CHAPTER 3

INTERACTION MODELING USING RELAY FEEDBACK TEST

The descriptions of various multivariable processes are discussed in the previous chapter. In this chapter these multivariable processes are used to formulate input-output time domain models using relay feedback responses are briefly explained. Generally the off-diagonal elements of transfer function matrix of MIMO system are responsible for interactions between inputs and outputs. When there is an excitation in the input signal, the process outputs get interaction effect became induced resulting rise in undesired responses. These responses can be analyzed by understanding the time domain dynamics of interactive (off-diagonal) elements of closed-loop transfer function. Hence, relay feedback responses are collected and modeled in time domain which can be used for analyzing cause of interaction.

3.1 INTRODUCTION

This section discusses the state of art review on system modeling using relay feedback for both SISO and MIMO systems. The relay feedback test is first proposed by Astrom and Hagglund (1984) for process control practitioners as a closed-loop tool for system identification and control. In this method periodic step functions are given as input to unknown process and outputs are collected (that lags behind the input by \( \pi \) radians) from where limit cycle data \((K_u\) and \(\omega_n\)) are found to approximate the process
model parameters. Luyben (1987) derived the transfer function models for highly non-linear distillation column using relay feedback method. The progress in the relay feedback autotuning is comprehensively documented in the book by Yu (1999). Wang et al (1997) derived expressions for periods and amplitudes of limit cycle under relay feedback. In this method, the time-domain information is then combined with frequency response point estimation using fourier series expansion of limit cycles to identify FOPDT model in single relay test. Atherton and Majhi (1998) derived a set of general expressions from a single asymmetrical relay feedback test for online process identification. These expressions are used to obtain the parameters of stable and unstable FOPDT and SOPDT transfer function models by using the initial limit cycle data. Thyagarajan and Yu (2003) reported mathematical model for stable and unstable FOPDT processes using relay feedback tests. Panda and Yu (2003) derived analytical expressions for the relay feedback responses of various benchmark processes based on the observation that a relay feedback test consists of series of step inputs and a stable limit cycle implies a convergent infinite series. Lee et al (2007) obtained the simple and accurate estimates of process ultimate information without modifying the original relay feedback system. This method uses various integral of relay responses instead of point data to reduce the effects of high harmonic terms. Majhi (2007) derived a set of general expressions from the relay feedback test for the identification of non-minimum phase SISO process.

Low order process modeling provides a basis for control system design and on-line autotuning in process control. Liu and Gao (2008) proposed a systematic on-line identification method to obtain SOPDT model from a single biased/unbiased relay feedback test. Exact expressions of the corresponding limit cycles are derived for assessing process response under the relay feedback test. But these processes are based on SISO where there
is no interaction between input and output. As chemical systems are dominated in time delay and most of them are multivariable in nature, Shen and Yu (1994) proposed automatic tuning of multivariable systems based on the concept of sequential identification-design. Wang et al (1997) presented relay feedback based parameter estimation for MIMO systems. Palmor et al (1995) and Wang and Cluett (2000) discussed about the stability of limit cycles in decentralized MIMO systems. These methods consider desired response (for example in case of 2-by-2 system) output \(y_1\) and input \(u_1\) for system identification and analysis neither discussed about the methods to reduce interactions nor giving any exact analytical expressions for the relay responses. This may be helpful in analyzing the interaction behavior between input/ output that provides information regarding closed-loop parameters of the MIMO system. Moreover, it has been felt that exact model parameters and information on interactions can be better obtained/ calculated from proper mathematical model of relay responses for MIMO systems. As the off-diagonal closed loop transfer functions contain information on interactions, it is better to analyze the control system based on their time domain characteristics. With this objective, time domain expressions of relay responses for interactive transfer functions are developed in this chapter. Relay tests are carried out on MIMO processes (2-by-2 and 3-by-3 MIMO systems with square matrix structures and subsystem transfer functions of FOPDT type) and undesired responses are modeled.

### 3.1.1 Relay Feedback Test on SISO Systems

When a relay output lags behind the input by \(\pi\) radians, the closed-loop system oscillates around set point with an ultimate period of \(P_u\). If relay of magnitude ‘h’ is inserted in the feedback loop, the input \(u(t)\) becomes ‘h’ which is shown in Figure 3.1.
The output starts increasing after a time delay of $D$. The controller output switches to opposite direction and becomes $u(t) = -h$ which is shown in Figure 3.2 (a). Figure 3.2 (b) shows the formation of limit cycle of amplitude $a$ with a phase lag of $\pi$ and the process variable signal crosses the set point. The value of amplitude is used to calculate ultimate gain, $(K_u)$ and the period of oscillation, $(P_u)$ is used to calculate ultimate frequency, $(\omega_u)$. Hence $K_u=4h/\pi a$ and $\omega_u=2\pi/ P_u$, these limit cycle data are used to calculate controller tuning parameters.

Initially to demonstrate the method, it is considered a FOPDT system and ideal relay. It is assumed that relay output is generated by a
series of small number of step changes in manipulated variable. Hence, the stabilized output is the sum of infinite terms of small responses due to those step changes and the actual relay output lags the input by an amount \( D \) (time delay). The shifted version of typical relay feedback response and a particular interval is designated as \([1 \text{ or } 2 \text{ or } 3, \text{ etc.}]\) which is shown in Figure 3.3. Responses in the first interval is \( y_1 \), second interval is \( y_2 \), etc. and found that the responses are upward when \( n \) is even and downward when it is odd.

Thus expressions are developed for \( y_1, \ y_2, \ y_3, \ldots, \ y_n \). The generalized expression for FOPDT systems using relay feedback test can be obtained as:

\[
y_n = k_p \left( 1 - e^{\frac{-2}{T_D}} \right) \left( \frac{2}{1 + e^{\frac{-2}{T_D}}} \right)
\]

In Figure 3.3 the stabilized oscillations give some informations like oscillation frequency, amplitude, etc. which are very useful in identifying FOPDT system parameters.

![Figure 3.3 Shifted version of relay response of a FOPDT system](image)
3.1.2 Relay Feedback Test on MIMO Systems

As shown in Figure 3.4, consider n-by-n MIMO systems with decentralized PI controller.

![Block diagram of n-by-n multivariable systems with decentralized PI controller](image)

**Figure 3.4** Block diagram of n-by-n multivariable systems with decentralized PI controller

The 2-by-2 MIMO systems described by:

\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} =
\begin{pmatrix}
g_{p11} & g_{p12} \\ g_{p21} & g_{p22}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
\]  

(3.2)

In autotuning procedure the initial relationship between \(y\) and \(u\) is simply as:

\[
\begin{pmatrix}
y_1 \\
u_1
\end{pmatrix}_{\text{OL}} = g_{p11}
\]  

(3.3)

where the subscript OL denotes the open-loop. When the loop is closed sequentially, the closed-loop relationship between \(y\) and \(u\) becomes...
complicated. Therefore it is considered that all the individual transfer functions of 2-by-2 MIMO systems are of FOPDT types.

