CHAPTER 7
FUZZY NUMBER INNER PRODUCT OVER FUZZY NUMBER SPACE

7.1 INTRODUCTION

The present chapter proposes a fuzzy version of real vectors and their inner product using fuzzy numbers. Fuzzy numbers having directions are considered as fuzzy number vectors with membership grade for each component. In this chapter, an inner product function is defined for membership values of the fuzzy number vectors. Moreover, extend the projection theorem on Hilbert spaces to its fuzzy version over fuzzy number space in connection with fuzzy number mapping.

7.2 FUZZY NUMBER INNER PRODUCT

Definition 7.2.1

Let $E = \{\mu/R: R \rightarrow [0,1]\}$ be a fuzzy number space, $\mu$ being a fuzzy number. Also let $\hat{A}$ be a fuzzy number vector space over $E$ and for fuzzy number vectors $[\mu_n]$, $[\mu_n] \in \hat{A}$, we can define a fuzzy number inner product $\langle , \rangle$ on $\hat{A}$ as follows:

$$\langle [\mu_n], [\mu_n] \rangle = (\Sigma_{i=0}^k x_i y_i, \forall_{i=0}^k \inf (\mu_m^{(i)}(x_i), \mu_n^{(i)}(x_i)))$$

where $[\mu_m] = \sum_i \mu_m^{(i)} = \frac{\mu_m(x_0)}{x_0} + \frac{\mu_m(x_1)}{x_1} + \frac{\mu_m(x_2)}{x_2} + \cdots$, for $x_0, x_1, x_2 \cdots \in R$ and for all $\mu_m^{(i)}$, $i = 0, 1, 2 \ldots k$. 
Then $\tilde{A}$ together with this inner product is called the **fuzzy number inner product space over fuzzy number space.**

Obviously

![Diagram](image)

**Figure 7.1 fuzzy number vectors**

Then $\tilde{A}$ together with this inner product is called the **fuzzy inner product space over fuzzy number space.**

**Note:** Fuzzy number vectors mentioned in this chapter is nothing but fuzzy numbers while ordinary vectors in the real number system are the real numbers with . To prove the properties and result of fuzzy number inner product over fuzzy number space, it is enough to prove the results for membership values of fuzzy numbers.

**Example 7.2.2:** Let and be two fuzzy number vectors (triangular fuzzy number) of
Take $[\mu_m] = [1,5] = \{(1,0), (2,0.5), (3,1), (4,0.5), (5,0)\}$

$[\mu_n] = [6,10] = \{(6,0), (7,0.5), (8,1), (9,0.5), (10,0)\}$

Clearly $[\mu_m] + [\mu_n] = [\mu_s]$

$$= [7,5] = \{(7,0), (9,0.5), (11,1), (13,0.5), (15,0)\}$$

is a fuzzy number vector.

Now scalar multiplication of fuzzy number vectors is defined as follows:

Consider a scalar $\alpha = 5$

$$\alpha [\mu_m] = 5 [1,5] = [5,5][1,5] = [5,25]$$

Or $\alpha [\mu_m] = \{(5,0), (10,0.5), (15,1), (20,0.5), (25,0)\}$, which is a fuzzy number vector.

Then $\langle [\mu_m], [\mu_n] \rangle = (\sum_{i=0}^{k} x_i y_i, \bigvee_{i=0}^{k} \inf_{(\mu_m^{(i)}, \mu_n^{(i)})})$

$$= (6 + 14 + 24 + 36 + 50, 1)$$

$$= (130,1) = 130.$$  

**Definition 7.2.3**

Let E be a fuzzy number space and $\tilde{A}$ be a fuzzy number inner product space over $E$ then we can define a **fuzzy number norm** $\| . \| : \tilde{A} \rightarrow R$ as

$$\| [\mu_m] \|^2 = \langle [\mu_m], [\mu_m] \rangle, \quad \text{for all} \quad [\mu_m] = \sum_{i} \mu_m^{(i)} \in \tilde{A}.$$
7.2.4 Properties of Fuzzy Number Inner Product Over Fuzzy Number Space

Let $\tilde{A}$ be a fuzzy number inner product space over $E$. Then for all $[\mu_m], [\mu_n] \in \tilde{A}$

(i) $\langle [\mu_m], [\mu_m] \rangle \geq 0$

(ii) $\| [\mu_m] \|^2 \geq 0$ and $\| [\mu_m] \| = \| \mu_n \|$ if and only if $[\mu_m] = [\mu_n]$

(iii) $\| \alpha [\mu_m] \|^2 = |\alpha|^2 \| [\mu_m] \|^2$ where $\alpha$ is a scalar.

