CHAPTER 6

INTEGRABILITY OF FUZZY NUMBER MAPPINGS

6.1 INTRODUCTION

The important kind of limit encountered in the calculus is the limit of a sum of elements. This method of finding the limit of a sum can be used to determine the lengths of curves, the areas bounded by curves, and the volumes of solids bounded by curved surfaces, and to solve other similar problems. An entirely different consideration of the problem of finding the area under a curve leads to a means of evaluating the integral. Historically ‘to integrate’ means ‘to give the sum of’ or ‘integration is the limit of sum’. In this chapter the integrability of fuzzy number mappings has been defined using this concept.

6.2 INTEGRABILITY OF CONVEX FUZZY NUMBER MAPPINGS

Definition 6.2.1

Let $E$ be a fuzzy number space and $F$ is a convex fuzzy number mapping defined over $E$. Let $\psi \equiv \{ \mu_0(r), \mu_1(r), \ldots, \mu_n(r) \}$ be a possible partition of the membership values (left part) of $[\mu]^r, \mu \in E, \ 0 \leq r \leq 1$.

Set up two fuzzy sums;
Upper Fuzzy sum

\[ S_k = \max_{0 \leq i \leq n} \{ d^{-}(M_i F(\mu_k)(r)) \} \].

Lower Fuzzy sum

\[ s_k = \max_{0 \leq i \leq n} \{ d^{-}(m_i F(\mu_k)(r)) \} \].

where \( M_i \) and \( m_i \) are the supremum and infimum membership values of the \( i^{th} \) sub interval of \([\mu]^r\).

\[ d^{-}(M_i F(\mu_k)(r)) = M_i \mid F(\mu_k)(r) - F(\mu_i)(r) \mid \], for all \( i < k = 0,1,2, \ldots, n \).

\[ d^{-}(m_i F(\mu_k)(r)) = m_i \mid F(\mu_k)(r) - F(\mu_i)(r) \mid \], for all \( i < k = 0,1,2, \ldots, n \).

Also \( d^{-} \) denotes the fuzzy metric defined for the left part membership values.

Now define \( I_F^V = \lim_{k \to n} \inf \{ S_k \} \)

and \( I_F^\wedge = \lim_{k \to n} \sup \{ s_k \} \).

\( I_F^V \) and \( I_F^\wedge \) are the Upper and Lower fuzzy integrals of \( F \) over \([\mu]^r\).

Thus \( F \) is said to be integrable over \([\mu]^r\) if \( I_F^V \) and \( I_F^\wedge \) exists.

Moreover if \( F \) is integrable over \([\mu]^r\), for all \( \mu \in E \) then \( F \) is integrable over \( E \).

**Note:** It is clear from the definitions that

\[ 0 \leq s_k \leq S_k \leq 1 \], for every \( k \).
Example 6.2.2

Let \( F(\mu)(x) = e^{x(\mu - 1)} \) be a fuzzy number function on the triangular fuzzy number interval [5, 45] with fuzzy sums \( S_k \) and \( S_{\bar{k}} \) as shown in the Table 6.1.

**Table 6.1 Upper and Lower Fuzzy sums**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \mu(x) )</th>
<th>( F(\mu)(x) )</th>
<th>( S_k )</th>
<th>( S_{\bar{k}} )</th>
</tr>
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<tbody>
<tr>
<td>5</td>
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<td>0.37</td>
<td>0.63</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0.25</td>
<td>0.47</td>
<td>0.53</td>
<td>0.158</td>
</tr>
<tr>
<td>15</td>
<td>0.5</td>
<td>0.61</td>
<td>0.39</td>
<td>0.195</td>
</tr>
<tr>
<td>20</td>
<td>0.75</td>
<td>0.78</td>
<td>0.22</td>
<td>0.165</td>
</tr>
<tr>
<td>25</td>
<td>1</td>
<td>1</td>
<td>0.22</td>
<td>0.165</td>
</tr>
<tr>
<td>30</td>
<td>0.75</td>
<td>0.78</td>
<td>0.39</td>
<td>0.195</td>
</tr>
<tr>
<td>35</td>
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<td>0.61</td>
<td>0.53</td>
<td>0.158</td>
</tr>
<tr>
<td>40</td>
<td>0.25</td>
<td>0.47</td>
<td>0.63</td>
<td>0</td>
</tr>
<tr>
<td>45</td>
<td>0</td>
<td>0.37</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Upper Fuzzy Sums