The sequential tuning (Yu, 1999) yields diagonal closed-loop transfer functions which are as follows:

\[
g_{p11,\text{CL}} = g_{p11} \left( 1 - \frac{g_{p12} g_{p21} G_{C2}}{g_{p11} g_{p22} (1 + g_{p22} G_{C2})} \right) \tag{3.4}
\]

\[
g_{p22,\text{CL}} = g_{p22} \left( 1 - \frac{g_{p12} g_{p21} G_{C1}}{g_{p11} g_{p22} (1 + g_{p11} G_{C1})} \right) \tag{3.5}
\]

In this case the off-diagonal closed-loop transfer function can be found as:

\[
g_{p12,\text{CL}} = \frac{g_{p12}}{1 + G_{C1} g_{p11}} \tag{3.6}
\]

\[
g_{p21,\text{CL}} = \frac{g_{p21}}{1 + G_{C2} g_{p22}} \tag{3.7}
\]

The above Equations (3.6) and (3.7) give interaction measure in closed-loop sense and the MIMO autotuning can be done by sequential tuning procedures. Consider a 2-by-2 MIMO system with a known pairing \((y_1- u_1); (y_2- u_2)\) under decentralized control. Since, the sequential autotuning approach has several advantages, firstly it makes the problem simple. The reason is proposed approach treats the MIMO system as a sequence of SISO systems where the relay feedback system is proven to be useful and reliable. Secondly, it operates in an efficient manner and thirdly it is a more accurate approach in terms of identification.
The procedure for MIMO autotuning is described as follows:

As shown in Figure 3.5(a), initially a relay is placed between $y_1$ and $u_1$, while loop 1 is on manual mode. Following the relay feedback test, a controller for loop 1 can be designed from the ultimate gain and ultimate frequency.

The next step as shown in Figure 3.5(b), is to perform relay feedback test between $y_2$ and $u_2$, while loop 1 is on auto mode. A controller can be designed for loop 2 following the relay feedback test.

As shown in Figure 3.5(c), once the controller on loop 2 is set on auto mode, another relay feedback test is formed between $y_1$ and $u_1$. In general, a new set of tuning constants is found for the controller in loop 1 and this procedure is repeated until the controller parameter converges. Typically, the controller parameter converges in 3 - 4 relay feedback tests in the case of 2-by-2 MIMO systems.
Figure 3.5 (Continued)

(a) Step 1 of sequential method of tuning for 2-by-2 multivariable system. (b) Step 2 of sequential method of tuning for 2-by-2 multivariable system. (c) Step 3 of sequential method of tuning for 2-by-2 multivariable system.

3.2 PROPOSED METHOD/ MODELING OF 2-BY-2 MIMO SYSTEMS

3.2.1 Modeling of 2-by-2 MIMO Systems Using Ideal Relay Feedback Test

The procedure for obtaining mathematical model of SISO systems are already explained in detail (Panda and Yu, 2002, 2003). The ideal relay response for Equation (3.7) is assumed to be formed by n-number of small
step changes. At the first instant, (after synchronizing input with output by time shift) the response can be described as:

\[
y_1 = k_{21} \left[ 1 - e^{- \frac{t}{\tau_{11}}} \right] - \frac{k_{22}K_{c2}}{\tau_{12}} \left[ 1 - \left( \tau_{12} - \tau_{22} \right) e^{- \frac{t}{\tau_{22}}} \right] y_1(t-1)
\]  \tag{3.8}

where the term \(y_1(t-1)\) is one step ahead prediction of \(y_1\).

At the second instant, the time is reset to zero at the initial point. The step response (relay output) can be given by: (i.e., introducing a time shift by D amount in the Equation (3.8))

\[
y_2 = k_{21} \left[ 1 - e^{- \frac{t + D_{21}}{\tau_{11}}} \right] - 2 \left( 1 - e^{- \frac{t}{\tau_{11}}} \right) \frac{k_{22}K_{c2}}{\tau_{12}} \left[ 1 + \left( \tau_{12} - \tau_{22} \right) e^{- \frac{t + D_{22}}{\tau_{22}}} \right] - 2 \left( 1 + \left( \tau_{12} - \tau_{22} \right) e^{- \frac{t}{\tau_{22}}} \right) y_2(t-1)
\]  \tag{3.9}

where \(D_{21}\) and \(D_{22}\) denotes the time delay of the individual transfer functions of the system.

Equation (3.9) can be simplified as follows:

\[
y_2 = k_{21} \left[ 1 - 2 \left( 1 - e^{- \frac{t + D_{21}}{\tau_{11}}} \right) \right] \frac{k_{22}K_{c2}}{\tau_{12}} \left[ 1 - 2 \left( 1 + \left( \tau_{12} - \tau_{22} \right) e^{- \frac{t + D_{22}}{\tau_{22}}} \right) \right] y_2(t-1)
\]  \tag{3.10}

The relay response at the third interval lags by an amount \(D_{21} + P_d/2\) and \(D_{22} + P_d/2\) from an input of I and II parts.
Thus the above equation can be given as:

\[
y_3 = k_{21} \left[ 1 - e^{-\frac{t}{\tau_{11}}} - 2 \left( 1 - e^{-\frac{t}{\tau_{22}}} \right) + 2 \left( 1 - e^{-\frac{t}{\tau_{33}}} \right) \right] - \frac{k_{22} K_{e2}}{\tau_{11}} \left[ 1 + (\tau_{11} - \tau_{22}) e^{-\frac{t}{\tau_{22}}} + (\tau_{22} - \tau_{33}) e^{-\frac{t}{\tau_{33}}} + (\tau_{33} - \tau_{11}) e^{-\frac{t}{\tau_{11}}} \right] y_3(t-1)
\]

(3.11)

Equation (3.11) can be easily simplified as:

\[
y_3 = k_{21} \left[ 1 - 2 + 2 \right] - e^{-\frac{t}{\tau_{11}}} \left[ \frac{\sigma_1}{\gamma_1} + \frac{\sigma_2}{\gamma_2} - 2 e^{-\frac{t}{\gamma_3}} + 2 \right] - \frac{k_{22} K_{e2}}{\tau_{11}} \left[ 1 - 2 + 2 \right] + \left( \tau_{11} - \tau_{22} \right) e^{-\frac{t}{\gamma_2}} \left[ \frac{\sigma_1}{\gamma_1} + \frac{\sigma_2}{\gamma_2} - 2 e^{-\frac{t}{\gamma_3}} + 2 \right] y_3(t-1)
\]