Proof

\[
\| \alpha [\mu_m] \|^2 = \langle \alpha [\mu_m], \alpha [\mu_m] \rangle = |\alpha|^2 \langle [\mu_m], [\mu_m] \rangle \\
= |\alpha|^2 \| [\mu_m] \|^2
\]

(iv) $\| [\mu_m] + [\mu_n] \|^2 \leq \| [\mu_m] \|^2 + \| [\mu_n] \|^2$.

Proof

\[
\| [\mu_m] + [\mu_n] \|^2 = \langle [\mu_m] + [\mu_n], [\mu_m] + [\mu_n] \rangle \\
= \bigvee_{i=1}^{k} \inf \left( [\mu_m]_{(i)} + [\mu_n]_{(i)} + [\mu_m]_{(i)} + [\mu_n]_{(i)} \right) \\
= \bigvee_{i=1}^{k} \inf \left( [\mu_m]_{(i)} + [\mu_n]_{(i)} + (\mu_m)_{(i)} + (\mu_n)_{(i)} \right) \\
= \bigvee_{i=1}^{k} \inf \left( [\mu_m]_{(i)} + [\mu_n]_{(i)} + 2 (\mu_m)_{(i)} + (\mu_n)_{(i)} \right) \\
= \langle [\mu_m], [\mu_m] \rangle + 2 \langle [\mu_m], [\mu_n] \rangle + \langle [\mu_n], [\mu_n] \rangle \\
= \| [\mu_m] \|^2 + 2 \| [\mu_m], [\mu_n] \| + \| [\mu_n] \|^2 \\
\leq \| [\mu_m] \|^2 + \| [\mu_n] \|^2.
\]
(v) \[ \| [\mu_m] + [\mu_n] \|^2 + \| [\mu_m] - [\mu_n] \|^2 = 2 \| [\mu_m] \|^2 + 2 \| [\mu_n] \|^2 \] (Parallelogram law).

**Proof**

From (iv) we have
\[ \| [\mu_m] + [\mu_n] \|^2 = \| [\mu_m] \|^2 + 2 \| [\mu_m], [\mu_n] \| + \| [\mu_n] \|^2. \]

Similarly,
\[ \| [\mu_m] - [\mu_n] \|^2 = \| [\mu_m] \|^2 - 2 \| [\mu_m], [\mu_n] \| + \| [\mu_n] \|^2. \]

Thus \[ \| [\mu_m] + [\mu_n] \|^2 + \| [\mu_m] - [\mu_n] \|^2 = 2 \| [\mu_m] \|^2 + 2 \| [\mu_n] \|^2. \]

**Definition 7.2.5**

Let \( \tilde{A} \) be a fuzzy number inner product space over \( E \). The set \( S = \{ [\mu_1], [\mu_2], \ldots, [\mu_k] \} \subset \tilde{A} \), for all \( [\mu_m] = \sum_i \mu_m^{(i)} \in \tilde{A}, m = 1, 2, \ldots, k \) is said to be **Fuzzy number Orthogonal set** if
\[ \langle [\mu_m], [\mu_n] \rangle = 0 \quad \text{for all } [\mu_m], [\mu_n] \in S \text{ and } [\mu_m] \neq [\mu_n]. \]

**Definition 7.2.6**

A Fuzzy number Orthogonal set \( S = \{ [\mu_1], [\mu_2], \ldots, [\mu_k] \} \subset \tilde{A} \) is said to be **Fuzzy number Orthonormal set** if
\[ \langle [\mu_m], [\mu_n] \rangle = \| [\mu_m] \|^2 = 1, \text{ for all } [\mu_m] \in S. \]
**Theorem 7.2.7**

Let $E$ be a fuzzy number space and $\tilde{A}$ be a fuzzy number inner product space over $E$. Then the fuzzy number orthonormal set of $\tilde{A}$ is linearly independent.

**Proof**

Let $S=\{[\mu_1], [\mu_2], \ldots, [\mu_k]\}$ be a fuzzy number orthonormal set of $\tilde{A}$.

