CHAPTER 1

PAIRWISE ALMOST CONTINUOUS FUNCTIONS

The first chapter is devoted to the study of almost continuous, almost $\beta$-continuous, almost precontinuous and almost semi-continuous functions in bitopological spaces. The concept of almost continuous mappings was introduced by Singal and Singal [15]. This class contains the class of continuous mappings and is contained in the class of weakly continuous mappings. The concept of $\beta$-open sets and $\beta$-continuous functions was introduced and investigated by Abd El- Monsef et al. [1]. Nasef and Noiri [11] was introduced a new class of functions called almost precontinuous functions and studied fundamental properties of almost $\beta$-continuous functions and Noiri and Popa [12] investigated further properties of almost $\beta$-continuous functions. Jafari and Noiri [5] investigated some more properties of almost precontinuous functions. Munshi and Bassan [9] introduced a new class of mappings called almost semi-continuous mappings. This class contains the class of almost continuous mappings Singal and Singal [15] and that of semi continuous mappings Levine [8].

Bose and Sinha [4] introduced the concept of pairwise almost continuous by using $(i, j)$ regular open set in place of $\sigma_i$-open sets in the range of space $(Y, \sigma_1, \sigma_2)$. Noiri and Popa [13] introduced the concept of almost precontinuous functions in bitopological spaces. Khedr and Noiri [7] introduced the concept of almost s-continuous functions in bitopological spaces.

In section 1, we gives further properties of almost continuity and closure continuity in bitopological spaces and gives almost continuity
imply closure continuity in bitopological spaces.

In section 2, we introduced pairwise almost \( \beta \)-continuous functions and investigated further properties of pairwise almost \( \beta \)-continuous functions. We gives several properties concerning pairwise \( \beta \)-continuity [14], pairwise almost \( \beta \)-continuity and pairwise weak \( \beta \)-continuity.

In section 3, we investigated some more properties of pairwise almost precontinuous functions. It turns out that pairwise almost precontinuity is stronger than pairwise almost weak continuity.

In section 4, we investigated some more properties of pairwise almost semi-continuous functions in bitopological spaces. This class contains the class of pairwise almost continuous mappings and that of pairwise semi-continuous mappings. We discuss the strength of pairwise almost semi-continuous vis-a-vis several other pairwise continuities such as pairwise feebly continuity, pairwise semi-continuity, pairwise weak continuity [4], pairwise almost continuity and pairwise \( \theta \)-continuity [3]. We give the composition and product of two almost semi continuous functions in bitopological spaces. We introduced pairwise pre semi-open and pairwise pre-semi \( \delta \)-open mapping and their properties.

1. PAIRWISE ALMOST CONTINUOUS FUNCTIONS :-

1.1. Definition [4]. A function \( f : ( X, \tau_1, \tau_2) \to ( Y, \sigma_1, \sigma_2) \) is said to be pairwise almost continuous ( p. a. c. ) if for each \( x \in X \) and each \( \sigma_i \)-open set \( V \) of \( Y \) containing \( f(x) \), there exists a \( \tau_i \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subset \sigma_i \cdot \text{Int}(\sigma_j \cdot \text{Cl}(V)) \).

A function \( f : ( X, \tau_1, \tau_2) \to ( Y, \sigma_1, \sigma_2) \) is pairwise almost continuous iff \( f^{-1}(V) \) is a \( \tau_i \)-open (resp., \( \tau_i \)-closed) in \( X \) for every \((i, j)\) regular open
( resp., (i, j) regular closed ) set V of Y.

1.2. Lemma. For a bitopological space \(( X, \tau_1, \tau_2)\), the following properties hold:

1. \((i, j)\alpha O(X) = (i, j) PO(X) \cap (i, j) SO(X)\).

2. \((i, j) PO(X) \cup (i, j) SO(X) \subset (i, j) SPO(X)\).

Proof. This follows easily from the definitions.

1.3. Theorem. For a function \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\), the following are equivalent:

1. \(f\) is pairwise almost continuous.

2. \(\tau_j - Cl(f^{-1}(V)) \subset f^{-1}(\sigma_j - Cl(V))\) for every \(V \in (i, j) SPO(Y)\).

3. \(\tau_j - Cl(f^{-1}(V)) \subset f^{-1}(\sigma_j - Cl(V))\) for every \(V \in (i, j) SO(Y)\).

4. \(f^{-1}(V) \subset \tau_i - Int(f^{-1}(\sigma_i - Int(\tau_j - Cl(V))))\) for every \(V \in (i, j) PO(Y)\).

Proof: (1) \(\Rightarrow\) (2). Let \(V \in (i, j) SPO(Y)\), we have \(\sigma_j - Cl(V)\) is a \((j, i)\)regular closed in Y. Since \(f\) is pairwise almost continuous, \(f^{-1}(\sigma_j - Cl(V))\) is a \(\tau_j - closed in X\) and we obtain \(\tau_j - Cl(f^{-1}(V)) \subset f^{-1}(\sigma_j - Cl(V))\).

(2) \(\Rightarrow\) (3). Since \((i, j) SO(Y) \subset (i, j) SPO(Y)\), this is obvious.

(3) \(\Rightarrow\) (1). Let \(F\) be any \((j, i)\) regular closed set of Y. Then \(F = \sigma_j - Cl(\tau_i - Int(F))\) and hence \(F \in (i, j) SO(Y)\). Therefore, we have \(\sigma_j - Cl(f^{-1}(F)) \subset f^{-1}(\sigma_j - Cl(F)) = f^{-1}(F)\). Hence \(f^{-1}(F)\) is a \(\tau_j -closed and f\) is pairwise almost continuous.
(1) \(\Rightarrow\) (4). Let \(V \in (i, j)\ PO(Y)\). Then \(V \subset \sigma_i\ -\ \text{Int}(\ \sigma_j\ -\ \text{Cl}(\ V))\) and \(\sigma_i\ -\ \text{Int}(\ \sigma_j\ -\ \text{Cl}(\ V))\) is an \((i, j)\) regular open. Since \(f\) is pairwise almost continuous, \(f^{-1}(\sigma_i\ -\ \text{Int}(\ \sigma_j\ -\ \text{Cl}(\ V)))\) is a \(\tau_i\)-open in \(X\) and hence \(f^{-1}(V) \subset f^{-1}(\sigma_i\ -\ \text{Int}(\ \sigma_j\ -\ \text{Cl}(\ V)))\).

(4) \(\Rightarrow\) (1). Let \(V\) be any \((i, j)\) regular open set of \(Y\). Then \(V \in (i, j)\ PO(Y)\) and hence \(f^{-1}(V) \subset \tau_i\ -\ \text{Int}(f^{-1}(\sigma_i\ -\ \text{Int}(\ \sigma_j\ -\ \text{Cl}(\ V)))) = \tau_i\ -\ \text{Int}(f^{-1}(\sigma_i\ -\ \text{Int}(\ \sigma_j\ -\ \text{Cl}(\ V))))\). Therefore, \(f^{-1}(V)\) is a \(\tau_i\)-open in \(X\) and hence \(f\) is pairwise almost continuous.

1.4. Lemma. For a subset \(V\) of \((Y, \sigma_1, \sigma_2)\), the following properties hold:

(1) \((i, j)\ \alpha\text{Cl}(\ V) = \sigma_i\ -\ \text{Cl}(\ V)\) for every \(V \in (j, i)\ SPO(Y)\).

(2) \((i, j)\ \rho\text{Cl}(\ V) = \sigma_i\ -\ \text{Cl}(\ V)\) for every \(V \in (j, i)\ SO(Y)\).

(3) \((i, j)\ \kappa\text{Cl}(\ V) = \sigma_j\ -\ \text{Int}(\ \sigma_i\ -\ \text{Cl}(\ V))\) for every \(V \in (j, i)\ PO(Y)\).

Proof. (1) Let \(V \in (j, i)\ SPO(Y)\). Then \(V \subset \sigma_i\ -\ \text{Cl}(\sigma_j\ -\ \text{Int}(\ \sigma_i\ -\ \text{Cl}(\ V)))\), we have \(\alpha\text{Cl}(\ V) = V \cup \sigma_i\ -\ \text{Cl}(\sigma_j\ -\ \text{Int}(\ \sigma_i\ -\ \text{Cl}(\ V))) = \sigma_i\ -\ \text{Cl}(\ V)\).

(2) Let \(V \in (j, i)\ SO(Y)\). Then \(V \subset \sigma_i\ -\ \text{Cl}(\sigma_j\ -\ \text{Int}(\ V))\), we have
\[(i, j)\rho\text{Cl}(\ V) = V \cup \sigma_i\ -\ \text{Cl}(\sigma_j\ -\ \text{Int}(\ V)) = \sigma_i\ -\ \text{Cl}(\ V)\).

(3) Let \(V \in (j, i)\ PO(Y)\). Then \(V \subset \sigma_j\ -\ \text{Int}(\ \sigma_i\ -\ \text{Cl}(\ V))\), we have
\[(i, j)\kappa\text{Cl}(\ V) = V \cup \sigma_j\ -\ \text{Int}(\ \sigma_i\ -\ \text{Cl}(\ V)) = \sigma_j\ -\ \text{Int}(\ \sigma_i\ -\ \text{Cl}(\ V))\).

1.5. Corollary. For a function \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\), the following are equivalent:

(1) \(f\) is pairwise almost continuous.

(2) \(\tau_j\ -\ \text{Cl}(f^{-1}(V)) \subset f^{-1}(\alpha\ \text{Cl}(\ V))\) for every \(V \in (i, j)\ SPO(Y)\).
(3) $\tau_j\text{-Cl}(f^{-1}( V )) \subset f^{-1}( (i, j)p\text{Cl}( V ))$ for every $V \in (i, j)\text{SO}(Y)$.

(4) $f^{-1}( V ) \subset \tau_i\text{-Int}(f^{-1}( (i, j)s\text{Cl}( V )))$ for every $V \in (i, j)\text{PO}(Y)$.