(3.12)

It can be seen from the above equation that the terms in the RHS (i.e., \( y_n = y_1 + y_2 + y_3 + \ldots \)) is the sum of the step responses at \( t_1, t_2, t_3, \ldots \) are slowly forming a series. As the time tends to infinity the response becomes stabilized and it can be described as \( y_n = y_1 + y_2 + y_3 + \ldots \). In general, the ultimate properties are found out in 3 - 4 iterations which are enough to get converged properties of ultimate parameters (\( K_u \) and \( P_u \)).

\[
y_n = k_{21} \left[ 1 - 2 + 2 - \ldots \right] - e^{-\frac{t}{\tau_{11}}} \left[ \frac{\sigma_1}{\gamma_1} + \frac{\sigma_2}{\gamma_2} - 2 e^{-\frac{t}{\gamma_3}} + \ldots + 2 e^{-\frac{t}{\gamma_n}} - 2 \right] - \frac{k_{22} K_{e2}}{\tau_{11}} \left[ 1 - 2 + 2 - \ldots \right] + \left( \tau_{11} - \tau_{22} \right) e^{-\frac{t}{\gamma_2}} \left[ \frac{\sigma_1}{\gamma_1} + \frac{\sigma_2}{\gamma_2} - 2 e^{-\frac{t}{\gamma_3}} + \ldots + 2 e^{-\frac{t}{\gamma_n}} - 2 \right] y_n(t-1)
\]

(3.13)
Let \( r_1 = e^{\frac{p_1}{\tau_1}} \) and \( r_2 = e^{\frac{p_2}{\tau_2}} \)

In Equation (3.13) RHS has 4 parts:

\[
y_n = k_{21} \{ \text{First part} - \text{Second part} \} - \frac{k_{22}K_c}{t} \{ \text{Third part} + \text{Fourth part} \}
\]

The RHS of the above series becomes

First part = \([1-2+2-2+\ldots, \text{upto } \infty] = 1\]

Similar way Third part = 1

Second part = \[2 \left[ e^{\frac{-p_1}{\tau_1}} r_1^n - 2r_1^{n-1} + 2r_1^{n-2} - \ldots + 2r_1 - 2 \right]

This second part can be substituted into following series

Second part = \[2 \left[ 1 - r_1 + r_1^2 - r_1^3 + \ldots \right] = \frac{2}{1 + r_1} = \frac{2}{1 + e^{\frac{p_1}{\tau_1}}}

Similar way Fourth part = \[\frac{2}{1 + r_2} = \frac{2}{1 + e^{\frac{p_2}{\tau_2}}}

Thus the generalized analytical expressions for Equation (3.7) using ideal relay feedback test for 2-by-2 MIMO system is given by:

\[
y_n = k_{21} \left( 1 - e^{-\frac{\tau_1}{\tau_2}} \left( \frac{2}{1 + e^{\frac{p_1}{\tau_1}}} \right) \right) - \frac{k_{22}K_c}{\tau_{12}} \left( 1 + (\tau_{12} - \tau_{22}) e^{-\frac{\tau_{12}}{\tau_{22}}} \right) \left( \frac{2}{1 + e^{\frac{p_2}{\tau_{22}}}} \right) y_n(t-1)
\] (3.14)

The term \( y_n(t-1) \) in Equation (3.14) is one step ahead prediction of \( y_n \).
Similarly, the generalized analytical expressions for Equation (3.6) using ideal relay feedback test for 2-by-2 MIMO system is given by:

\[
y_n = k_{12} \left( 1 - e^{-\frac{\tau_{12}}{\tau_{21}}} \right) \frac{k_{11}K_{c1}}{\tau_{11}} \left( 1 + \left( \tau_{11} - \tau_{11} \right)e^{-\frac{\tau_{11}}{\tau_{21}}} \right) \frac{2}{1 + e^{-\frac{\tau_{11}}{\tau_{21}}} \left( \tau_{11} \right) e^{-\frac{\tau_{11}}{\tau_{21}}} \right) y_n(t-1)
\]

(3.15)

The term \(y_n(t-1)\) in Equation (3.15) is one step ahead prediction of \(y_n\).

3.2.2 Modeling of 2-by-2 MIMO Systems Using Biased Relay Feedback Test

The biased relay response for Equation (3.7) is assumed to be formed by \(n\)-number of small step changes. Let \(\mu_c = \mu_0 + \mu\), \(\mu_c = \mu_0 - \mu\). The process input in the relay feedback test consists of a series of step changes with down amplitude, \(\mu_d\) and up amplitude, \(\mu_u\). At the first instant, (after synchronizing input with output by time shift) the response can be described as:

\[
y_1 = k_{21} \mu_0 \left[ 1 - e^{-\frac{\tau_{12}}{\tau_{12}}} \right] - \frac{k_{22}K_{c2}}{\tau_{12}} \mu_c \left[ t - \left( \tau_{12} + \tau_{22} \right) (1 - e^{-\frac{\tau_{12}}{\tau_{22}}} \right) y_1(t-1)
\]

(3.16)

At the second instant, the time is reset to zero at the initial point. The step response (relay output) can be given by: (i.e., introducing a time shift by \(D\) amount in the Equation (3.16))
\[ y_2 = k_{21} \mu_0 \left[ 1 - e^{-\frac{t + D_{21}}{\tau_{21}}} \right] - 2 \left( 1 - e^{-\frac{t}{\tau_{21}}} \right) - \\
- \frac{k_{22} K_{C2}}{\tau_{22}} \mu_L \left[ (t + D_{22}) - (\tau_{22} + \tau_{22}) \left( 1 - e^{-\frac{t + D_{22}}{\tau_{22}}} \right) \right] - 2 \left[ t - (\tau_{22} + \tau_{22}) \left( 1 - e^{-\frac{t}{\tau_{22}}} \right) \right] y_2 (t - 1) \]

\[ (3.17) \]

where \( D_{21} \) and \( D_{22} \) denotes time delay of the individual transfer functions of the system.

The above Equation (3.17) can be simplified as:

\[ y_2 = k_{21} \mu_0 \left[ 1 - 2 \right] - e^{-\frac{t}{\tau_{21}}} \left[ e^{-\frac{D_{21}}{\tau_{21}}} - 2 \right] - \\
- \frac{k_{22} K_{C2}}{\tau_{22}} \mu_L \left[ t \left[ 1 - 2 \right] + (\tau_{22} - \tau_{22}) \left( 1 - e^{-\frac{t}{\tau_{22}}} \right) \left[ e^{-\frac{D_{22}}{\tau_{22}}} - 2 \right] \right] y_2 (t - 1) \]

\[ (3.18) \]

Let \( p_+ \) and \( p_- \) be the positive and negative half cycle periods.