Assume that $[\lambda] \in \tilde{A}$ can be expressed as the linear combination of the vectors of $S$.

ie, $[\lambda] = \sum_{m=1}^{k} r_m [\mu_m]$, where $r_m$ is a scalar

Now $\langle [\lambda], [\lambda] \rangle = \langle \sum_{m=1}^{k} r_m [\mu_m], \sum_{m=1}^{k} r_m [\mu_m] \rangle$

$= \sum_{m=1}^{k} |r_m|^2 \langle [\mu_m], [\mu_m] \rangle$

$= \sum_{m=1}^{k} |r_m|^2$, since $[\mu_m] \in S$.

If $[\lambda] = \sum_{m=1}^{k} r_m [\mu_m] = 0$

$\Rightarrow \sum_{m=1}^{k} |r_m|^2 = 0$

$\Rightarrow r_m = 0$, for each $1 \leq m \leq k$.

Hence $S$ is linearly independent.

**Theorem 7.2.8**

Let $\tilde{A}$ be a fuzzy number inner product space over a fuzzy number space $E$ and $S = \{ [\mu_1], [\mu_2], \ldots, [\mu_k] \}$ be a fuzzy number orthonormal set of $\tilde{A}$. Then for $[\lambda] \in \tilde{A}$,

$$\sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle^2 = ||[\lambda]||^2 \iff [\lambda] = \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle [\mu_m].$$
Proof

Suppose $[\lambda] = \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m]$, for $[\lambda] \in \tilde{A}$ be a fuzzy number vector and $[\mu_m] \in S$.

Now 

$$ ||[\lambda]||^2 = \langle [\lambda], [\lambda] \rangle $$

$$ = \langle \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m], \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m] \rangle $$

$$ = \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m] \ [\mu_m] \rangle $$

$$ = \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m] \ [\mu_m] \rangle, \text{ since } [\mu_m] \in S. $$

Thus 

$$ \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m] \ [\mu_m] \rangle = ||[\lambda]||^2. $$

Conversely assume that 

$$ ||[\lambda]||^2 = \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m] \ [\mu_m] \rangle. $$

$$ = \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m] \ [\mu_m] \rangle, \text{ since } [\mu_m] \in S $$

$$ = \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m] \ [\mu_m] \rangle $$

$$ = \langle \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m], \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m] \rangle $$

$$ \Rightarrow \langle [\lambda], [\lambda] \rangle = \langle \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m], \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m] \rangle $$

$$ [\lambda] = \sum_{m=1}^{k} \langle [\lambda], [\mu_m] \rangle \ [\mu_m]. $$

Hence the proof.
Lemma 7.2.9

Let $\tilde{\mathcal{A}}$ be a fuzzy number inner product space over $E$. Then for $[\mu], [\lambda] \in \tilde{\mathcal{A}}$,

$$\langle [\mu], [\lambda] \rangle = V_{i=1}^k \inf \mu^{(i)}, \lambda^{(i)} \text{ where } [\mu] = \sum_i \mu^{(i)}, \ [\lambda] = \sum_i \lambda^{(i)}.$$ 

We have $|||[\mu]|||^2 = \langle [\mu], [\mu] \rangle = V_{i=1}^k \inf \mu^{(i)}, \mu^{(i)}$ and $|||[\lambda]|||^2 = \langle [\lambda], [\lambda] \rangle = V_{i=1}^k \inf \lambda^{(i)}, \lambda^{(i)}$.

Now $|||[\mu]|||^2 |||[\lambda]|||^2 = V_{i=1}^k \inf \mu^{(i)}, \mu^{(i)} V_{i=1}^k \inf \lambda^{(i)}, \lambda^{(i)}$

$$= V_{i=1}^k \inf (\mu^{(i)}, \mu^{(i)})(\lambda^{(i)}, \lambda^{(i)})$$

$$= V_{i=1}^k \inf (\mu^{(i)}, \lambda^{(i)})(\lambda^{(i)}, \mu^{(i)})$$

$$= V_{i=1}^k \inf (\lambda^{(i)}, \mu^{(i)})(\mu^{(i)}, \lambda^{(i)})$$

$$= V_{i=1}^k \inf (\mu^{(i)}, \lambda^{(i)})(\lambda^{(i)}, \lambda^{(i)})$$

$$= V_{i=1}^k \inf (\mu^{(i)}, \lambda^{(i)}) V_{i=1}^k \inf (\mu^{(i)}, \lambda^{(i)})$$

$$= \langle [\mu], [\lambda] \rangle \langle [\mu], [\lambda] \rangle.$$ 

$$\Rightarrow |\langle [\mu], [\lambda] \rangle|^2 = |||[\mu]|||^2 |||[\lambda]|||^2.$$ 