**Proof.** This is an immediate consequence of **Theorem 1.3** and **Lemma 1.4**.

1.6. **Definition.** A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is **pairwise closure continuous** (p. c. c) at $x \in X$ if, given any $\sigma_i$-open set $V$ in $Y$ containing $f(x)$, there exists a $\tau_i$-open set $U$ in $X$ containing $x$ such that $f(\tau_j\text{-Cl}( U )) \subset \sigma_j\text{-Cl}( V )$. If this condition is satisfied at each $x \in X$, then $f$ is said to be **pairwise closure continuous**.

1.7. **Theorem.** Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be pairwise almost continuous. Then $f$ is pairwise closure continuous.

**Proof.** Let $x \in X$ and let $V$ be a $\sigma_i$-open set containing $f(x)$. By pairwise almost continuity of $f$, there exists a $\tau_i$-open set $U$ containing $x$ such that $f(U) \subset \sigma_i\text{-Int}(\sigma_j\text{-Cl}( V ))$. Let $y \in \tau_j\text{-Cl}( U )$. For any $\sigma_i$-open set $W$ containing $f(y)$ there exists, by pairwise almost continuity of $f$, a $\tau_i$-open set $A$ containing $y$ such that $f(A) \subset \sigma_i\text{-Int}(\sigma_j\text{-Cl}( W ))$. Since $y \in \tau_j\text{-Cl}( U )$, we have $A \cap U \neq \emptyset$. Therefore,

$$\emptyset \neq f(A \cap U) \subset \sigma_i\text{-Int}(\sigma_j\text{-Cl}( V )) \cap \sigma_i\text{-Int}(\sigma_j\text{-Cl}( W )) \subset \sigma_j\text{-Cl}( W ).$$

Since $\sigma_i\text{-Int}(\sigma_j\text{-Cl}( V )) \cap \sigma_i\text{-Int}(\sigma_j\text{-Cl}( W ))$ is a $\sigma_i$-open, we have $\sigma_i\text{-Int}(\sigma_j\text{-Cl}( V )) \cap \sigma_i\text{-Int}(\sigma_j\text{-Cl}( W )) \cap W \neq \emptyset$, that is $\sigma_i\text{-Int}(\sigma_j\text{-Cl}( V )) \cap W \neq \emptyset$.

Since this is true for every $\tau_i$-open set containing $f(y)$ we have $f(y) \in \sigma_j\text{-Cl}( V )$. Also since this is true for every $y \in \tau_j\text{-Cl}( U )$ we obtain $f(\tau_j\text{-Cl}( U )) \subset \sigma_j\text{-Cl}( V )$, that is $f$ is pairwise closure continuous.

1.8. **Definition.** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called **pairwise closure compact** if every $\tau_i$-open cover of $A$ has a finite subcollection whose $\tau_j$-closure cover $A$. 
1.9. Theorem. Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be pairwise closure continuous and let \( K \) be a pairwise closure compact subset of \( X \). Then \( f(K) \) is a pairwise closure compact subset of \( Y \).

**Proof.** Let \( \nu \) be a \( \sigma_i \)-open cover of \( f(K) \). For each \( k \in K \), \( f(k) \in V_k \) for some \( V_k \in \nu \). By pairwise closure continuity of \( f \), there exists a \( \tau_i \)-open set \( U_k \) containing \( x \) such that \( f(\tau_j-\text{Cl}(U_k)) \subseteq \sigma_j-\text{Cl}(V_k) \). The collection \( \{ U_k : k \in K \} \) is a \( \tau_i \)-open cover of \( K \) and so, since \( K \) is pairwise closure compact, there is a finite subcollection \( \{ U_k : k \in K_0 \} \), where \( K_0 \) is a finite subset of \( K \), and \( \{ \tau_j-\text{Cl}(U_k) : k \in K_0 \} \) covers \( K \). Clearly \( \{ \sigma_j-\text{Cl}(V_k) : k \in K_0 \} \) covers \( f(K) \) and thus \( f(K) \) is a pairwise closure compact subset of \( Y \).

1.10. Corollary. Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be pairwise almost continuous and let \( K \) be a pairwise closure compact subset of \( X \). Then \( f(K) \) is a pairwise closure compact subset of \( Y \).

2. PAIRWISE ALMOST \( \beta \)-CONTINUOUS FUNCTIONS :-

2.1. Lemma. Let \( A \) be a subset of a bitopological space \( (X, \tau_1, \tau_2) \), then

(1) \( (i, j) \beta\text{Cl}(X - A) = X - (i, j) \beta\text{Int}(A) \).

(2) \( x \in (i, j) \beta\text{Cl}(A) \) iff \( A \cap U \neq \emptyset \) for each \( U \in (i, j)\beta\text{O}(X, x) \).

(3) \( A \) is an \( (i, j) \beta \)-closed in \( X \) iff \( A = (i, j) \beta\text{Cl}(A) \).

(4) \( (i, j) \beta\text{Cl}(A) \) is an \( (i, j) \beta \)-closed in \( X \).
2.2. **Definition.** A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be

1. **pairwise weakly \( \beta \)-continuous (p. w. \( \beta \). c.)** at \( x \in X \) if for each \( \sigma_i \)-open set \( V \) of \( Y \) containing \( f(x) \), there exists \( U \in (i, j) \beta O(X, x) \) such that \( f(U) \subseteq \sigma_j Cl(V) \),

2. **pairwise almost \( \beta \)-continuous (p. a. \( \beta \). c.)** at \( x \in X \) if for each \( \sigma_i \)-open set \( V \) of \( Y \) containing \( f(x) \), there exists \( U \in (i, j) \beta O(X, x) \) such that \( f(U) \subseteq \sigma_i Int(\sigma_j Cl(V)) \).

2.3. **Theorem.** For a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \), the following are equivalent:

(i) \( f \) is pairwise almost \( \beta \)-continuous at \( x \in X \).

(ii) For each \( \sigma_i \)-neighbourhood \( V \) of \( f(x) \), \( x \in \tau_j Cl(\tau_i Int(\tau_j Cl(f^{-1}((i, j) sCl(V)))))) \).

(iii) For each \( \sigma_i \)-neighbourhood \( V \) of \( f(x) \), and each \( \tau_i \)-neighbourhood \( U \) of \( x \), there exists a nonempty \( \tau_i \)-open set \( G \subseteq U \) such that \( G \subseteq \tau_j Cl(f^{-1}((i, j) sCl(V))) \).

(iv) For each \( \sigma_i \)-neighbourhood \( V \) of \( f(x) \), there exists \( U \in (i, j) SO(X, x) \) such that \( U \subseteq \tau_j Cl(f^{-1}((i, j) sCl(V))) \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( V \) be any \( \sigma_i \)-neighbourhood of \( f(x) \). Then there exists \( U \in (i, j) \beta O(X, x) \) such that \( f(U) \subseteq (i, j) sCl(V) \). Then \( U \subseteq f^{-1}((i, j) sCl(V)) \). Since \( U \) is an \( (i, j) \beta \)-open, \( x \in U \subseteq \tau_j Cl(\tau_i Int(\tau_j Cl(U))) \subseteq \tau_j Cl(\tau_i Int(\tau_j Cl(f^{-1}((i, j) sCl(V))))) \).

(2) \( \Rightarrow \) (3). Let \( V \) be any \( \sigma_i \)-neighbourhood of \( f(x) \) and \( U \) a \( \tau_i \)-open set of \( X \) containing \( x \). Since \( x \in \tau_j Cl(\tau_i Int(\tau_j Cl(f^{-1}((i, j) sCl(V))))), \) we have \( U \cap \tau_i Int(\tau_j Cl(f^{-1}((i, j) sCl(V)))) \neq \emptyset \).

Put \( G = U \cap \tau_i Int(\tau_j Cl(f^{-1}((i, j) sCl(V)))) \), then \( G \) is a nonempty \( \tau_i \)-open set.
G ⊆ U and G ⊆ τ_i-Int(τ_j-Cl( f^{-1}((i, j) sCl( V ))))) ⊆ τ_j-Cl( f^{-1}((i, j) sCl( V ))).

(3) ⇒ (4). Let U_x be the system of a τ_i-open sets containing x. For any τ_i-open set U ∈ U_x and any σ_i-neighbourhood V of f(x), there exists a nonempty τ_i-open set G_U ⊆ U such that G_U ⊆ τ_j-Cl( f^{-1}((i, j) sCl( V ))). Let G = ∪{ G_U : U ∈ U_x }. Then G is a τ_i-open, x ∈ τ_j-Cl(G) and G ⊆ τ_j-Cl( f^{-1}((i, j) sCl( V ))). Put U_0 = G ∪{ x }, then G ⊆ U_0 ⊆ τ_j-Cl(G) and U_0 ∈ (i, j)SO(X, x). And also we have U_0 ⊆ τ_j-Cl( f^{-1}((i, j) sCl( V ))).

(4) ⇒ (1). Let V be any σ_i-open neighbourhood of f(x). There exists G ∈ (i, j) SO(X, x) such that G ⊆ τ_j-Cl( f^{-1}((i, j) sCl( V ))). Then we have x ∈ f^{-1}(( V ) ∩ G ⊆ f^{-1}((i, j) sCl( V )) ∩ τ_j-Cl(τ_i-Int( G )) ⊆ f^{-1}((i, j) sCl( V )) ∩ τ_j-Cl( τ_i-Int(τ_j-Cl( f^{-1}((i, j) sCl( V )))))) = (i, j) βInt( f^{-1}(( i, j ) sCl( V )))). Therefore, put U = (i, j) βInt ( f^{-1}((i, j)sCl( V ))), then U ∈ (i, j) βO(X, x) and f(U) ⊆ (i, j) sCl( V ) = σ_i-Int(σ_j-Cl( V )). This shows that f is a pairwise almost β-continuous at x.

2.4. Theorem. For a function f : ( X, τ_1, τ_2 ) → ( Y, σ_1, σ_2 ), the following are equivalent:

(1) f is pairwise almost β-continuous.