The relay response at the third interval lags by an amount of \( D_{21} + P_+ \) and \( D_{22} + P_- \) from input of I and II parts and it can be given as:

\[ y_3 = k_{21} \mu_0 \left[ 1 - e^{-\frac{t + D_{21}}{\tau_{21}}} \right] - 2 \left( 1 - e^{-\frac{t}{\tau_{21}}} \right) - 2 \left( 1 - e^{-\frac{t}{\tau_{21}}} \right) - \\
- \frac{k_{22} K_{C2}}{\tau_{22}} \mu_L \left[ \left( t + D_{22} + \frac{P_-}{2} \right) + (\tau_{22} - \tau_{22}) \left( 1 - e^{-\frac{t + D_{22}}{\tau_{22}}} \right) \left( 1 - e^{-\frac{\tau_{22}}{\tau_{22}}} \right) \right] y_3 (t - 1) \]

\[ (3.19) \]
Equation (3.19) can be easily simplified as:

\[
y_3 = k_{21} \mu_0 \left\{ 1 - 2 + 2 \right\} - e^{\frac{-t}{\tau_{21}}} \left[ \frac{D_{21}^2}{\tau_{21}^2} - \frac{P_0}{\tau_{21}} \right] + \left( \frac{P_0}{\tau_{21}} + 2 \right)
\]

\[
- k_{22} K c_2 \mu_0 \frac{\mu_0}{\tau_{12}} \left\{ 1 - 2 + 2 \right\} + \left( \tau_{12} + \tau_{22} \right) \left\{ 1 - e^{\frac{-t}{\tau_{22}}} \right\} \left[ \frac{D_{22}^2}{\tau_{22}^2} - \frac{P_0}{\tau_{22}} \right] + \left( \frac{P_0}{\tau_{22}} + 2 \right)\right)\right)\} y_3 (t-1)
\]

(3.20)

It can be seen from the above equation that the terms in the RHS are slowly forming a series. As time tends to infinity, the response becomes stabilized and it can be described as:

\[
y_n = k_{21} \mu_0 \left\{ 1 - 2 + 2 - \ldots \right\} - e^{\frac{-t}{\tau_{21}}} \left[ \frac{\sum_{i=1}^{\infty} \frac{P_i}{\tau_{21}^i}}{\tau_{21}^i} - \frac{\sum_{i=1}^{\infty} \frac{P_i}{\tau_{21}^i}}{\tau_{21}^i} + \ldots + e^{\frac{-t}{\tau_{21}}} \right] - \frac{k_{22} K c_2 \mu_0 \frac{\mu_0}{\tau_{12}}}{\tau_{12}} \left\{ 1 - 2 + 2 - \ldots \right\} + \left( \tau_{12} - \tau_{22} \right) \left\{ 1 - e^{\frac{-t}{\tau_{22}}} \right\} \left[ \frac{\sum_{i=1}^{\infty} \frac{P_i}{\tau_{22}^i}}{\tau_{22}^i} - \frac{\sum_{i=1}^{\infty} \frac{P_i}{\tau_{22}^i}}{\tau_{22}^i} + \ldots + e^{\frac{-t}{\tau_{22}}} \right]
\]

\[
\left[ y_1 (t-1) + y_2 (t-1) + y_3 (t-1) + \ldots + y_n (t-1) \right]
\]

(3.21)

It can be seen from Equation (3.21) the RHS has 4 parts:

\[
y_n = k_{21} \mu_0 \left\{ \text{First part} - \text{Second part} \right\} - \frac{k_{22} K c_2 \mu_0 \frac{\mu_0}{\tau_{12}}}{\tau_{12}} \left\{ \text{Third part} + \text{Fourth part} \right\}
\]
The RHS of the above series becomes

First part = [1-2+2-2+...upto \( \infty \)] = 1

As

\[
\sum_{n=0}^{\infty} e^{-\frac{np}{\tau_{21}}} = \frac{1}{1-e^{-\frac{p}{\tau_{21}}}}
\]

Second part = \[
\frac{1}{1-e^{-\frac{p}{\tau_{21}}}} \left(1-e^{-\frac{lp}{\tau_{21}}} \right) = \frac{1}{1-e^{-\frac{p}{\tau_{21}}}}
\]

\[
\sum_{n=0}^{\infty} x_1^n x_2^n - x_2 \sum_{n=0}^{\infty} x_1^{n-1} x_2^{n-1} = \lim_{n \to \infty} (n-x) x_1^n x_2^n (\mu - \mu_0) e^{-\frac{p}{\tau_{22}}} = \frac{1-x_1}{1-x_1 x_2}
\]

As \( n \to \infty \),

\[
\frac{1-x_1}{1-x_1 x_2} = \frac{1-e^{-\frac{p}{\tau_{22}}}}{1-e^{-\frac{lp}{\tau_{21}}}}
\]

Thus, the generalized analytical expressions of Equation (3.7) using biased relay feedback test for 2-by-2 MIMO system is given by:

\[
y_n = \left( k_{21} \mu - 2k_{21} \mu_0 e^{-\frac{p}{\tau_{21}}} \right) \frac{1}{1-e^{-\frac{p}{\tau_{21}}}} - \frac{k_{22} K c_2}{\tau_{12}} \left( \mu t_l + \mu_0 \right) - \frac{k_{22} K c_2}{\tau_{12}} \left( \tau_{22} - \tau_{12} \right) \left( \mu - 2\mu_0 e^{-\frac{p}{\tau_{22}}} \right) \frac{1-e^{-\frac{p}{\tau_{22}}}}{1+e^{-\frac{p}{\tau_{22}}}} y_n(t-1)
\]

(3.22)

The term \( y_n(t-1) \) in Equation (3.22) is one step ahead prediction of \( y_n \).

Similarly, the generalized analytical expressions for Equation (3.6) using biased relay feedback test for 2-by-2 MIMO system is given by:
The term $y_n(t-1)$ in Equation (3.23) is one step ahead prediction of $y_n$.

### 3.3 PROPOSED METHOD/ MODELING OF 3-BY-3 MIMO SYSTEMS

Based on the concept of sequential autotuning method (Shen and Yu, 1994) each controller is designed in sequence. As shown in Figure 3.6, considering 3-by-3 MIMO system with known pairing $(y_1-u_1)$, $(y_2-u_2)$ and $(y_3-u_3)$ under decentralized PI control. An ideal relay is placed initially between $y_1$ and $u_1$ while loops 2 and 3 are on manual modes. Following the relay-feedback test, a controller can be designed from the ultimate gain and ultimate frequency. The next step is to perform relay-feedback test between $y_2$ and $u_2$ while loop 1 on auto and loop 3 is on manual modes. Lastly, relay-feedback test between $y_3$ and $u_3$ while loops 1 and 2 are in auto modes a controller can be designed for loop 3 following the relay-feedback test. This procedure is repeated until the controller parameter converges. Typically, the controller parameter converges in 3 - 4 relay-feedback tests for 3-by-3 MIMO systems.