\[
S_1 = \max\{0.25 \times 0.1, 0.5 \times 0.24, 0.75 \times 0.41, 1 \times 0.63\} = 0.63
\]

\[
S_2 = \max\{0.5 \times 0.14, 0.75 \times 0.31, 1 \times 0.53\} = 0.53
\]

\[
S_3 = \max\{0.75 \times 0.17, 1 \times 0.39\} = 0.39
\]

\[
S_4 = \max\{1 \times 0.22\} = 0.22
\]

Lower Fuzzy Sums

\[
s_1 = \max\{0 \times 0.1, 0.24, 0 \times 0.41, 0 \times 0.63\} = 0
\]
\[ s_2 = \max\{0.25 \times 0.14, 0.25 \times 0.31, 0.25 \times 0.41, 0.25 \times 0.63\} = 0.158 \]

\[ s_3 = \max\{0.5 \times 0.17, 0.5 \times 0.39\} = 0.195 \]

\[ s_4 = \max\{0.75 \times 0.22\} = 0.165 \]

Then Upper and Lower integrals are

\[ I_F \supseteq \lim_{k \to \infty} \inf\{S_k\} = 0.22 \sim 0.2 \]

and \[ I_F \supseteq \lim_{k \to \infty} \sup\{s_k\} = 0.195 \sim 0.2. \]

**Definition 6.2.3 (Oscillatory Fuzzy Sum---OFS)**

Oscillatory Fuzzy Sum for a convex fuzzy number mapping is defined as,

\[ \omega_F([\mu]^r) = \min_{0 \leq k \leq n} \sup\{s_k, S_k\}, \]

for every upper and lower fuzzy sums of the partition

\[ \psi = \{\mu_0(r), \mu_1(r), \ldots, \mu_n(r)\} \]

of the membership values (left part) of \([\mu]^r, \mu \in E, 0 \leq r \leq 1\).

**Theorem 6.2.4**

The necessary and sufficient condition for the integrability of a convex fuzzy number mapping \( F \) over \( E \) is, For every \( \epsilon > 0 \) there is a \( \delta > 0 \) (very small) and for every partition \( \psi = \{\mu_0(r), \mu_1(r), \ldots, \mu_n(r)\} \) of the membership values (left part) of \([\mu]^r, \mu \in E, 0 \leq r \leq 1\), the Oscillatory Fuzzy Sum (OFS), \( \omega_F([\mu]^r) < \epsilon \).
Proof

Necessary

Suppose $F$ is integrable over $E$.

Let $[\mu]^r$ be a fuzzy number interval $\mu \in E$, $0 \leq r \leq 1$ and $\psi = \{\mu_0(r), \mu_1(r), \ldots, \mu_n(r)\}$ be a possible partition of the membership values (left part) of $[\mu]^r$.

Thus $I_F^\vee$ and $I_F^\wedge$ exists on $[\mu]^r$.

Then there is a $\delta > 0$ such that the upper fuzzy sum $S_K \leq \delta$ and thus the lower fuzzy sum $s_K \leq \delta$ for every $k$ so that

$$\min_{0 \leq k \leq n} \sup \{s_k, S_k\} \leq \varepsilon, \text{ where } \varepsilon > 0$$

$$\Rightarrow \omega_F([\mu]^r) \leq \varepsilon.$$

Sufficient

Assume that $\omega_F([\mu]^r) \leq \varepsilon$ where $\varepsilon > 0$

ie, $\min_{0 \leq k \leq n} \sup \{s_k, S_k\} \leq \varepsilon$, for every $k$

Since $0 \leq s_k \leq S_k \leq 1$, for every,

$$\min \sup \{s_k, S_k\} = \min_{0 \leq k \leq n} \{S_k\}$$

Thus $\min_{0 \leq k \leq n} \{S_k\} \leq \varepsilon$

$$\Rightarrow \lim_{k \to n} \inf \{S_k\} \leq \varepsilon.$$
\[ \Rightarrow l_F^Y \text{ exists on } [\mu]^R. \]