(2) For each x ∈ X and each V ∈ σ_i-open containing f(x), there exists U ∈ (i, j) βO(X) containing x such that f ( U ) ⊆ σ_i-Int(σ_j-Cl( V )).

(3) f^{-1}( F ) ∈ (i, j)βC( X ) for every F ∈ (i, j)RC( Y ).

(4) f^{-1}( V ) ∈ (i, j) βO( X ) for every V ∈ (i, j) RO( Y ).

Proof. The proof is obvious and is thus omitted.

2.5. Theorem. For a function f : ( X, τ_1, τ_2 ) → ( Y, σ_1, σ_2 ), the following are equivalent:
(1) $f$ is pairwise almost $\beta$-continuous.

(2) $f((i, j)\beta \text{Cl}( A )) \subset (i, j) \delta \text{Cl}( f( A ))$ for every subset $A$ of $X$.

(3) $(i, j) \beta \text{Cl}(f^{-1}( B )) \subset f^{-1}((i, j)\delta \text{ Cl}( B ))$ for every subset $B$ of $Y$.

(4) $f^{-1}( F ) \in (i, j)\beta C( X )$ for every $(i, j)$ $\delta$-closed set $F$ of $Y$.

(5) $f^{-1}( V ) \in (i, j)\beta O( X )$ for every $(i, j)$ $\delta$-open set $V$ of $Y$.

**Proof.** (1) $\Rightarrow$ (2). Let $A$ be a subset of $X$. Since $(i, j) \delta \text{Cl}(f( A ))$ is an $(i, j) \delta$-closed in $Y$, it is denoted by $\cap \{ F_\alpha : F_\alpha \in (i, j) \text{RC}( Y ), \alpha \in \Lambda \}$, where $\Lambda$ is an index set. By Theorem 2.4, we have

$$A \subset f^{-1}((i, j) \delta \text{Cl}(f( A ))) = \cap \{ f^{-1}( F_\alpha ) : \alpha \in \Lambda \} \subset (i, j) \beta \text{C}(X)$$

and hence

$$(i, j) \beta \text{Cl}(A) \subset f^{-1}((i, j) \delta \text{Cl}(f(A))).$$

Therefore, we obtain

$$f((i, j)\beta \text{Cl}( A )) \subset (i, j)\delta \text{Cl}(f( A )).$$

(2) $\Rightarrow$ (3). Let $B$ be a subset of $Y$. We have

$$f((i, j) \beta \text{Cl}( f^{-1}( B ))) \subset (i, j) \delta \text{Cl}(f( f^{-1}( B )))) \subset (i, j) \delta \text{Cl}( B)$$

and hence

$$(i, j) \beta \text{Cl}(f^{-1}( B )) \subset f^{-1}((i, j)\delta \text{Cl}( B )).$$}

(3) $\Rightarrow$ (4). Let $F$ be any $(i, j)$ $\delta$-closed set of $Y$. We have $f^{-1}( (i, j) \beta \text{Cl}( f^{-1}( F ))) \subset f^{-1}((i, j)\delta \text{Cl}( F )) = f^{-1}( F )$ and hence $f^{-1}( F )$ is an $(i, j)$ $\beta$-closed in $X$.

(4) $\Rightarrow$ (5). Let $V$ be any $(i, j)$ $\delta$-open set of $Y$. We have

$$f^{-1}( Y - V ) = X - f^{-1}( V ) \in (i, j) \beta \text{C}( X )$$

and hence $f^{-1}( V ) \in (i, j) \beta O( X )$.

(5) $\Rightarrow$ (1). Let $V$ be any $(i, j)$ regular open set of $Y$. Since $V$ is an $(i, j)$ $\delta$-open in $Y$, $f^{-1}( V ) \in (i, j) \beta O(X)$ and hence by Theorem 2.4, $f$ is pairwise almost $\beta$-continuous.

**2.6. Theorem.** For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following are equivalent:
(1) f is pairwise almost $\beta$-continuous.

(2) $(i, j) \beta Cl( f^{-1}(\sigma_i-Cl(\sigma_j-Int(\sigma_i-Cl( B ))))) \subset f^{-1}(\sigma_i-Cl( B ))$ for every subset B of Y.

(3) $(i, j) \beta Cl(f^{-1}(\sigma_i-Cl(\sigma_j-Int( F )))) \subset f^{-1}(\sigma_i-Cl( F ))$ for every $\sigma_i$-closed set of Y.

(4) $(i, j) \beta Cl(f^{-1}(\sigma_i-Cl( V ))) \subset f^{-1}(\sigma_i-Cl( V ))$ for every $\sigma_j$-open set V of Y.

(5) $f^{-1}( V ) \subset (i, j) \beta Int(f^{-1}((i, j)sCl( V )))$ for every $\sigma_i$-open set V of Y.

(6) $f^{-1}( V ) \subset \tau_j-Cl(\tau_i-Int(\tau_j-Cl( f^{-1}((i, j)sCl( V )))))$ for every $\sigma_i$-open set V of Y.

**Proof.** (1) $\Rightarrow$ (2). Let B be any subset of Y. Assume that $x \in X - f^{-1}(\sigma_i-Cl( B ))$. Then $f(x) \in Y-\sigma_i-Cl(B)$ and there exists a $\sigma_i$-open set V containing $f(x)$ such that $V \cap B = \emptyset$, hence $\sigma_i-Int(\sigma_j-Cl( V )) \cap \sigma_i-Cl(\sigma_j-Int(\sigma_i-Cl( B ))) = \emptyset$. Since f is pairwise almost $\beta$-continuous, there exists $U \in (i, j) \beta O( X, x )$ such that $f(U) \subset \tau_i-Int(\tau_j-Cl( V ))$. Therefore, we have

$$U \cap f^{-1}(\sigma_i-Cl(\sigma_j-Int(\sigma_i-Cl( B )))) = \emptyset$$

and hence

$$x \in X - (i, j) \beta Cl( f^{-1}(\sigma_i-Cl(\sigma_j-Int(\sigma_i-Cl( B ))))).$$

Thus we obtain $(i, j) \beta Cl( f^{-1}(\sigma_i-Cl(\sigma_j-Int(\sigma_i-Cl( B ))) \subset f^{-1}(\sigma_i-Cl( B )))$.

(2) $\Rightarrow$ (3). Let F be any $\sigma_i$-closed set of Y. Then we have

$$(i, j) \beta Cl( f^{-1}(\sigma_i-Cl(\sigma_j-Int( F ))))$$

$$= (i, j) \beta Cl( f^{-1}(\sigma_i-Cl(\sigma_j-Int(\sigma_i-Cl( F )))))$$

$$\subset f^{-1}(\sigma_i-Cl(\sigma_j-Int( F ))) \subset f^{-1}( F ).$$

(3) $\Rightarrow$ (4). For any $\sigma_j$-open set V of Y. $\sigma_i-Cl(V)$ is an $(i, j)$ regular closed in Y and we have

$$(i, j) \beta Cl( f^{-1}(\sigma_i-Cl( V ))) = (i, j) \beta Cl( f^{-1}(\sigma_i-Cl(\sigma_j-Int(\sigma_i-Cl( V ))) \subset f^{-1}(\sigma_i-Cl( V )))$$

(4) $\Rightarrow$ (5). Let V be any $\sigma_i$-open set Y. Then $Y-\sigma_j-Cl(V)$ is a $\sigma_j$-open in Y by
using Lemma 2.1 and 1.4(3), we have
\[ X - (i, j) \beta \text{Int}(f^{-1}((i, j)\text{sCl}(V))) = (i, j) \beta \text{Cl}(f^{-1}(\sigma_i-\text{Int}(\sigma_j-\text{Cl}(V)))) \]
\[ \subset f^{-1}(\sigma_i-\text{Cl}(\sigma_j-\text{Cl}(V))) \subset X - f^{-1}(V). \]
Therefore, we obtain
\[ f^{-1}(V) \subset (i, j) \beta \text{Int}(f^{-1}((i, j)\text{sCl}(V))). \]

(5) \Rightarrow (6). Let V be any \( \sigma_i \)-open set of Y. We have
\[ f^{-1}(V) \subset (i, j) \beta \text{Int}(f^{-1}((i, j)\text{sCl}(V))) \subset \tau_j-\text{Cl}(\tau_i-\text{Int}(\tau_j-\text{Cl}(f^{-1}((i, j)\text{sCl}(V))))). \]

(6) \Rightarrow (1). Let x be any point of X and V any \( \sigma_i \)-open set Y containing f(x). Then x \in f^{-1}(V) \subset \tau_j-\text{Cl}(\tau_i-\text{Int}(\tau_j-\text{Cl}(f^{-1}((i, j)\text{sCl}(V))))). \) It follows from Theorem 2.3, that f is pairwise almost \( \beta \)-continuous at any point x of X. Therefore, f is pairwise almost \( \beta \)-continuous.

2.7. Theorem. For a function f : (X, \( \tau_1, \tau_2 \)) \rightarrow (Y, \sigma_1, \sigma_2), the following are equivalent :

(1) f is pairwise almost \( \beta \)-continuous.

(2) (i, j) \( \beta \text{Cl}(f^{-1}(V)) \subset f^{-1}(\sigma_i-\text{Cl}(V)) \) for each V \( \in (j, i) \beta \text{O}(Y) \).

(3) (i, j) \( \beta \text{Cl}(f^{-1}(V)) \subset f^{-1}(\sigma_i-\text{Cl}(V)) \) for each V \( \in (j, i) \text{SO}(Y) \).

(4) \( f^{-1}(V) \subset (i, j)\beta \text{Int}(f^{-1}(\tau_j-\text{Int}(\tau_i-\text{Cl}(V)))) \) for each V \( \in (i, j) \text{PO}(Y) \).