Figure 3.6 shows the structure of the 3-by-3 MIMO system with decentralized control. The system can be described as:
The decentralized controller is described by:

\[
G_C(s) = \begin{pmatrix}
G_{C1} & 0 & 0 \\
0 & G_{C2} & 0 \\
0 & 0 & G_{C3}
\end{pmatrix}
\]  
(3.25)

Figure 3.6  Closed loop representation of 3-by-3 MIMO process with decentralized control structure

It is evident from closed-loop sequential autotuning for 3-by-3 system, a diagonalised transfer functions as \(g_{p11,CL}\), \(g_{p22,CL}\) and \(g_{p33,CL}\) (that govern basic transfer functions for controller design) and interactive transfer functions as \(g_{p12,CL}\), \(g_{p13,CL}\), \(g_{p21,CL}\), \(g_{p23,CL}\), \(g_{p31,CL}\) and \(g_{p32,CL}\) are observed. Thus, the effective closed-loop relation between \(y_2\) and \(u_1\) for 3-by-3 MIMO system is:
\[ g_{p21,CL} = \frac{g_{p21}}{1 + G_{C2}g_{p22}} - \frac{g_{p23}g_{p31}}{G_{C3}g_{p33}(1 + G_{C3}g_{p33})} \]  

(3.26)

where,

\[ g_{p21} = \frac{k_{21}e^{-D_{21}s}}{\tau_{21}s + 1} \quad g_{p22} = \frac{k_{22}e^{-D_{22}s}}{\tau_{22}s + 1} \quad G_{C2} = K_{C2}\left(1 + \frac{1}{\tau_{22}s}\right) \]

\[ g_{p31} = \frac{k_{31}e^{-D_{31}s}}{\tau_{31}s + 1} \quad g_{p23} = \frac{k_{23}e^{-D_{23}s}}{\tau_{23}s + 1} \quad G_{C3} = K_{C3}\left(1 + \frac{1}{\tau_{23}s}\right) \]

3.3.1 Modeling of 3-by-3 MIMO Systems Using Ideal Relay Feedback Test

Mathematical models are developed to represent relay responses produced by closed-loop interaction transfer function for 3 input 3 output systems (Equation (3.26)). It is assumed that relay output is generated by series of small number of step changes in manipulated variable hence the stabilized output is a sum of infinite terms of small responses due to those step changes. For 3-by-3 systems the relay responses obtained from interactive transfer functions are modeled as follows. The relay response is assumed to be formed by n-number of small step changes. The first interval response can be expressed as:

\[ y_1 = \left( k_{21}\left[1 - e^{-\frac{t}{\tau_{21}}}\right] - \frac{k_{22}K_{C2}}{\tau_{12}}\left[1 + (\tau_{12} - \tau_{22})e^{-\frac{t}{\tau_{22}}}\right]\right)y_1(t-1) - \left( \frac{k_{23}k_{31}}{K_{C3}k_{33}} \left[1 - \left(a_1e^{-\frac{t}{\tau_{13}}} + a_2e^{-\frac{t}{\tau_{23}}} + a_3e^{-\frac{t}{\tau_{33}}}\right)\right] \right) \frac{k_{33}K_{C3}}{\tau_{13}}\left[1 + (\tau_{13} - \tau_{33})e^{-\frac{t}{\tau_{33}}}\right]y_1(t-1) \]

(3.27)
where,

\[ a_1 = \frac{\tau_{13} (\tau_{13} - \tau_{33})}{(\tau_{13} - \tau_{23})(\tau_{13} - \tau_{33})}; \quad a_2 = \frac{\tau_{23} (\tau_{23} - \tau_{33})}{(\tau_{23} - \tau_{13})(\tau_{23} - \tau_{31})}; \quad a_3 = \frac{\tau_{31} (\tau_{13} - \tau_{33})}{(\tau_{31} - \tau_{13})(\tau_{31} - \tau_{23})} \]

Let \( D = D_{23} + D_{31} - D_{33} \), at the second instant the relay output can be given by:

\[
y_2 = k_{21} \left[ 1 - \frac{1}{e^{\tau_{21}}} - 2 - e^{-\tau_{21}} \right] \frac{k_{23} K_{C2}}{\tau_{21}} \left[ 1 - \frac{1}{e^{\tau_{21}}} - 2 - e^{-\tau_{21}} \right] y_2(t-1) - \frac{k_{23} K_{C3}}{K_{C1} K_{33}} \left[ 1 - \frac{1}{e^{\tau_{31}}} - 2 - e^{-\tau_{31}} \right] y_2(t-1)
\]

\[(3.28)\]

Equation (3.28) can be simplified as:

\[
y_2 = k_{21} \left[ 1 - 2 \right] \frac{1}{e^{\tau_{21}}} \left[ \frac{D_{12}}{\tau_{21}} \left[ 2 \right] \right] \frac{k_{23} K_{C2}}{\tau_{21}} \left[ 1 - 2 \right] \left[ \frac{D_{12}}{\tau_{22}} \left[ 2 \right] \right] y_2(t-1) - \frac{k_{23} K_{C3}}{K_{C1} K_{33}} \left[ 1 - \frac{1}{e^{\tau_{31}}} - 2 - e^{-\tau_{31}} \right] y_2(t-1)
\]

\[(3.29)\]
At the third instant,

\[ y_3 = k_{21} \left[ \left( 1 - e^{\frac{t - D_2}{\tau_{22}}} \right) - 2 \left( 1 - e^{\frac{t - D_2}{\tau_{21}}} \right) + 2 \left( 1 - e^{\frac{t}{\tau_{21}}} \right) \right] - \]

\[ \frac{k_{23}K_c}{\tau_{12}} \left[ \left( 1 - (\tau_{12} - \tau_{22})e^{\frac{t - D_2}{\tau_{22}}} \right) - 2 \left( 1 - (\tau_{12} - \tau_{22})e^{\frac{t - D_2}{\tau_{21}}} \right) + 2 \left( 1 - (\tau_{12} - \tau_{22})e^{\frac{t}{\tau_{21}}} \right) \right] y_3(t-1) - \]

\[ \frac{k_{33}K_c}{\tau_{13}} \left[ \left( 1 - (\tau_{13} - \tau_{33})e^{\frac{t - D_3}{\tau_{33}}} \right) - 2 \left( 1 - (\tau_{13} - \tau_{33})e^{\frac{t - D_3}{\tau_{31}}} \right) + 2 \left( 1 - (\tau_{13} - \tau_{33})e^{\frac{t}{\tau_{31}}} \right) \right] \]

\[(3.30)\]