Now $|||[\mu]|||^2 |||[\lambda]|||^2 = V_{i=1}^k \inf \mu^{(i)}, \mu^{(i)} V_{i=1}^k \inf \lambda^{(i)}, \lambda^{(i)}$

$$= V_{i=1}^k \mu^{(i)} V_{i=1}^k \lambda^{(i)}.$$
\[ = \bigvee_{i=1}^{k} \mu(i) \lambda(i) = \bigvee_{i=1}^{k} (\mu(i) \lambda(i), \mu(i) \lambda(i)) \]

\[ = \langle [\mu \lambda], [\mu \lambda] \rangle \]

\[ = \langle [\mu] [\lambda], [\mu] [\lambda] \rangle \]

\[ = \| [\mu] [\lambda] \|^2 . \]

Thus \[ |\langle [\mu], [\lambda] \rangle|^2 = \| [\mu] \|^2 \| [\lambda] \|^2 = \| [\mu] [\lambda] \|^2 . \]

### 7.3 FUZZY HILBERT SPACE OVER FUZZY NUMBER INNER PRODUCT SPACE

**Definition 7.3.1 (Fuzzy Hilbert space over Fuzzy number inner product space)**

Let \( E \) be a fuzzy number space, \( \tilde{A} \) be a fuzzy number inner product space over \( E \) and \( H \subset \tilde{A} \) is said to be fuzzy Hilbert space over \( E \) if it is fuzzy complete with respect to the fuzzy norm

\[ \| . \| : H \rightarrow R, \text{ with } \| [\mu_m] \|^2 = \langle [\mu_m], [\mu_m] \rangle, \quad [\mu_m] = \sum_i \mu_m(i) \in H \]

Moreover \( H \) satisfies Parallelogram law with respect to the fuzzy norm.

**Theorem 7.3.2**

Let \( F: E \rightarrow E \) be a fuzzy number mapping and \( H \) be a fuzzy Hilbert space over \( E \). Then \( F(H) \) defined as \( F(H) = \{ [\gamma] ; F([\mu]) = [\gamma], \ [\mu] \in H \} \) and \( F([\mu]) \neq F([\lambda]) \) for all \( [\mu] \neq [\lambda] \in H \) is also a fuzzy Hilbert space over \( E \).
Proof

Let $E$ be a fuzzy number space and $\tilde{A}$ be a fuzzy number inner product space over $E$ with $H \subset \tilde{A}$, $H$ being a fuzzy Hilbert space over $E$,

$$F(H) = \{[\gamma] : F([\mu]) = [\gamma] \text{ for all } [\mu] \in H \} \text{ and } F([\mu]) \neq F([\lambda])$$

for all $[\mu] \neq [\lambda] \in H$.

Since, $F([\mu]) = [\gamma]$ for all $[\mu] \in H$, define $\gamma = \sum_i \gamma^{(i)}$.

Then clearly $F(H) \subset \tilde{A}$ over $E$.

Thus it is possible to define the same inner product function and norm function (Definition 7.2.1, and 7.2.3) on $F(H)$.

Now to prove that $F(H)$ is fuzzy complete.

Let $\{F([\mu_n])\}$ be a Cauchy sequence in $F(H)$,

where $\{[\mu_n]\}$ is Cauchy in $H$.

Thus for very small $\delta, \epsilon > 0$ and $\| [\mu_n] - [\mu_m] \| < \delta$, \n
$$\| F([\mu_n]) - F([\mu_m]) \| < \epsilon.$$ 

Since $\{[\mu_n]\}$ is Cauchy in $H$, $\lim_{n \to \infty}[\mu_n] = [\mu]$.

Now for $[\mu] \in H$, $F([\mu]) = F(\lim_{n \to \infty}[\mu_n]) = \lim_{n \to \infty}[F([\mu_n])]$.

This shows that every Cauchy sequence in $F(H)$ is convergent.

Moreover it is clear that $F(H)$ satisfies Parallelogram law.