Proof. (1) \Rightarrow (2). Let V be any \( (j, i) \beta \)-open set of Y, we have \( \sigma_i-\text{Cl}(V) \) is an \( (i, j) \) regular closed in Y. Since f is pairwise almost \( \beta \)-continuous, \( f^{-1}(\sigma_i-\text{Cl}(V)) \) is an \( (i, j) \beta \)-closed in X. Therefore, we obtain
\[ (i, j) \beta \text{Cl}(f^{-1}(V)) \subset f^{-1}(\sigma_i-\text{Cl}(V)). \]

(2) \Rightarrow (3). This is obvious since \( (j, i) \text{SO}(Y) \subset (j, i) \beta \text{O}(Y) \).

(3) \Rightarrow (1). Let F be any \( (i, j) \) regular closed set of Y. Then F is a \( (j, i) \) semi-open in Y and hence \( (i, j) \beta \text{Cl}(f^{-1}(F)) \subset f^{-1}(\sigma_i-\text{Cl}(F)) = f^{-1}(F). \)
shows that $f^{-1}(F)$ is an $(i, j) \beta$-closed. Therefore, $f$ is pairwise almost $\beta$-continuous.

(1) $\Rightarrow$ (4). Let $V$ be any $(i, j)$ preopen set of $Y$. Then $V \subset \tau_i\text{-Int}(\tau_j\text{-Cl}(V))$ and $\tau_i\text{-Int}(\tau_j\text{-Cl}(V))$ is an $(i, j)$ regular open in $Y$. Since $f$ is pairwise almost $\beta$-continuous $f^{-1}(\tau_i\text{-Int}(\tau_j\text{-Cl}(V)))$ is an $(i, j) \beta$-open in $X$ and hence we obtain that $f^{-1}(V) \subset f^{-1}(\sigma_i\text{-Int}(\sigma_j\text{-Cl}(V))) \subset (i, j) \beta\text{Int}(f^{-1}(\tau_i\text{-Int}(\tau_j\text{-Cl}(V))))$.

(4) $\Rightarrow$ (1). Let $V$ be any $(i, j)$ regular open set of $Y$. Then $V$ is an $(i, j)$ preopen and $f^{-1}(V) \subset (i, j) \beta\text{Int}(f^{-1}(\sigma_i\text{-Int}(\sigma_j\text{-Cl}(V)))) = (i, j) \beta\text{Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is an $(i, j) \beta$-open in $X$ and hence $f$ is pairwise almost $\beta$-continuous.

2.8. Corollary. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following are equivalent:

(1) $f$ is pairwise almost $\beta$-continuous.

(2) $(i, j) \beta\text{Cl}(f^{-1}(V)) \subset f^{-1}((j, i)\alpha\text{Cl}(V))$ for each $V \in (j, i) \beta\text{O}(Y)$.

(3) $(i, j) \beta\text{Cl}(f^{-1}(V)) \subset f^{-1}((i, j) p\text{Cl}(V))$ for each $V \in (j, i) \text{SO}(Y)$.

(4) $f^{-1}(V) \subset (i, j) \beta\text{Int}(f^{-1}((i, j) s\text{Cl}(V)))$ for each $V \in (i, j) \text{PO}(Y)$.

2.9. Definition. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be pairwise faintly $\beta$-continuous (p. f. $\beta$. c.) if for each $x \in X$ and each $(i, j) \theta$-open set $V$ of $Y$ containing $f(x)$, there exists $U \in (i, j) \beta\text{O}(X, x)$ such that $f(U) \subset V$.

2.10. Lemma [10]. Let $(X, \tau_1, \tau_2)$ be a bitopological space. If $U$ is a $\tau_j$-open set of $X$, then $(i, j) \delta\text{Cl}(U) = \tau_i\text{-Cl}(U) = (i, j) \theta\text{Cl}(U)$.

2.11. Theorem. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) hold for
the following properties of a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \):

(1) \( f \) is pairwise \( \beta \)-continuous.

(2) \( f^{-1}(\delta \text{Cl}(B)) \) is an \((i, j)\) \( \beta \)-closed in \( X \) for every subset \( B \) of \( Y \).

(3) \( f \) is pairwise almost \( \beta \)-continuous.

(4) \( f \) is pairwise weakly \( \beta \)-continuous.

(5) \( f \) is pairwise faintly \( \beta \)-continuous.

If, in addition, \((Y, \sigma_1, \sigma_2)\) is pairwise regular then the five properties are equivalent.

**Proof.** (1) \( \Rightarrow \) (2). Since \((i, j)\) \( \delta \text{Cl}(B) \) is a \( \sigma_i \)-closed in \( Y \) for every subset \( B \) of \( Y \), then \( f^{-1}((i, j) \delta \text{Cl}(B)) \) is an \((i, j)\) \( \beta \)-closed in \( X \).

(2) \( \Rightarrow \) (3). For any subset \( B \) of \( Y \), \( f^{-1}((i, j) \delta \text{Cl}(B)) \) is an \((i, j)\) \( \beta \)-closed in \( X \) and hence we have

\[
(i, j) \beta \text{Cl}(f^{-1}(B)) \subset (i, j)\beta \text{Cl}(f^{-1}((i, j) \delta \text{Cl}(B))) = f^{-1}((i, j) \delta \text{Cl}(B)).
\]

It follows from **Theorem 2.6**, that \( f \) is pairwise almost \( \beta \)-continuous.

(3) \( \Rightarrow \) (4). This is obvious.

(4) \( \Rightarrow \) (5). Let \( F \) be any \((i, j)\) \( \theta \)-closed set of \( Y \). We have

\[
(i, j) \beta \text{Cl}(f^{-1}(F)) \subset f^{-1}((i, j) \theta \text{Cl}(F)) = f^{-1}(F).
\]

Therefore, \( f^{-1}(F) \) is an \((i, j)\) \( \beta \)-closed in \( X \) and hence \( f \) is pairwise faintly \( \beta \)-continuous.

Suppose that \( Y \) is pairwise regular. We prove that (5) \( \Rightarrow \) (1). Let \( V \) be any \( \sigma_i \)-open set of \( Y \). Since \( Y \) is pairwise regular, \( V \) is an \((i, j)\) \( \theta \)-open in \( Y \). By the pairwise faint \( \beta \)-continuity of \( f \), \( f^{-1}(V) \) is an \((i, j)\) \( \beta \)-open in \( X \). Therefore, \( f \) is pairwise \( \beta \)-continuous.
2.12. **Definition.** A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be **pairwise almost \( \beta \)-open** if \( f(U) \subseteq \sigma_i \text{-Int}(\sigma_j \text{-Cl}(f(U))) \), for every \((i, j) \beta\)-open set \( U \) of \( X \).

2.13. **Theorem.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a pairwise almost \( \beta \)-open and pairwise weakly \( \beta \)-continuous function, then \( f \) is pairwise almost \( \beta \)-continuous.

**Proof.** Let \( x \in X \) and let \( V \) be a \( \sigma_i \)-open set of \( Y \) containing \( f(x) \). Since \( f \) is pairwise weakly \( \beta \)-continuous, there exists \( U \in (i, j) \beta O(X, x) \) such that \( f(U) \subseteq \sigma_j \text{-Cl}(V) \). Since \( f \) is pairwise almost \( \beta \)-open, \[ f(U) \subseteq \sigma_i \text{-Int}(\sigma_j \text{-Cl}(f(U))) \subseteq \sigma_i \text{-Int}(\sigma_j \text{-Cl}(V)) \] and hence \( f \) is pairwise almost \( \beta \)-continuous.

2.14. **Definition.** A bitopological space \( (X, \tau_1, \tau_2) \) is said to be

(1) **pairwise almost regular** [16] if for any \((i, j)\) regular closed set \( F \) of \( X \) and any point \( x \in X - F \) there exist disjoint \( \tau_i \)-open set \( U \) and \( \tau_j \)-open set \( V \) such that \( x \in U \) and \( F \subseteq V \).

(2) **pairwise semi-regular** [16] if for any \( \tau_i \)-open set \( U \) of \( X \), and each point \( x \in U \) there exists an \((i, j)\) regular open set \( V \) of \( X \) such that \( x \in V \subseteq U \).

2.15. **Theorem.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a pairwise weakly \( \beta \)-continuous function and \( Y \) is pairwise almost regular, then \( f \) is pairwise almost \( \beta \)-continuous.

**Proof.** Let \( x \in X \) and let \( V \) be any \( \sigma_i \)-open set of \( Y \) containing \( f(x) \). By the pairwise almost regularity of \( Y \), there exists an \((i, j)\) regular open set \( G \) of \( Y \) such that \( f(x) \in G \subseteq \sigma_j \text{-Cl}(G) \subseteq \sigma_i \text{-Int}(\sigma_j \text{-Cl}(V)) \). Since \( f \) is pairwise weakly \( \beta \)-continuous, there exists \( U \in (i, j) \beta O(X, x) \) such that \( f(U) \subseteq \sigma_j \text{-Cl}(G) \subseteq \sigma_i \text{-Int}(\sigma_j \text{-Cl}(V)) \). Therefore, \( f \) is pairwise almost \( \beta \)-continuous.
2.16. **Theorem.** If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is an pairwise almost \( \beta \)-continuous function and \( Y \) is pairwise semi-regular, then \( f \) is pairwise \( \beta \)-continuous.

**Proof.** Let \( x \in X \) and \( V \) be a \( \sigma_i \)-open set of \( Y \) containing \( f(x) \). By the pairwise semi regularity of \( Y \), there exists an \((i, j)\) regular open set \( G \) of \( Y \) such that \( f(x) \in G \subseteq V \). Since \( f \) is pairwise almost \( \beta \)-continuous, there exists \( U \in (i, j) \beta O(X, x) \) such that \( f(U) \subseteq \sigma_i \text{-Int}(\sigma_j \text{-Cl}(G)) = G \subseteq V \) and hence \( f \) is pairwise \( \beta \)-continuous.

2.17. **Corollary.** For a function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \). If \( Y \) is a pairwise regular space, the following are equivalent:

1. \( f \) is pairwise \( \beta \)-continuous.
2. \( f \) is pairwise almost \( \beta \)-continuous.
3. \( f \) is pairwise weakly \( \beta \)-continuous.