Equation (3.30) can be simplified as:

\[ y_3 = k_{31} \left[ 1 - 2 \right] - e^{\frac{t}{\tau_{11}}} \left[ \frac{\tau_{31} \delta_1}{\tau_{11}} + 2 \frac{\tau_{31} \delta_1}{\tau_{11}} \right] - \frac{k_{33}K_c}{\tau_{13}} \left[ 1 - 2 \right] \left( \tau_{13} - \tau_{33} \right) e^{\frac{t - D_3}{\tau_{33}}} + \frac{k_{33}k_{31}}{K_c\tau_{13}} \left[ 1 - 2 \right] \left( \tau_{22} - \tau_{33} \right) e^{\frac{t - D_2}{\tau_{22}}} + 2 \left( \tau_{22} - \tau_{33} \right) e^{\frac{t}{\tau_{22}}} \right] y_3(t-1) - \]

\[ \left[ 1 - 2 \right] - e^{\frac{t}{\tau_{11}}} \left[ \frac{\tau_{31} \delta_1}{\tau_{11}} + 2 \frac{\tau_{31} \delta_1}{\tau_{11}} \right] - \frac{k_{33}K_c}{\tau_{13}} \left[ 1 - 2 \right] \left( \tau_{13} - \tau_{33} \right) e^{\frac{t - D_3}{\tau_{33}}} + \frac{k_{33}k_{31}}{K_c\tau_{13}} \left[ 1 - 2 \right] \left( \tau_{22} - \tau_{33} \right) e^{\frac{t - D_2}{\tau_{22}}} + 2 \left( \tau_{22} - \tau_{33} \right) e^{\frac{t}{\tau_{22}}} \right] y_3(t-1) - \]

\[ \left[ 1 - 2 \right] - e^{\frac{t}{\tau_{11}}} \left[ \frac{\tau_{31} \delta_1}{\tau_{11}} + 2 \frac{\tau_{31} \delta_1}{\tau_{11}} \right] - \frac{k_{33}K_c}{\tau_{13}} \left[ 1 - 2 \right] \left( \tau_{13} - \tau_{33} \right) e^{\frac{t - D_3}{\tau_{33}}} + \frac{k_{33}k_{31}}{K_c\tau_{13}} \left[ 1 - 2 \right] \left( \tau_{22} - \tau_{33} \right) e^{\frac{t - D_2}{\tau_{22}}} + 2 \left( \tau_{22} - \tau_{33} \right) e^{\frac{t}{\tau_{22}}} \right] y_3(t-1) - \]

\[(3.31)\]
As time tends to infinity, the response becomes stabilized and the overall response can be expressed as:

\[
y_n = k_{21} \left[ 1 - 2 + 2 - \ldots \right] - e^{\frac{t}{\tau_{21}}} \left[ \frac{d_{21} \sum_{i=1}^{n} P_{21}}{\tau_{21}} - 2e^{\frac{t}{\tau_{21}}} + \ldots + 2e^{\frac{t}{\tau_{21}}} - 2 \right]
\]

\[
- \frac{k_{22} K_{c2}}{\tau_{12}} \left[ 1 - 2 + 2 - \ldots \right] - \left( \tau_{12} - \tau_{22} \right) e^{\frac{t}{\tau_{22}}} \left[ \frac{d_{22} \sum_{i=1}^{n} P_{22}}{\tau_{22}} - 2e^{\frac{t}{\tau_{22}}} + \ldots + 2e^{\frac{t}{\tau_{22}}} - 2 \right]
\]

\[
y_i(t-1) - y_j(t-1) + y_k(t-1) = \ldots + y_n(t-1)
\]

\[
\left\{ \begin{array}{c}
\frac{k_{22} K_{c2}}{K_{c1} K_{c3}} \left[ 1 - 2 + 2 - \ldots \right] + a \frac{t}{\tau_{22}} \left[ \frac{d_{22} \sum_{i=1}^{n} P_{22}}{\tau_{22}} - 2e^{\frac{t}{\tau_{22}}} + \ldots + 2e^{\frac{t}{\tau_{22}}} - 2 \right] + \\
\frac{k_{22} K_{c2}}{K_{c1} K_{c3}} \left[ 1 - 2 + 2 - \ldots \right] - a \frac{t}{\tau_{22}} \left[ \frac{d_{22} \sum_{i=1}^{n} P_{22}}{\tau_{22}} - 2e^{\frac{t}{\tau_{22}}} + \ldots + 2e^{\frac{t}{\tau_{22}}} - 2 \right] + \\
- \frac{k_{22} K_{c2}}{\tau_{12}} \left[ 1 - 2 + 2 - \ldots \right] - \left( \tau_{12} - \tau_{33} \right) e^{\frac{t}{\tau_{33}}} \left[ \frac{d_{22} \sum_{i=1}^{n} P_{22}}{\tau_{22}} - 2e^{\frac{t}{\tau_{22}}} + \ldots + 2e^{\frac{t}{\tau_{22}}} - 2 \right]
\end{array} \right.
\]

\[
y_i(t-1) - y_j(t-1) + y_k(t-1) = \ldots + y_n(t-1)
\]

(3.32)

The above Equation (3.32) can be written as:

\[
y_n = k_{21} \{ \text{First part} - \text{Second part} \} - \frac{k_{22} K_{c2}}{\tau_{12}} \{ \text{Third part} + \text{Fourth part} \} -
\]
\[-\frac{k_{23}k_{31}}{K_{c3}k_{33}} \{ 1-(\text{Fifth part} + \text{Sixth part} + \text{Seventh part}) \} - \frac{k_{33}K_{c3}}{\tau_{13}} \{ \text{Eighth part} + \text{Ninth part} \} \]

Let \( r_1 = e^{-\frac{P_2}{\tau_{52}}} \); \( r_2 = e^{-\frac{P_2}{2\tau_{52}}} \); \( r_3 = e^{-\frac{P_2}{3\tau_{52}}} \); \( r_4 = e^{-\frac{P_2}{4\tau_{52}}} \) and \( r_5 = e^{-\frac{P_2}{5\tau_{52}}} \)

The RHS of the above series becomes

First part = \([1-2+2-2+\ldots] = 1\)

Similar way Third part = 1

Second part = \(2 \left[ \frac{D_{r_1}}{e^{-\frac{P_2}{\tau_{52}}} r_1^n - 2r_1^{n-1} - 2r_1^{n-2} - \ldots - 2r_1 - 2} \right] \)

This second part can be put into following series

Third part = \(2 \left[ 1 - r_1 + r_1^2 - r_1^3 + \ldots \right] = \frac{2}{1 + r_1} = \frac{2}{1 + e^{-\frac{P_2}{\tau_{52}}}} \)