It is obvious from the definitions of \( S_k \) and \( s_k \) that
\[ \sup_{0 \leq k \leq n} \{ s_k \} \leq \varepsilon \]
\[ \Rightarrow \lim_{k \to n} \sup \{ s_k \} \leq \varepsilon \]
\[ \Rightarrow l_F^\Lambda \text{ exists on } [\mu]^R. \]

Hence \( F \) is integrable on \([\mu]^R\) for every \( \mu \in E \) and therefore \( F \) is integrable on \( E^\cdot \)

**Theorem 6.2.5**

Let \( F: E \to E \) be a convex fuzzy number mapping. If \( F \) is continuous on \( E \) then it is integrable on \( E^\cdot \).

**Proof**

Suppose \( F \) is continuous on \( E \).

This implies \( F \) is continuous on every fuzzy number interval \([\mu]^R\), \( \mu \in E^\cdot \), \( 0 \leq r \leq 1 \).

Let \( \psi = \{ \mu_0(r), \mu_1(r), \ldots, \mu_n(r) \} \) be a possible partition of the membership values (left part) of \([\mu]^R\).

Since \( F \) is continuous, for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) (very small \( \varepsilon, \delta \)) such that
\[ |F(\mu)(r) - F(\mu_\varepsilon)(r)| < \varepsilon \text{ whenever } |\mu(r) - \mu_\varepsilon(r)| < \delta, \text{ for every } \mu(r), \mu_\varepsilon(r) \in \psi. \]
In particular,

\[ |F(\mu_k)(r) - F(\mu_l)(r)| < \varepsilon \quad \text{whenever} \quad |\mu_k(r) - \mu_l(r)| < \delta, \text{ for every} \; \mu_i(r), \; \mu_i(r) \in \psi \; \text{and} \; i < k \]

\Rightarrow M_i |F(\mu_k)(r) - F(\mu_l)(r)| < M_i \varepsilon ,

where \( M_i \) is the supremum membership function of the \( i^{th} \) interval

\Rightarrow \max_{0 \leq i \leq n} M_i |F(\mu_k)(r) - F(\mu_l)(r)| < \left( \max_{0 \leq i \leq n} M_i \right) \varepsilon = M \varepsilon \; \text{(say)}

\Rightarrow S_k < M \varepsilon = \eta , \; \text{clearly} \; \eta > 0 \; \text{(very small)}, \; \text{since} \; 0 \leq M \leq 1 .

Thus OFS, \( \omega_F([\mu]^r) \leq \eta . \)

Hence the proof is completed.

**Theorem 6.2.6**

Let \( F: E \to E \) be a convex fuzzy number mapping. If \( F \) is a monotonic function on \( E \) then it is integrable on \( E . \)

**Proof**

Let \( F \) be a monotonic function on \( E \) and assume that \( F \) is monotonically increasing on the left part of \([\mu]^r, \mu \in E , \; 0 \leq r \leq 1 .\)

ie. \( F(\mu_k)(r) - F(\mu_l)(r) < \delta , \) for every \( \mu_i(r), \mu_k(r) \in \psi \) with

\( \mu_i(r) < \mu_k(r) \; , \text{where} \; \delta > 0 \; \text{(very small)} \)

\Rightarrow |F(\mu_k)(r) - F(\mu_l)(r)| < \delta
\[ \Rightarrow \max_{0 \leq i \leq n} M_i \left| F(\mu_k)(r) - F(\mu_i)(r) \right| < \left( \max_{0 \leq i \leq n} M_i \right) \delta \]

\[ \Rightarrow \max_{0 \leq i \leq n} d^-[M_i F(\mu_k)(r)] < \alpha \text{ where } \alpha = \left( \max_{0 \leq i \leq n} M_i \right) \delta > 0 \]

\[ \Rightarrow S_k < \alpha \text{ for all } k. \]

Similarly, \( S_k < \beta \text{ for all } k \), where \( \beta > 0 \).