**Proof.** This is an immediate consequence of **Theorems 2.15 and 2.16.**

2.18. **Definition.** The \((i, j) \beta \)-frontier of a subset \( A \) of \((X, \tau_1, \tau_2)\), denoted by \((i, j) \beta Fr(A)\), is defined by
\[
(i, j) \beta Fr(A) = (i, j) \beta Cl(A) \cap (i, j) \beta Cl(X - A) = (i, j) \beta Cl(A) - (i, j) \beta Int(A).
\]

2.19. **Theorem.** The set of all points \( x \) of \( X \) at which a function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is not pairwise almost \( \beta \)-continuous is identical with the union of the \((i, j) \beta \)-frontiers of the inverse images of an \((i, j)\) regular open sets containing \( f(x) \).

**Proof.** Let \( x \) be a point of \( X \) at which \( f \) is not pairwise almost \( \beta \)-continuous. Then, there exists an \((i, j)\) regular open set \( V \) of \( Y \) containing \( f(x) \) such that \( U \cap (X - f^{-1}(V)) \neq \emptyset \) for every \( U \subseteq (i, j) \beta O(X, x) \). Therefore, we have
x ∈ (i, j) βCl(X - f^{-1}(V)) = X - (i, j) βInt(f^{-1}(V)) and x ∈ f^{-1}(V). Thus, we obtain x ∈ (i, j) βFr(f^{-1}(V)). Conversely, suppose that f is pairwise almost β-continuous at x ∈ X and let V be an (i, j) regular open set containing f(x). Then there exists U ∈ (i, j) βO(X, x) such that U ⊆ f^{-1}(V), hence x ∈ (i, j) βInt(f^{-1}(V)). Therefore, x ∈ X - (i, j) βFr(f^{-1}(V)). This completes the proof.

2.20. Definition. A function f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is said to be

(1) pairwise complementary almost β-continuous (p. co. a. β. c.) if for each (i, j) regular open set V of Y, f^{-1}((i, j) Fr(V)) is an (i, j) β-closed in X, where (i, j) Fr(V) denotes the (i, j) frontier of V.

(2) pairwise weakly α-continuous (p. w. α. c.) if for each point x ∈ X and each σ_i-open set V of Y containing f(x), there exists U ∈ (i, j) αO(X, x) such that f(U) ⊆ σ_j-Cl(V).

2.21. Theorem. If f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is pairwise weakly α-continuous and pairwise complementary almost β-continuous, then f is pairwise almost β-continuous.

Proof. Let x ∈ X and let V be an (i, j) regular open set of Y containing f(x). Then f(x) ∈ (Y - (i, j) Fr(V)) and hence x ∈ X - f^{-1}((i, j) Fr(V)). Since f is pairwise weakly α-continuous, there exists G ∈ (i, j) αO(X, x) such that f(G) ⊆ σ_j-Cl(V). Put U = G ∩ (X - f^{-1}((i, j) Fr(V))). Then U ∈ (i, j)βO(X, x) and f(U) ⊆ f(G) ∩ (Y - (i, j) Fr(V)) ⊆ σ_j-Cl(V) ∩ (Y - (i, j) Fr(V)) = V. This shows that f is pairwise almost β-continuous.

2.22. Definition [6]. A bitopological space (X, τ_1, τ_2) is said to be pairwise Housdorff space if for each pair of distinct points x any y of X, there exists a τ_i-open set U containing x and a τ_j-open set V containing y such that U ∩ V = φ.
2.23. **Theorem.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise almost \( \beta \)-continuous, \( g : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise weakly \( \alpha \)-continuous and \( Y \) is pairwise Hausdorff, then the set \( \{x \in X : f(x) = g(x)\} \) is \((i, j)\) \( \beta \)-closed in \( X \).

**Proof.** Let \( A = \{x \in X : f(x) = g(x)\} \) and \( x \in X - A \). Then \( f(x) \neq g(x) \). Since \( Y \) is pairwise Hausdorff, there exist \( \sigma_i \)-open set \( V \) and \( \sigma_j \)-open set \( W \) of \( Y \) such that \( f(x) \in V, g(x) \in W \) and \( V \cap W = \emptyset \), hence \( \sigma_i\text{-Int}(\sigma_j\text{-Cl}(V)) \cap \sigma_i\text{-Cl}(W) = \emptyset \). Since \( f \) is pairwise almost \( \beta \)-continuous, there exists \( G \in (i, j)\betaO(X, x) \) such that \( f(G) \subseteq \sigma_i\text{-Int}(\sigma_j\text{-Cl}(V)) \). Since \( g \) is pairwise weakly \( \alpha \)-continuous, there exists an \((i, j)\) \( \alpha \)-open set \( H \) of \( X \) containing \( x \) such that \( g( H ) \subseteq \sigma_i\text{-Cl}( W ) \). Now, put \( U = G \cap H \), then \( U \in (i, j)\betaO(X, x) \) and \( f( U ) \cap g( U ) \subseteq \sigma_i\text{-Int}(\sigma_j\text{-Cl}(V)) \cap \sigma_i\text{-Cl}(W) = \emptyset \). Therefore, we obtain \( U \cap A = \emptyset \) and hence \( x \in X - (i, j)\betaCl( A ) \). This shows that \( A \) is an \((i, j)\) \( \beta \)-closed in \( X \).

2.24. **Theorem.** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a function and \( g : (X, \tau_1, \tau_2) \rightarrow (X \times Y, \mathcal{J}_1, \mathcal{J}_2) \) the graph function defined by \( g(x) = (x, f(x)) \) for every \( x \in X \). Then \( g \) is pairwise almost \( \beta \)-continuous iff \( f \) is pairwise almost \( \beta \)-continuous.

**Proof.** **Necessity.** Let \( x \in X \) and \( V \in (i, j)\text{RO}(Y) \) containing \( f(x) \). Then we have \( g(x) = (x, f(x)) \in X \times V \in (i, j)\text{RO}(X \times Y) \). Since \( g \) is pairwise almost \( \beta \)-continuous, there exists an \((i, j)\) \( \beta \)-open set \( U \) of \( X \) containing \( x \) such that \( g( U ) \subseteq X \times V \). Therefore, we obtain \( f( U ) \subseteq V \) and hence \( f \) is pairwise almost \( \beta \)-continuous.

**Sufficiency.** Let \( x \in X \) and \( W \) be an \((i, j)\) regular open set of \((X \times Y)\) containing \( g(x) \). There exist \( U_1 \in (i, j)\text{RO}(X) \) and \( V \in (i, j)\text{RO}(Y) \) such that \((x, f(x)) \in U_1 \times V \subseteq W \). Since \( f \) is pairwise almost \( \beta \)-continuous, there exists \( U_2 \in (i, j)\betaO(X) \) such that \( x \in U_2 \) and \( f( U_2 ) \subseteq V \). Put \( U = U_1 \cap U_2 \), then we obtain \( x \in U \in (i, j)\betaO(X) \) and \( g( U ) \subseteq U_1 \times V \subseteq W \). This shows that \( g \) is pairwise almost \( \beta \)-continuous.
2.25. Theorem. If \( f_1 : (X_1, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise weakly \( \beta \)-continuous, \( f_2 : (X_2, \tau'_1, \tau'_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise almost \( \beta \)-continuous and \( Y \) is pairwise Hausdorff, then the set \{ \((x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\) \} is an \((i, j)\) \( \beta \)-closed in \((X_1 \times X_2, \mathcal{I}_1, \mathcal{I}_2)\).

Proof. Let \( A = \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\} \) and \((x_1, x_2) \in X_1 \times X_2 - A\). Then \( f_1(x_1) \neq f_2(x_2) \) and there exist \( \sigma_1 \)-open set \( V_1 \) and \( \sigma_j \)-open set \( V_2 \) of \( Y \) such that \( f_1(x_1) \in V_1, f_2(x_2) \in V_2 \) and \( V_1 \cap V_2 = \emptyset \), hence \( \sigma_j^{-\text{Cl}}(V_1) \cap \sigma_j^{-\text{Int}}(\sigma_i^{-\text{Cl}}(V_2)) = \emptyset \). Since \( f_1 \) (resp., \( f_2 \)) is pairwise weakly \( \beta \)-continuous (resp., pairwise almost \( \beta \)-continuous), there exists \( U_1 \in (i, j) \beta O(X_1, x_1) \) such that \( f_1(U_1) \subset \sigma_j^{-\text{Cl}}(V_1) \) (resp., \( U_2 \in (i, j) \beta O(X_2, x_2) \) such that \( f_2(U_2) \subset \sigma_j^{-\text{Int}}(\sigma_i^{-\text{Cl}}(V_2)) \)). Therefore, we obtain \((x_1, x_2) \in U_1 \times U_2 \subseteq X_1 \times X_2 - A \) and \( U_1 \times U_2 \in (i, j) \beta O(X_1 \times X_2) \). This shows that \( A \) is an \((i, j)\) \( \beta \)-closed in \((X_1 \times X_2, \mathcal{I}_1, \mathcal{I}_2)\).

2.26. Theorem. If \( g : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise almost \( \beta \)-continuous and \( S \) is an \((i, j)\) \( \delta \)-closed set of \((X \times Y, \mathcal{I}_1, \mathcal{I}_2)\) then \( P_\lambda(S \cap G(g)) \) is an \((i, j)\) \( \beta \)-closed in \( X \), where \( P_\lambda \) represents the projection of \( X \times Y \) onto \( X \) and \( G(g) \) denotes the graph of \( g \).