Similar way Fourth part = \( \frac{2}{1 + r_2} = \frac{2}{1 + e^{-\frac{P_2}{2\tau_{52}}}} \)

Fifth part = \( \frac{2}{1 + r_3} = \frac{2}{1 + e^{-\frac{P_2}{3\tau_{52}}}} \); Sixth part = \( \frac{2}{1 + r_4} = \frac{2}{1 + e^{-\frac{P_2}{4\tau_{52}}}} \)

Seventh part = \( \frac{2}{1 + r_5} = \frac{2}{1 + e^{-\frac{P_2}{5\tau_{52}}}} \)
Therefore, the generalized analytical expressions for \( g_{p21,\text{CL}} \) of 3-by-3 MIMO system is given by:

\[
y_n = \left( k_{21} \left[ 1 - e^{-\tau_{21}/21} \left( \frac{2}{1 + e^{-\tau_{21}/22}} \right) \right] + \frac{k_{22}K_{C2}}{\tau_{12}} \left[ 1 + (\tau_{12} - \tau_{22}) e^{-\tau_{22}/22} \left( \frac{2}{1 + e^{-\tau_{22}/22}} \right) \right] \right) y_n (t-1) - \\
- \left( \frac{k_{23}k_{31}T_{33}}{K_{C3}k_{33}} \left[ 1 - \left( a_1 e^{-\tau_{13}/T_{13}} \left( \frac{2}{1 + e^{-\tau_{13}/T_{23}}} \right) + a_2 e^{-\tau_{23}/T_{23}} \left( \frac{2}{1 + e^{-\tau_{23}/T_{23}}} \right) + a_3 e^{-\tau_{33}/T_{33}} \left( \frac{2}{1 + e^{-\tau_{33}/T_{33}}} \right) \right] \right) y_n (t-1) \\
\left( \frac{k_{33}K_{C3}}{\tau_{13}} \left[ 1 + (\tau_{13} - \tau_{33}) e^{-\tau_{33}/T_{33}} \left( \frac{2}{1 + e^{-\tau_{33}/T_{33}}} \right) \right] \right)
\]

(3.33)

The term \( y_n(t-1) \) in Equation (3.33) is one step ahead prediction of \( y_n \).

Similarly, the generalized analytical expressions for other off-diagonal interactive closed-loop transfer functions can be obtained (\( g_{p12,\text{CL}} \), \( g_{p13,\text{CL}} \), \( g_{p23,\text{CL}} \), \( g_{p31,\text{CL}} \) and \( g_{p32,\text{CL}} \)).

### 3.3.2 Modeling of 3-by-3 MIMO Systems Using Biased Relay Feedback Test

For 3-by-3 MIMO systems, the relay responses are obtained from interactive transfer function (Equation (3.26)) is modeled as follows. The procedure for obtaining the mathematical model of SISO systems using biased relay is explained in detail by (Gu et al 2008 and Liu et al 2008).

The relay response is assumed to be formed by n-number of small step changes. At the first interval, the response can be expressed as:
\[ y_1 = \left( k_{21} \mu_0 \left[ 1 - e^{-\tau_{21}/\tau_{12}} \right] - \frac{k_{22} K_{C_2}}{\tau_{12}} \mu \left[ t - \left( \tau_{12} - \tau_{22} \right) \left( 1 - e^{-\tau_{22}/\tau_{12}} \right) \right] \right) y_1(t-1) - \]

\[ - \left( \frac{k_{23} \mu_0}{K_{C_3}} \frac{\tau_{13}}{\tau_{13}} \left[ 1 - \left( a_1 e^{-\tau_{13}/\tau_{13}} + a_2 e^{-\tau_{23}/\tau_{23}} + a_3 e^{-\tau_{33}/\tau_{33}} \right) \right] \right) y_1(t-1) \]

\[
\begin{align*}
&= \left( \frac{k_{23} \mu_0 \tau_{13}}{K_{C_3} k_{33}} \left[ 1 - \left( a_1 e^{-\tau_{13}/\tau_{13}} + a_2 e^{-\tau_{23}/\tau_{23}} + a_3 e^{-\tau_{33}/\tau_{33}} \right) \right] \right) y_1(t-1) \]
\end{align*}
\]

(3.34)

where \( a_1 = \frac{\tau_{13} (\tau_{13} - \tau_{33})}{(\tau_{13} - \tau_{13}) (\tau_{13} - \tau_{33})} \), \( a_2 = \frac{\tau_{23} (\tau_{23} - \tau_{33})}{(\tau_{23} - \tau_{13}) (\tau_{23} - \tau_{33})} \), \( a_3 = \frac{\tau_{31} (\tau_{31} - \tau_{31})}{(\tau_{31} - \tau_{13}) (\tau_{31} - \tau_{33})} \)

Let \( D = D_{23} + D_{31} - D_{33} \), at the second instant, the relay output can be given by:

\[ y_2 = k_{21} \mu_0 \left[ 1 - 2 - e^{-\tau_{21}/\tau_{11}} \left[ e^{-\tau_{21}/\tau_{11}} - 2 \right] \right] - \frac{k_{22} K_{C_2}}{\tau_{12}} \mu \left[ t \left[ 1 - 2 \right] + \left( \tau_{12} - \tau_{22} \right) e^{-\tau_{22}/\tau_{12}} \left[ e^{-\tau_{22}/\tau_{12}} - 2 \right] \right] y_2(t-1) - \]

\[ - \left( \frac{k_{23} \mu_0}{K_{C_3} k_{33}} \left[ 1 - \left( a_1 e^{-\tau_{13}/\tau_{13}} + a_2 e^{-\tau_{23}/\tau_{23}} + a_3 e^{-\tau_{33}/\tau_{33}} \right) \right] \right) y_2(t-1) \]

\[
\begin{align*}
&= \left( \frac{k_{23} \mu_0 \tau_{13}}{K_{C_3} k_{33}} \left[ 1 - \left( a_1 e^{-\tau_{13}/\tau_{13}} + a_2 e^{-\tau_{23}/\tau_{23}} + a_3 e^{-\tau_{33}/\tau_{33}} \right) \right] \right) y_2(t-1) \]
\end{align*}
\]

(3.35)

As time tends to infinity, the response becomes stabilized and the overall response can be expressed as \( y_n = y_1 + y_2 + \ldots \).