Hence $F(H)$ is a fuzzy Hilbert space over $E$. 
Definition 7.3.3

Let $\tilde{A}$ be a fuzzy number inner product space over $E$. Then $\tilde{B} \subset \tilde{A}$ is said to be a **fuzzy number orthogonal** set if for $[\mu_m] \in \tilde{B}$,
\[
\langle [\mu_m], [\mu_m] \rangle = \overline{0}, \forall [\mu_n] \in \tilde{A}.
\]

It is denoted by $[\mu_m] \perp [\mu_n]$ ($[\mu_m]$ is orthogonal to $[\mu_n]$).

Definition 7.3.4

Let $\tilde{B} \subset \tilde{A}$ be a fuzzy number orthogonal set if for all $[\mu_n] \in \tilde{B}$, $||[\mu_n]|| = 1$ then $\tilde{B}$ is called a **fuzzy number orthonormal set** of $\tilde{A}$.

Definition 7.3.5

Let $E$ be a fuzzy number space and $H$ be a fuzzy Hilbert space over $E$. A complete fuzzy number orthonormal set of $H$ is a called **fuzzy number orthonormal basis** of $H$.

Proposition 7.3.6

Let $\{[\mu_n]\}$ be a sequence of the fuzzy Hilbert space $H$ over $E$. Then the following statements are equivalent.

(i) $\{[\mu_n]\}$ is complete

(ii) If $\langle [\mu_n], [\mu] \rangle = \overline{0}$ then $[\mu] = \overline{0}$.

Definition 7.3.7

(This is the same concept as described in Definition 7.3.3 but defined for fuzzy number mapping.)
Let $E$ be a fuzzy number space and $F$ is a fuzzy number mapping defined on $E$. A fuzzy subset $F(\bar{B})$ of the fuzzy Hilbert space $F(H)$ is said to be fuzzy number orthogonal with respect to every fuzzy number orthogonal set $\bar{B}$ in $H$ if for every $[F(\mu_n)] \in F(\bar{B})$,

$$\langle [F(\mu_m)], [F(\mu_n)] \rangle = \overline{0}, \forall [F(\mu_n)] \in F(H).$$

It is denoted by $[F(\mu_m)] \perp [F(\mu_n)]$.

**Definition 7.3.8**

A fuzzy number orthonormal set $F(\bar{B})$ in a fuzzy Hilbert space is a fuzzy number orthonormal basis for $F(H)$ if it is maximal in $F(H)$ ($F(\bar{B})$ maximal means that if $F(\bar{B}) \subset F(\bar{A})$ for some $F(\bar{A}) \subset F(H)$ then $F(\bar{A}) = F(\bar{B})$).

Obviously every non-zero fuzzy Hilbert space has a fuzzy orthonormal basis.

**Definition 7.3.9**

If $F(\bar{B})$ is fuzzy number orthonormal basis for $F(H)$ then for $F([\mu]) \in F(\bar{B})$, we can find some $F([\gamma]) \in F(H)$ such that $F([\mu])$ converges to $F([\gamma])$.

Now for any closed fuzzy subset $F(\bar{A})$ of the fuzzy Hilbert space $F(H)$ has a fuzzy orthonormal basis same as that of $F(H)$.

Then for $([\mu]) \in F(\bar{A})$, it is possible to find some $F([\gamma]) \in F(H)$ such that $F([\mu])$ converges to $F([\gamma])$ and $F([\mu]) - F([\gamma]) = F([\sigma]) \in F(\bar{A})^\perp$ where $F(\bar{A})^\perp$ is the fuzzy number vector orthogonal to $F(\bar{A})$. 
Theorem 7.3.10

[Projection theorem for Fuzzy Hilbert space over fuzzy number space with fuzzy number mapping]

Let $H$ be a fuzzy Hilbert space over a fuzzy number space $E$ and $F: E \to E$ be a fuzzy number mapping so that $F(H)$ is a fuzzy Hilbert space over $E$, then there exists a non-empty closed bounded fuzzy subset $F(\tilde{A})$ of $F(H)$ such that $F(H) = F(\tilde{A}) + F(\tilde{A})^\perp$. Moreover $F(\tilde{A})^\perp = F(\tilde{A})$.

Proof

Let $E$ be a fuzzy number space and $F: E \to E$ be a fuzzy number mapping.

Consider a non-empty closed bounded subset $\tilde{A}$ of $H$.

Then $F(\tilde{A}) = \{F([\mu]) : [\mu] \in \tilde{A}\}$ is a bounded closed fuzzy subset of the fuzzy Hilbert space $F(H)$ over $E$.

Let $F([\mu]) \in F(H)$ is arbitrary.