Proof. Let \( S \) be an \((i, j)\) \( \delta \)-closed set of \( X \times Y \) and \( x \in (i, j) \beta \text{Cl}(P_\lambda(S \cap G(g))) \). Let \( U \) be any \( \tau_i \)-open set of \( X \) containing \( x \) and \( V \) any \( \sigma_1 \)-open set of \( Y \) containing \( g(x) \). Since \( g \) is pairwise almost \( \beta \)-continuous by Theorem 2.6, we have \( x \in g^{-1}(V) \subset (i, j) \beta \text{Int}(g^{-1}(\sigma_i^{-\text{Int}}(\sigma_j^{-\text{Cl}}(V)))) \) and \( U \cap (i, j) \beta \text{Int}(g^{-1}(\sigma_i^{-\text{Int}}(\sigma_j^{-\text{Cl}}(V)))) \in (i, j) \beta O(X, x) \). Since \( x \in (i, j) \beta \text{Cl}(P_\lambda(S \cap G(g))) \), \([U \cap (i, j) \beta \text{Int}(g^{-1}(\sigma_i^{-\text{Int}}(\sigma_j^{-\text{Cl}}(V))))] \cap P_\lambda(S \cap G(g))\) contains some point \( u \) of \( X \). This implies that \((u, g(u)) \in S \) and \( g(u) \in \sigma_i^{-\text{Int}}(\sigma_j^{-\text{Cl}}(V)) \). Thus, we have \( \phi \neq [U \times \sigma_i^{-\text{Int}}(\sigma_j^{-\text{Cl}}(V))] \cap S \subseteq \mathcal{I}_i^{-\text{Int}}(\mathcal{I}_j^{-\text{Cl}}(U \times V)) \cap S \) and \((x, g(x)) \in (i, j) \delta \text{Cl}(S) \). Since \( S \) is an \((i, j)\) \( \delta \)-closed, we have \((x, g(x)) \in S \cap G(g) \) and \( x \in P_\lambda(S \cap G(g)) \). Then \( P_\lambda(S \cap G(g)) \) is an \((i, j)\) \( \beta \)-closed in \( X \).
2.27. Corollary. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ has an $(i, j)$ $\delta$-closed graph and $g : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise almost $\beta$-continuous, then the set $\{ x \in X : f(x) = g(x) \}$ is an $(i, j)$ $\beta$-closed in $X$.

Proof. Since $G(f)$ is $(i, j)$ $\delta$-closed and $p_\beta(G(f) \cap G(g)) = \{ x \in X : f(x) = g(x) \}$, it follows from Theorem 2.26, that $\{ x \in X : f(x) = g(x) \}$ is $(i, j)$ $\beta$-closed in $X$.

2.28. Definition. A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise $\beta$-$T_2$ if for any distinct points $x, y$ of $X$, there exist disjoint $(i, j)$ $\beta$-open set, $(j, i)$ $\beta$-open set $U, V$ of $X$ such that $x \in U$ and $y \in V$.

2.29. Theorem. If for each pair of distinct points $x_1$ and $x_2$ in a bitopological space $(X, \tau_1, \tau_2)$ there exists a function $f$ of $(X, \tau_1, \tau_2)$ into a pairwise Hausdorff space $(Y, \sigma_1, \sigma_2)$ such that

1. $f(x_1) \neq f(x_2)$.

2. $f$ is pairwise weakly $\beta$-continuous at $x_1$.

3. pairwise almost $\beta$-continuous at $x_2$, then $X$ is pairwise $\beta T_2$.

Proof. Since $Y$ is pairwise Hausdorff, there exist $\tau_i$-open $V_1$ and $\tau_j$-open $V_2$ of $Y$ such that $f_1(x_1) \in V_1$, $f_2(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Since $f$ is pairwise weakly $\beta$-continuous at $x_1$, there exists $U_1 \in (i, j) \beta O(X, x_1)$ such that $f(U_1) \subset \sigma_j-Cl(\sigma_i-Cl(V_1)) \neq \emptyset$. Since $f$ is pairwise almost $\beta$-continuous at $x_2$ there exists $U_2 \in (j, i) \beta O(X, x_2)$ such that $f(U_2) \subset \sigma_j-Int(\sigma_i-Cl(V_2))$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$. This shows that $X$ is pairwise $\beta-T_2$.

2.30. Definition. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ has a pairwise $\beta$-closed graph if for each $(x, y) \in X \times Y - G(f)$, there exist $U \in (i, j) \beta O(X, x)$ and a $\sigma_i$-open set $V$ of $Y$ containing $y$ such that $[U \times \sigma_j-Cl(V)] \cap G(f) = \emptyset$. 
2.31. **Lemma.** A function \( f : ( X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) has a pairwise \( \beta \)-closed graph iff for each \( (x, y) \in (X \times Y, \mathcal{J}_1, \mathcal{J}_2) \) such that \( y \neq f(x) \), there exist an \((i, j)\) \( \beta \)-open set \( U \) and a \( \sigma_i \)-open set \( V \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap \sigma_j \text{-Cl}(V) = \phi \).

2.32. **Theorem.** If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is a pairwise almost \( \beta \)-continuous function and \( Y \) is pairwise Hausdorff then \( f \) has an \((i, j)\) \( \beta \)-closed graph.

**Proof.** Let \( (x, y) \in X \times Y \) such that \( y \neq f(x) \). Then there exist \( \sigma_i \)-open set \( V \) and \( \sigma_j \)-open set \( W \) such that \( f(x) \in V \), \( y \in W \) and \( V \cap W = \phi \), hence \( V \cap \sigma_i \text{-Cl}(W) = \phi \). Then \( f(x) \in Y - \sigma_i \text{-Cl}(W) \) and \( Y - \sigma_i \text{-Cl}(W) \) is an \((i, j)\) regular open in \( Y \). There exists \( U \in (i, j) \beta \text{O}(X, x) \) such that \( f(U) \subseteq Y - \sigma_i \text{-Cl}(W) \) and hence \( f(U) \cap \sigma_i \text{-Cl}(W) = \phi \). Therefore, by **Lemma 2.31**, \( f \) has a pairwise \( \beta \)-closed graph.

2.33. **Corollary.** If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is a pairwise \( \beta \)-continuous function and \( Y \) is pairwise Hausdorff, then \( f \) has a pairwise closed graph.

3. **PAIRWISE ALMOST PRECONTINUOUS FUNCTIONS :-

3.1. **Definition.** A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is called **pairwise almost precontinuous** (p. a. p. c.) at \( x \in X \) if for each \((i, j)\) regular open set \( V \subseteq Y \) containing \( f(x) \), there exists \( U \in (i, j) \text{PO}(X, x) \) such that \( f(U) \subseteq V \). If \( f \) is pairwise almost pre continuous at every point of \( X \), then it is called **pairwise almost precontinuous**.

3.2. **Theorem.** For a function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \), the following are equivalent:
(1) \( f \) is pairwise almost precontinuous.

(2) For each \( x \in X \) and each \( V \in \sigma_i\)-open containing \( f(x) \), there exists \( U \in (i, j) \text{PO}(X) \) containing \( x \) such that \( f(U) \subseteq \sigma_i\text{-Int}(\sigma_j\text{-Cl}(V)) \).

(3) \( f^{-1}(F) \in (i, j)\text{PC}(X) \) for every \( F \in (i, j)\text{RC}(Y) \).

(4) \( f^{-1}(V) \in (i, j)\text{PO}(X) \) for every \( V \in (i, j)\text{RO}(Y) \).

**Proof.** The proof is obvious and is thus omitted.

**3.3. Theorem.** For a function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \), the following are equivalent:

(1) \( f \) is pairwise almost precontinuous.

(2) \( f((i, j)p\text{Cl}(A)) \subseteq (i, j)\delta\text{Cl}(f(A)) \) for every subset \( A \) of \( X \).

(3) \( (i, j)p\text{Cl}(f^{-1}(B)) \subseteq f^{-1}((i, j)\delta\text{Cl}(B)) \) for every subset \( B \) of \( Y \).

(4) \( f^{-1}(F) \in (i, j)\text{PC}(X) \) for every \( (i, j)\delta\)-closed set \( F \) of \( Y \).

(5) \( f^{-1}(V) \in (i, j)\text{PO}(X) \) for every \( (i, j)\delta\)-open set \( V \) of \( Y \).

**Proof.** The proof is similar as **Theorem 2.6**.

**3.4. Definition [16].** Let \( B_1 \) be the family of all \( (i, j) \) regular open subsets of \( X \) and let \( B_2 \) be the family of all \( (j, i) \) regular open subsets of \( X \). Since the intersection of two \( (i, j) \) regular open subsets of \( X \) is again \( (i, j) \) regular open set. Therefore \( B_1 \) and \( B_2 \) both generate topologies for \( X \) say \( \tau^i_1 \) and \( \tau^i_2 \) respectively. Thus with every bitopological space \( (X, \tau_1, \tau_2) \) there is associated another bitopological space \( (X, \tau^i_1, \tau^i_2) \), called the **pairwise semiregularization of** \( \tau_1 \) and \( \tau_2 \). The space \( (X, \tau_1, \tau_2) \) is said to be pairwise
semiregular if \( \tau_1^s = \tau_1 \) and \( \tau_2^s = \tau_2 \).

3.5. Remark. Between pairwise almost precontinuity and pairwise precontinuity, we have the following relationship: A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise almost precontinuous iff \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1^s, \sigma_2^s) \) is pairwise precontinuous.

3.6. Definition. A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called **pairwise almost weakly continuous** if \( f(V) \subseteq \tau_i - \text{Int}(\tau_j - \text{Cl}(f^{-1}(\sigma_j - \text{Cl}(V)))) \) for every \( \sigma_i \)-open set \( V \) of \( Y \).

3.7. Remark. Pairwise precontinuity implies pairwise almost precontinuity, pairwise almost precontinuity implies pairwise almost weak continuity. However, the converse are not true as the following examples show.

3.8. Example. Let \( X = X_1 = X_2 = \{ a, b, c, d \} \), \( \tau_1 = \{ \phi, X, \{a\}, \{b\}, \{a, b\} \} \), \( \tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b, c\}, \{a, b, c\}, \{a, b, d\} \} \) and \( \sigma_1 = \{ \phi, X, \{a\}, \{a, c\}, \{a, d\} \} \). Let \( f : (X_1, \tau_1, \tau_2) \rightarrow (X_2, \sigma_1, \sigma_2) \) be the identity mapping. Then \( f \) is a pairwise almost continuous and hence pairwise almost precontinuous function which not pairwise precontinuous because, there exists \( \{a\} \in \sigma_i \)-open such that \( f^{-1}(\{a\}) = \{a\} \notin (i, j) \text{PO}(X) \).