The generalized analytical expressions for 3-by-3 MIMO system using biased relay feedback test is given by:
\[ y_n = \left( k_{21}\mu_1 - 2k_{21}\mu_0 e^{-\frac{t}{\tau_n}} \left( \frac{p}{1 - e^{-\frac{\tau_n}{\tau_1}}} \right) - \left( \frac{p}{1 + e^{-\frac{2\tau_n}{\tau_2}}} \right) \right) 
- \frac{k_{22}K_{C2}}{\tau_{12}}(\mu_1 + \mu \mu) \frac{k_{22}K_{C2}}{\tau_{12}}(\tau_{12} + \tau_{22}) \mu_+ - 2\mu e^{-\frac{t}{\tau_n}} \left( \frac{p}{1 - e^{-\frac{\tau_n}{\tau_1}}} \right) \right) y_n(t - 1) 
- \left( \frac{2\mu_0 a_1 e^{-\frac{t}{\tau_n}}}{1 - e^{-\frac{\tau_n}{\tau_1}}} \right) + 2\mu_0 a_2 e^{-\frac{t}{\tau_n}} \left( \frac{p}{1 + e^{-\frac{2\tau_n}{\tau_2}}} \right) \n- \left( \frac{k_{33}K_{C3}}{\tau_{13}}(\mu_1 + \mu \mu) \frac{k_{33}K_{C3}}{\tau_{13}}(\tau_{13} + \tau_{33}) \mu_+ - 2\mu e^{-\frac{t}{\tau_n}} \left( \frac{p}{1 + e^{-\frac{2\tau_n}{\tau_2}}} \right) \right) \right) y_n(t - 1) \]

(3.36)

The term \( y_n(t-1) \) in Equation (3.36) is one step ahead prediction of \( y_n \).

Similarly, the generalized analytical expressions for other off-diagonal interactive closed-loop transfer functions can be obtained \((g_{p12,CL}, g_{p13,CL}, g_{p23,CL}, g_{p31,CL} \text{ and } g_{p32,CL})\).

### 3.4 MODEL VALIDATION FOR 2-BY-2 MIMO SYSTEMS

The analytical expressions (developed in the section 3.3.1) provide the mathematical models for undesirable relay responses of MIMO system. These theoretical equations are validated for four distillation columns having 2-by-2 MIMO structure and coupled tanks system using ideal and biased relay feedback tests respectively.
From Figures 3.7, 3.8, 3.9, 3.10, 3.11 and 3.13 it can be seen that the validation of the derived mathematical models where theoretical response matches exactly with experimental one for ideal and biased relay feedback tests.

Figure 3.7 Validation of analytical expression (dashed) of 2×2 interactive cross TF with experimental ideal relay response (solid) for WB process
Figure 3.8 Validation of analytical expression (dashed) of $2 \times 2$ interactive cross TF with experimental biased relay response (solid) for WB process

Figure 3.9 Validation of analytical expression (dashed) of $2 \times 2$ interactive cross TF with experimental ideal relay response (solid) for VL process
Figure 3.10 Validation of analytical expression (dashed) of 2×2 interactive cross TF with experimental ideal relay response (solid) for WW process

Figure 3.11 Validation of analytical expression (dashed) of 2×2 interactive cross TF with experimental ideal relay response (solid) for TS process
Figure 3.12 Desired experimental relay response for coupled tanks system (with $u_1$ as control signal)

Figure 3.13 Validation of analytical expression (dotted) of $2 \times 2$ interactive cross TF (of undesired) with real-time experimental relay response (solid) for coupled tanks system (with $u_1$ as control signal)
Undesired relay responses obtained after simulating WB, VL, WW, TS columns and coupled tanks system are compared with its original ideal and biased relay responses respectively.

In Figure (3.12), having two signals one is sinusoidal oscillation for the level in tank 4, \( h_4 \) and other is series of step inputs for a change in \( u_1 \) switching between 2.7V and 2.9V. In this case the other control signal \( u_2 \) is kept constant.

### 3.5 MODEL VALIDATION FOR 3-BY-3 MIMO SYSTEMS

The analytical expressions developed in section 3.3.2 provide the mathematical models for undesired relay responses of MIMO system. These theoretical equations are validated for Orgunaikke and Ray (section 2.2.2.2) and Shell control problem (as described below) and found both are having FOPDT structures for their subsystems of 3-by-3 MIMO structure (as mentioned in the scope of the thesis in chapter 1) using ideal relay feedback tests.

![Validation of analytical expression (dashed) of 3x3 interactive cross TF with experimental ideal relay response (solid) for OR process](image)

Figure 3.14 Validation of analytical expression (dashed) of 3x3 interactive cross TF with experimental ideal relay response (solid) for OR process
The process transfer function matrix of Shell Control Problem (SCP) 3-by-3 MIMO system having all the subsystems are of FOPDT model structures which are given by:

\[
G(s) = \begin{bmatrix}
4.05e^{-27s} & 1.77e^{-28s} & 5.88e^{-27s} \\
50s + 1 & 60s + 1 & 50s - 1 \\
5.39e^{-18s} & 5.72e^{-14s} & 6.9e^{-15s} \\
50s + 1 & 60s + 1 & 40s + 1 \\
4.38e^{-20s} & 4.42e^{-22s} & 7.2e^{-19s} \\
33s + 1 & 44s + 1 & 19s + 1
\end{bmatrix}
\]

Figure 3.15 Validation of analytical expression (dashed) of 3x3 interactive cross TF with experimental ideal relay response (solid) for Shell control process (SCP)

Undesired relay responses obtained after simulating OR and SCP are compared with its original relay responses.

Figure 3.14 and 3.15 shows the validation of derived mathematical models where the theoretical response matches exactly with the experimental one for relay feedback tests.
It is observed that for 2-by-2 MIMO systems, the responses of WB, VL, WW and TS column are having sharp/ saw-toothed peaks as they have $D/\tau$ ratios 0.6, 0.1, 0.2 and 0.4 respectively.

It is observed that for 3-by-3 MIMO systems the responses of OR column and SCP are exhibiting exponentially developed periodic responses.

Thus, it can be concluded that with higher $D/\tau$ ratio processes show exponentially or fully developed periodic peaks whereas with lower $D/\tau$ ratio limit cycle oscillations become sharp/ saw toothed peaks.

3.6 SUMMARY

Off-diagonal elements of transfer function matrix of MIMO system represent interactions between inputs and outputs. Relay inputs are fed to these transfer functions and responses are obtained. These responses are analyzed by formulating the time domain dynamics of interactive (off-diagonal) elements of closed-loop transfer function. The analytical expression representing the general trend of oscillatory response is obtained for biased input relay which is applicable for industrial situation. A close look to these expressions reveal that it contains few segmental terms in the right hand side of the equation and these segments show increasing number of terms containing exponential stabilizing (decaying) components giving rise to overall stable (oscillatory) response. These expressions are obtained from interactive closed-loop transfer functions that are formed from individual transfer function components (of FOPDT type with low $D/\tau$ ratio) of open-loop MIMO systems. The output relay responses are characterized by their $D/\tau$ ratio. The relay feedback responses are helpful to understand the causes of interaction which are dealt in next chapter.