Obviously, for some small \( \varepsilon > 0 \) the OFS,

\[ \omega_F([\mu]^r) = \min_{0 \leq k \leq n} \sup \{ s_k, S_k \} \leq \varepsilon. \]

Similar proof follows for the right part by assuming \( F \) is monotonically decreasing with respect to right part membership values.

Thus \( F \) is integrable on \( E \).

**Definition 6.2.7**

The **Integral Membership Function (IMF)** of a fuzzy number interval \([\mu]^r, \mu \in E, 0 \leq r \leq 1\) with respect to the convex fuzzy number mapping \( F \) is defined as

\[ I_F([\mu]^r) = \max(I_F^V, I_F^L). \]

**Proposition 6.2.8**

Let \( F: E \rightarrow E \) be a convex fuzzy number mapping and \( F \) is integrable over \( E \). Then there exists a membership value \( \mu(x) \) of any \( x \in [\mu]^r \) such that \( I_F([\mu]^r) = \mu(x) (\mu_h(r) - \mu_0(r)) \) with \( \mu_0(r) \) and \( \mu_h(r) \) are the boundaries of the membership values.
6.3 RIEMANN INTEGRABILITY OF CONVEX FUZZY NUMBER MAPPINGS

Definition 6.3.1

If the upper and lower fuzzy integrals (definition 6.2.1) of $F$ are equal, for every fuzzy number interval then $F$ is said to be **Riemann fuzzy integrable over $E$**.

In other words, $F$ is said to be Riemann Fuzzy Integrable over $E$ if the IMF, $I_F([\mu]^r) = I_F$ where $I_F = I_F^V = I_F^\wedge$, for every fuzzy number interval $[\mu]^r$.

Theorem 6.3.2

Let $E$ be a fuzzy number space and $F$ be a convex fuzzy number mapping. The condition for $F$ to be fuzzy integrable on every fuzzy number interval $[\mu]^r, \mu \in E, \ 0 \leq r \leq 1$ is that the OFS, $\omega_F([\mu]^r) = 0$, for every partition $\psi$ of $[\mu]^r$.

Proof

Let $\psi = \{\mu_0(r), \mu_1(r), \ldots, \mu_n(r)\}$ be a possible partition of the membership values (left part) of $[\mu]^r, \mu \in E, \ 0 \leq r \leq 1$.

Assume that the OFS, $\omega_F([\mu]^r) = 0$

$$\Rightarrow \min_{0 \leq k \leq n} \sup \{s_k, S_k\} = 0$$

$$\Rightarrow \min_{0 \leq k \leq n} \{S_k\} = 0, \text{ since } 0 \leq s_k \leq S_k \leq 1, \text{for every } k$$

Thus $\max_k \{s_k\} = 0$
\[ \Rightarrow s_k = 0 \text{ for every } k. \]

This implies that the lower fuzzy sum is equal to zero and hence the lower fuzzy integral is also zero.

On the other hand \( \omega_F([\mu]^r) = 0 \) implies the existence of upper fuzzy integrals only otherwise upper and lower fuzzy integrals will be coincide.

Thus \( F \) is Fuzzy Riemann integrable over \( E \).

\[ \textbf{6.4 \hspace{1em} CONCLUSION} \]

In this chapter, it is found that the integrability of fuzzy number mappings explained on the basis of upper and lower fuzzy sums is definitely an easy application. The highlight of this chapter is the introduction of Oscillatory Fuzzy Sum. The necessary and sufficient conditions for fuzzy integrability have been developed using Oscillatory Fuzzy Sum. Moreover this chapter generated a theorem for Riemann integrability in connection with Oscillatory Fuzzy Sum.