3.9. Example. Let \( X = \{a, b, c, d\} \) and \( \tau_1 = \tau_2 = \sigma_1 = \sigma_2 = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\} \). Define a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) as follows:

\[
f(a) = c, f(b) = d, f(c) = b \text{ and } f(d) = a.
\]

Then \( f \) is a pairwise almost weakly continuous. However, \( f \) is not pairwise almost precontinuous, because there exists an \( (i, j) \) regular open set \( \{c\} \) of \( X \) such that \( f^{-1}(\{c\}) \notin (i, j) \text{PO}(X) \).

3.10. Definition. A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called **pairwise M-preopen** if the image of each \( (i, j) \) preopen set is an \( (i, j) \) preopen.
3.11. **Theorem.** If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise M-preopen pairwise almost weakly continuous. Then \( f \) is pairwise almost precontinuous.

**Proof.** Suppose that \( x \in X \) and \( V \) is a \( \sigma_i \)-open set containing \( f(x) \). Since \( f \) is pairwise almost weakly continuous, then there exists \( U \in (i, j)PO(X, x) \) such that \( f(U) \subseteq \sigma_j\text{-Cl}(V) \). Since \( f \) is pairwise M-preopen, \( f(U) \) is an \((i, j)\) preopen in \( Y \) and hence \( f(U) \subseteq \sigma_i\text{-Int}(\sigma_j\text{-Cl}(f(U))) \subseteq \sigma_i\text{-Int}(\sigma_j\text{-Cl}(\sigma_i\text{-Int}(V))) = \sigma_i\text{-Int}(\sigma_j\text{-Cl}(V)) \). It follows that \( f(U) \subseteq \sigma_i\text{-Int}(\sigma_j\text{-Cl}(V)) \). Hence \( f \) is pairwise almost precontinuous.

3.12. **Definition.** A bitopological space \( (X, \tau_1, \tau_2) \) is called **pairwise submaximal** if every \( \tau_j \)-dense subset of \( X \) is \( \tau_i \)-open in \( X \).

3.13. **Definition.** The filter base \( F \) is called **pairwise \( \delta \)-convergent** (resp., pairwise \( p \)-convergent) to a point \( x \) in \( (X, \tau_1, \tau_2) \) if for any \( \tau_i \)-open set \( U \) containing \( x \) (resp., any \( U \in (i, j)\) PO\( (X, x) \)) there exists \( B \in F \) such that \( B \subseteq \tau_i\text{-Int}(\tau_j\text{-Cl}(U)) \) (resp., \( B \subseteq U \)).

3.14. **Theorem.** If a function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise almost precontinuous, then for each point \( x \in X \) and each filter base \( F \) in \( X \) pairwise \( p \)-converging to \( x \), the filter base \( f(F) \) is pairwise \( \delta \)-convergent to \( f(x) \). If \( X \) is pairwise submaximal, then the converse also holds.

**Proof.** Suppose that \( x \in X \) and \( F \) is any filter base in \( X \) pairwise \( p \)-converging to \( x \). By the pairwise almost precontinuity of \( f \), for any \((i, j)\) regular open set \( V \) in \( Y \) containing \( f(x) \), there exists \( U \in (i, j)\) PO\( (X, x) \) such that \( f(U) \subseteq V \). But \( F \) is pairwise \( p \)-convergent to \( x \) in \( X \), then there exists \( B \in F \) such that \( B \subseteq U \). It follows that \( f(B) \subseteq V \). This means that \( f(F) \) is pairwise \( \delta \)-convergent to \( f(x) \).

Now suppose that \( X \) is pairwise submaximal. Let \( x \) be a point in \( X \) and \( V \) any \((i, j)\) regular open set containing \( f(x) \). Since \( X \) is pairwise...
submaximal, every (i, j) preopen set of X is \(\tau_i\)-open. If we set \(F = (i, j) \text{ PO}(X, x)\), then \(F\) will be a filter base which pairwise p-converges to \(x\). So there exists \(U\) in \(F\) such that \(f(U) \subset V\). This completes the proof.

3.15. Corollary. Let \((X, \tau_i, \tau_j)\) be a pairwise submaximal space. Then a function \(f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_1, \sigma_2)\) is pairwise almost precontinuous iff \(f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma^s_1, \sigma^s_2)\) is pairwise continuous.

3.16. Definition [13]. A bitopological space \((X, \tau_i, \tau_j)\) is said to be **pairwise pre-\(T_2\)** if for each pair of distinct points \(x\) and \(y\) in \(X\), there exist an \((i, j)\) preopen set \(U\) containing \(x\) and a \((j, i)\) preopen set \(V\) containing \(y\) such that \(U \cap V = \phi\).

3.17. Theorem. If \(f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_1, \sigma_2)\) is a pairwise almost precontinuous injection and \(Y\) is pairwise Hausdorff, then \(X\) is pairwise pre-\(T_2\).

**Proof.** Since \(f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_1, \sigma_2)\) is pairwise almost precontinuous injective, \(f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma^s_1, \sigma^s_2)\) is a pairwise precontinuous injection and \((Y, \sigma^s_1, \sigma^s_2)\) is pairwise Hausdorff. Let \(x\) and \(y\) be any distinct points of \(X\). Since \(f\) is injective, \(f(x) \neq f(y)\) and hence there exist disjoint \(\sigma^s_1\)-open set \(V\) and \(\sigma^s_2\)-open \(W\) of \((Y, \tau^s_1, \tau^s_2)\) such that \(f(x) \in V\) and \(f(y) \in W\). Therefore, we obtain \(f^{-1}(V) \in (i, j) \text{ PO}(X, x), f^{-1}(W) \in (j, i) \text{ PO}(X, y)\) and \(f^{-1}(V) \cap f^{-1}(W) = \phi\). This shows that \(X\) is pairwise pre-\(T_2\).

3.18. Definition. A bitopological space \((X, \tau_i, \tau_j)\) is a **door space** if every subset of \(X\) is either \((\tau_i, \tau_j)\)open or \((\tau_i, \tau_j)\)closed.

3.19. Lemma. If \((X, \tau_i, \tau_j)\) is a door space, then every \((i, j)\) preopen set in \(X\) is a \(\tau_i\)-open.

3.20. Theorem. Let \(f, g : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_1, \sigma_2)\) be functions, \(Y\) pairwise Hausdorff and \(X\) is door space. If \(f\) and \(g\) are pairwise almost precontinuous functions, then the set \(E = \{x \in X : f(x) = g(x)\}\) is \(\tau_i\)-closed in \(X\).
Proof. Let \( x \in X - E \). It follows that \( f(x) \neq g(x) \). Since \( Y \) is pairwise Hausdorff, then there exist \( \tau_i \)-open set \( V_1 \) and \( \tau_j \)-open set \( V_2 \) in \( Y \) such that \( f(x) \in V_1 \), \( g(x) \in V_2 \) and \( V_1 \cap V_2 = \emptyset \). Since \( V_1 \) and \( V_2 \) are disjoint, we obtain \( \tau_i \)-Int(\( \tau_j \)-Cl( \( V_1 ) \)) \( \cap \) \( \tau_j \)-Int(\( \tau_i \)-Cl( \( V_2 ) \)) = \( \emptyset \). Since \( f \) and \( g \) are pairwise almost precontinuous, there exist \( (i, j) \) preopen sets \( U_1 \) and \( (j, i) \) preopen \( U_2 \) in \( X \) containing \( x \) such that \( f(U_1) \subset \tau_i \)-Int(\( \tau_j \)-Cl( \( V_1 ) \)) and \( g(U_2) \subset \tau_j \)-Int(\( \tau_i \)-Cl( \( V_2 ) \)). Put \( U = U_1 \cap U_2 \). So by Lemma 3.19, \( U \) is a \( \tau_i \)-open set in \( X \) containing \( x \). Thus we have \( f(U) \cap g(U) = \emptyset \). It follows that \( x \notin \tau_i \)-Cl( \( E \) ). Hence \( \tau_i \)-Cl( \( E \) ) \( \subset \) \( E \) and \( E \) is a \( \tau_i \)-closed in \( X \).

3.21. Lemma. If \( A \) is an \( (i, j) \) \( \alpha \)-open set of a bitopological space \( (X, \tau_i, \tau_2) \) and \( B \in (i, j) \text{PO}(X) \) then \( A \cap B \in (i, j) \text{PO}(X) \).

3.22. Theorem. Let \( f, g : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be functions and \( Y \) is pairwise Hausdorff. If \( f \) is pairwise weakly \( \alpha \)-continuous and \( g \) is pairwise almost precontinuous, then the set \( E = \{ x \in X : f(x) = g(x) \} \) is \( (i, j) \) preclosed in \( X \).

Proof. Suppose that \( x \notin E \). Then \( f(x) \neq g(x) \). Since \( Y \) is pairwise Hausdorff, there exist \( \tau_i \)-open set \( V \) and \( \tau_j \)-open set \( W \) of \( Y \) such that \( f(x) \in V \), \( g(x) \in W \), and \( V \cap W = \emptyset \), hence \( \tau_j \)-Cl( \( V \) ) \( \cap \) \( \tau_j \)-Int(\( \tau_i \)-Cl( \( W \)) = \( \emptyset \). Since \( f \) is pairwise weakly \( \alpha \)-continuous there exists an \( (i, j) \alpha \)-open set \( U \) containing \( x \) such that \( f(U) \subset \tau_j \)-Cl(\( V \)). Since \( g \) is pairwise almost precontinuous there exists \( G \in (j, i) \text{PO}(X, x) \) such that \( g(G) \subset \tau_j \)-Int(\( \tau_i \)-Cl( \( W \)))). Put \( O = U \cap G \), then \( O \in (i, j) \text{PO}(X, x) \) by Lemma 3.21, and \( O \cap E = \emptyset \). Therefore, we obtain \( x \notin (i, j) \text{pCl}(E) \). This shows that \( E \) is an \( (i, j) \) preclosed in \( X \).