If $F(\tilde{A}) = \{\tilde{0}\}$ then $F(\tilde{A})^\perp = F(H)$.

If $F(\tilde{A}) \neq \{\tilde{0}\}$ then the proof as follows:

Since $F(\tilde{A})$ is a closed subset of $F(H)$, then $F(\tilde{A})$ is a fuzzy Hilbert space by theorem (Since every closed subset of a fuzzy Hilbert space is also fuzzy Hilbert).

Also every fuzzy Hilbert space has a fuzzy number orthonormal basis (Definition 7.3.8).
Then for $F([\mu]) \in F(H)$, it is possible to find some $F([\gamma]) \in F(H)$ such that $F([\mu])$ converges to $F([\gamma])$.

Since $F(\tilde{A})$ is a fuzzy Hilbert space and has a fuzzy (number) orthonormal basis same as that of $F(H)$, then

$$F([\mu]) - F([\gamma]) \in F(\tilde{A})^\perp,$$

for $[\mu], [\gamma] \in \tilde{A}$.

Assume $F([\mu]) - F([\gamma]) = F([\vartheta])$, for $[\vartheta] \in H$.

Clearly $F([\vartheta]) \in F(\tilde{A})^\perp$.

Now $F([\mu]) = F([\vartheta]) + F([\gamma])$ with $F([\gamma]) \in F(\tilde{A})$ and $F([\vartheta]) \in F(\tilde{A})^\perp$.

Since $F([\mu])$ is arbitrary, this is true for every $F([\mu])$ in $F(H)$.

Thus

$$F(H) = F(\tilde{A}) + F(\tilde{A})^\perp$$  \hspace{1cm} (7.1)

Let $F([\mu]) \in F(\tilde{A})$.

Then for every $F([\gamma]) \in F(\tilde{A})^\perp$, $\langle [F(\mu)], [F(\gamma)] \rangle = \overline{0}$.

But by the definition of fuzzy number inner product,

$$\langle [F(\gamma)], [F(\mu)] \rangle = \overline{0}.$$

Thus

$$[F(\mu)] \in F(\tilde{A})^{\perp\perp} = F(\tilde{A})^{\perp\perp},$$

for every $F([\mu]) \in F(\tilde{A})$.

Hence $[F(\tilde{A})]^{\perp\perp} \subseteq F(\tilde{A})$.  \hspace{1cm} (7.2)

Now let $F([\mu]) \in F(\tilde{A})^{\perp\perp}$.

By Definition 7.3.9, for $F([\vartheta]) \in F(\tilde{A})^\perp$, it is possible to write
\[ F([\vartheta]) = F([\alpha]) - F([\beta]) \text{ for } F([\alpha]), F([\beta]) \in F(\bar{A}). \]

Therefore \( F([\mu]) = F([\vartheta]) - F([\gamma]) \).

where \( F([\vartheta]), F([\gamma]) \in F(\bar{A})^\perp \) (since \( F([\mu]) \in [F(\bar{A})]^{\perp\perp} \)).

This implies \( F([\mu]) + F([\gamma]) = F([\vartheta]) \in F(\bar{A})^\perp \),

where \( F([\mu]) \in F(\bar{A})^{\perp\perp} \) and \( F([\gamma]) \in F(\bar{A})^\perp \).

Also \( F([\vartheta]) \in F(\bar{A})^\perp \subset F(H) \) and hence \( F([\mu]) + F([\gamma]) = F([\vartheta]) \) with \( F([\mu]) \in F(\bar{A}) \) and \( F([\gamma]) \in F(\bar{A})^\perp \).

In both cases \( F([\gamma]) \in F(\bar{A})^\perp \).

This shows that \( F([\mu]) \in F(\bar{A}) \) as well as \( F([\mu]) \in F(\bar{A})^{\perp\perp} \).

Since \( F([\mu]) \) is arbitrary, \( F(\bar{A})^{\perp\perp} = F(\bar{A}) \).

Hence the proof is completed.

7.4 CONCLUSION

This chapter made a new approach for inner product spaces over fuzzy number space named fuzzy number inner product. This fuzzy number inner product space has been developed using fuzzy numbers as fuzzy number vectors. Also the properties connecting fuzzy number inner products and fuzzy number norms are generated. Definitions and theorems for fuzzy number inner product, orthogonality and orthonormality are established for fuzzy number mappings. Moreover the extension of projection theorem over fuzzy number space has been proved.