 3\sqrt{3} \ln \left(1 - e^{-\frac{C_g}{T}}\right) \tag{4.11}
\end{align*}
\]

Using binomial expansion

\[
(2n + 3)^{3/2} = (2n + 1)^{3/2} + 3 (2n + 1)^{1/2} + \frac{3}{2} (2n + 1)^{-1/2} - \frac{1}{2} (2n + 1)^{-3/2} + ...
\]

and neglecting the higher order terms we get

\[
\begin{align*}
P \frac{T}{T} &= \frac{C^3_gI}{6\pi^2} \sum_{l=1}^{\infty} \frac{1}{l} \sum_{n=0}^{\infty} (3(2n + 1)^{1/2} + \frac{3}{2} (2n + 1)^{-1/2}) \, e^{-\frac{C_gI}{T} \sqrt{2n+1}} \\
&\quad + \frac{C^3_gI}{12\pi^2} 3\sqrt{3} \ln \left(1 - e^{-\frac{C_g}{T}}\right) \tag{4.12}
\end{align*}
\]

Putting \(2n = x\) the above expression becomes
\[
\frac{P}{T} = \frac{C_g^3 g_I}{6\pi^2} \sum_{l=1}^{\infty} \frac{1}{l} \int_{n=0}^{\infty} \left( 3(x + 1)^{1/2} + \frac{3}{2}(x + 1)^{-1/2} \right) e^{-\frac{C_g}{T}\sqrt{x+1}} dx \\
\quad + \frac{C_g^3 g_I}{12\pi^2} 3\sqrt{3} \ln \left( 1 - e^{-\frac{C_g}{T}} \right)
\]  

(4.13)

Using the value for the standard integral

\[
\int_0^{\infty} \frac{x^{\nu-1} e^{-\beta\sqrt{1+x}}}{\sqrt{1+x}} dx = \frac{2}{\sqrt{\pi}} \left( \frac{\beta}{2} \right)^{\frac{1}{2}-\nu} \Gamma(\nu) K_{\frac{1}{2}-\nu}(\beta)
\]

(4.14)

the above equation becomes

\[
\frac{P}{T} \approx \frac{g_I}{\pi^2} \sqrt{\frac{2}{\pi}} (C_g T)^{3/2} \sum_{l=1}^{\infty} \frac{1}{15/2} \left[ K_{3/2}(\frac{C_g}{T}l) + 3 \frac{C_g}{4} T \frac{K_{1/2}(\frac{C_g}{T}l)}{K_{1/2}(\frac{C_g}{T})} \right] \\
\quad + \frac{C_g^3 g_I}{12\pi^2} (3\sqrt{3} - 1) \ln(1 - e^{-\frac{C_g}{T}})
\]

(4.15)

for gluon plasma, in terms of modified Bessel functions \(K_\nu(x)\). This is the EoS for gluon plasma in quantum RHO. At extremely high temperature, confinement effects may be negligible and we expect to get EoS of relativistic free gas which we indeed get as shown below. As \(T \to \infty\),

\[
P \to \frac{g_I \pi^2}{90} T^4 \left( 1 + \frac{45}{4\pi^2} \frac{C_g^2}{T^2} + \ldots \right)
\]

(4.16)

\[
\to \frac{g_I \pi^2}{90} T^4 \equiv P_s
\]

Let us compare our results with a similar work in [1] which claims a \(T^7\) law for pressure. However, it is easy to see, from their calculation with infinite volume, that the \(T^7\) law is for the logarithm of partition function and pressure, in fact, is zero as in any harmonic oscillator with infinite volume.
They have put an additional statistical weights $g_v$, involving volume, which gives nonzero pressure. In fact, in recent similar calculations in BEC [4], there is no term involving volume and they assume that the system is of infinite extent. Therefore, one way to study SM of a system with finite volume with particles subjected to external confining potential is one we used here, borrowed from the theory of diamagnetism and de Hass Alphen effect. Next let us compare our EoS with the results of lattice simulation of QGP or lattice gauge theory (LGT).

### 4.3 Comparison with LGT

We can see, from Eq. (4.16), that our EoS of QGP is temperature dependent modification of normal $T^4$ law. Similar modifications is seen in LGT calculations and only at $T \to \infty$, one expects $T^4$ law. As an example, let us consider pure gauge QGP or gluon plasma and the pressure is, Eq. (4.4),

$$
\frac{P}{T} = -\frac{C_g^3 g_I}{6\pi^2} \sum_{n=1}^{\infty} ((2n + 3)^{3/2} - (2n + 1)^{3/2}) \ln(1 - e^{-\frac{C_g}{T} \sqrt{2n+1}}), \quad (4.17)
$$

which may be written as

$$
p \equiv \frac{P}{P_s} = -\frac{a^3 g_I}{6\pi^2} \frac{1}{t^3} \sum_{n=1}^{\infty} ((2n + 3)^{3/2} - (2n + 1)^{3/2}) \ln(1 - e^{-\frac{a}{t} \sqrt{2n+1}}), \quad (4.18)
$$

where $P_s$ is the Stefan-Boltzmann limit pressure and $t \equiv T/T_c$. $T_c$ is the critical temperature of the transition of hadrons to QGP and the parameter $a$ is defined as, $a \equiv C_g/T_c$.

Above equation may be compared with LGT results by varying the
Figure 4.1: Plots of $\frac{P}{P_s}$ as a Function of $\frac{T}{T_c}$ in Quantum RHO Potential and Lattice Results(+) for Pure Gauge

parameter $a$ as discussed at the end. Knowing $P$ or partition function, all other thermodynamic quantities like energy density, entropy, sound speed etc. may be derived. Further extension of our calculations for QGP with finite number of quark flavours, without or with finite chemical potential, is straightforward.

4.4 Results

On summarizing our results, in Fig. (4.1), we plotted our results for pressure in terms of $p \equiv P/P_s$ as function of $t \equiv T/T_c$ along with LGT results. By varying the parameter $a$, so as to get a similar behaviour as LGT results as function of $t$, we get $a = 2.3$. However, the curve lies slightly above the lattice data. We may correct our pressure by subtracting a constant $\Delta$ such
that both the curve matches at large $t$ values and we get good fit except near to $T = T_c$. The value of $\Delta$ here is 0.1. It is not unreasonable to subtract by $\Delta$ because we have neglected the mutual interactions between particles, mainly, attractive colour-coulombic interactions. We know that attractive mutual interactions among particles reduces the pressure. From the expression for $a$, we may estimate $T_c = C_g/a \approx 200\,\text{MeV}$ for $C_g = 2.35\,\text{fm}^{-1}$. Once pressure or partition function is obtained all other thermodynamic functions like $\varepsilon$, $C_v^2$ etc. may be easily evaluated. Similar comparison of our results and LGT results may be carried for other systems with flavoured QGP.
Bibliography