3.23. Corollary. Let \( f, g : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be functions and \( Y \) is pairwise Hausdorff. If \( f \) is pairwise continuous and \( g \) is pairwise precontinuous, then the set \( E = \{ x \in X : f(x) = g(x) \} \) is an \( (i, j) \) preclosed in \( X \).
3.24. **Theorem.** Let \( f : (X_1, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( g : (X_2, \tau'_1, \tau'_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be two pairwise almost precontinuous functions. If \( Y \) is a pairwise Hausdorff space, then the set \( \{ (x_1 \times x_2) \in X_1 \times X_2 : f(x_1) = g(x_2) \} \) is an \((i, j)\) preclosed in \((X_1 \times X_2, \mathcal{J}_1, \mathcal{J}_2)\).

**Proof.** Let \((x_1, x_2) \notin E\). Then \(f(x_1) \neq g(x_2)\). Since \(Y\) is pairwise Hausdorff, there exist disjoint \(\sigma_i\)-open neighbourhood \(V\) and \(\sigma_j\)-open neighbourhood \(W\) of \(f(x_1)\) and \(g(x_2)\), respectively. Since \(V\) and \(W\) are disjoint, we have \(\sigma_i\)-Int(\(\sigma_j\)-Cl(\(V\))) \cap \sigma_j\)-Int(\(\sigma_i\)-Cl(\(W\))) = \(\phi\). Since \(f\) and \(g\) are pairwise almost precontinuous, there exists \(U \in (i, j)\text{PO}(X_1, x_1)\) and \(G \in (j, i)\text{PO}(X_2, x_2)\) such that \(f(U) \subset \sigma_i\)-Int(\(\sigma_j\)-Cl(\(V\))) and \(g(G) \subset \sigma_j\)-Int(\(\sigma_i\)-Cl(\(W\))), respectively. Put \(O = U \times G\), then \((x_1, x_2) \in O\), \(O\) is \((i, j)\) preopen in \(X_1 \times X_2\) and \(O \cap E = \phi\). Therefore, we obtain \((x_1, x_2) \in (i, j)p\text{Cl}(E)\). This shows that \(E\) is an \((i, j)\) preclosed in \(X_1 \times X_2\).

3.25. **Corollary.** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a pairwise almost precontinuous function, if \(Y\) is pairwise Hausdorff, then the set \(E = \{ (x, y) : f(x) = f(y) \}\) is an \((i, j)\) preclosed in \((X \times X, \mathcal{J}_1, \mathcal{J}_2)\).

**Proof.** By setting \(X = X_1 = X_2\) and \(g = f\) in **Theorem 3.24**, the result follows.

3.26. **Corollary.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a pairwise precontinuous function and \(Y\) is pairwise Hausdorff, then the set \( \{ (x, y) : f(x) = f(y) \} \) is an \((i, j)\) preclosed in \((X \times X, \mathcal{J}_1, \mathcal{J}_2)\).

3.27. **Corollary.** Let \( f : (X_1, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( g : (X_2, \tau'_1, \tau'_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be two pairwise precontinuous functions. If \(Y\) is a pairwise Hausdorff space, then the set \( \{ (x, y) : f(x) = g(y) \} \) is an \((i, j)\) preclosed in \((X_1 \times X_2, \mathcal{J}_1, \mathcal{J}_2)\).

3.28. **Definition.** A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be **pairwise strongly compact relative to** \(X\) (resp., **pairwise N-closed relative to** \(X\)) if every cover of \(A\) by \((i, j)\) preopen (resp., \((i, j)\) regular open) sets of \(X\) has a finite subcover.
3.29. Definition [10]. A bitopological space \((X, \tau_1, \tau_2)\) is called **pairwise strongly compact** (resp., **pairwise nearly compact**) if every \((i, j)\) preopen (resp., \((i, j)\) regular open) cover of \(X\) has a finite subcover.

3.30. Theorem. If \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is pairwise almost precontinuous and \(K\) is a pairwise strongly compact relative to \(X\), then \(f(K)\) is pairwise N-closed relative to \(Y\).

**Proof.** Let \(\{G_\alpha : \alpha \in A\}\) be any cover of \(f(K)\) by \((i, j)\) regular open sets of \(Y\). Then, \(\{f^{-1}(G_\alpha) : \alpha \in A\}\) is a cover of \(K\) by \((i, j)\) preopen sets of \(X\). Since \(K\) is pairwise strongly compact relative to \(X\), there exists a finite subset \(A_0\) of \(A\) such that \(K \subseteq \bigcup \{f^{-1}(G_\alpha) : \alpha \in A_0\}\). Therefore, we obtain \(f(K) \subseteq \bigcup \{G_\alpha : \alpha \in A_0\}\). This shows that \(f(K)\) is pairwise N-closed relative to \(Y\).

3.31. Corollary. If \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is a pairwise almost precontinuous surjection and \(X\) is pairwise strongly compact, then \(Y\) is pairwise nearly compact.

3.32. Definition. A function \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is said to be **pairwise \(\delta\)-continuous** if for each \(x \in X\) and each \(\sigma_i\)-open set \(V\) of \(Y\) containing \(f(x)\), there exists a \(\tau_i\)-open set \(U\) in \(X\) containing \(x\) such that \(f(\tau_i-\text{Int}(\tau_j-\text{Cl}(U))) \subseteq \sigma_i-\text{Int}(\sigma_j-\text{Cl}(V))\).

3.33. Theorem. If \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is pairwise almost precontinuous and \(g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)\) is pairwise \(\delta\)-continuous, then \(g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)\) is pairwise almost precontinuous.

**Proof.** The proof is obvious and is omitted.

3.34. Theorem. If \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is a pairwise M-preopen surjection and \(g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)\) is a function such that \(g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)\) is pairwise almost precontinuous, then \(g\) is pairwise almost precontinuous.
Proof. Let \( y \in Y \) and \( x \in X \) such that \( f(x) = y \). Let \( G \) be an \((i, j)\) regular open set containing \((\text{gof})(x)\). Then there exists \( U \in (i, j)PO(X, x)\) such that \( g(f(U)) \subset G \). Since \( f \) is pairwise M-preopen, \( f(U) \in (i, j)\ PO(Y, y) \) such that \( g(f(U)) \subset G \). This shows that \( g \) is pairwise almost precontinuous at \( y \).

3.35. Lemma. If \( A \in (i, j)PO(X) \) and \( B \in (i, j)SO(X) \), then \( A \cap B = (i, j)PO(X) \).

3.36. Theorem. If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a pairwise almost precontinuous and \( A \) is an \((i, j)\) semi-open set of \( X \), then the restriction \( f/A : (X, \tau_{1/A}, \tau_{2/A}) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise almost precontinuous.

Proof. Let \( V \) be any \((i, j)\) regular open set of \( Y \). Since \( f \) is pairwise almost precontinuous, the inverse image of \( V \) is an \((i, j)\) preopen in \( X \) and \((f/A)^{-1}(V) = A \cap f^{-1}(V) \). Since \( A \) is an \((i, j)\) semi-open in \( X \). By the Lemma 3.35, that \( A \cap f^{-1}(V) \in (i, j)PO(A) \). Therefore \( f/A \) is pairwise almost precontinuous.

3.37. Lemma. If \( U \in (i, j)PO(X) \) and \( V \in (i, j)PO(U) \), then \( V \in (i, j)PO(X) \).

3.38. Theorem. Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a function and \( x \in X \). If there exists \( U \in (i, j)PO(X, x) \) such that the restriction of \( f \) to \( U \) is pairwise almost precontinuous at \( x \), then \( f \) is pairwise almost precontinuous at \( x \).

Proof. Suppose that \( V_2 \) is any \((i, j)\) regular open set containing \( f(x) \). Since \( f/U \) is pairwise almost precontinuous at \( x \), there exists \( V_1 \in (i, j)PO(U, x) \) such that \( f(V_1) = f/U(V_1) \subset V_2 \). Since \( U \in (i, j)PO(X, x) \). By Lemma 3.37, \( V_1 \in (i, j)PO(X, x) \). This shows clearly that \( f \) is pairwise almost precontinuous at \( x \).

3.39. Definition [13]. A be a subset of \((X_1, \tau_1, \tau_2)\). The \((i, j)\) prefrontier of \( A \) is defined by
\[
(i, j) pFr(A) = (i, j) pCl(A) \cap (i, j)pCl(X - A) = (i, j) pCl(A) - (i, j) pInt(A).
\]
3.40. **Theorem.** The set of all points \( x \) of \( X \) at which \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is not pairwise almost precontinuous is identical with the union of the \((i, j)\) prefrontiers of the inverses images of an \((i, j)\) regular open subsets of \( Y \) containing \( f(x) \).

**Proof.** If \( f \) is not pairwise almost precontinuous at \( x \in X \), then there exists an \((i, j)\) regular open set \( V \) containing \( f(x) \) such that for every \( U \in(i, j)PO(X, x), f(U) \cap (Y - V) \neq \emptyset \). This means that for every \( U \in(i, j)PO(X, x) \), we must have \( U \cap (X - f^{-1}(V)) \neq \emptyset \). Hence, it follows that \( x \in (i, j)pCl(X - f^{-1}(V)) \). But \( x \in f^{-1}(V) \) and hence \( x \in (i, j)pCl(f^{-1}(V)) \). This means that \( x \) belongs to the \((i, j)\) prefrontier of \( f^{-1}(V) \). Suppose that \( x \) belongs to the \((i, j)\) prefrontier of \( f^{-1}(V_1) \) for some \((i, j)\) regular open subset \( V_1 \) of \( Y \) such that \( f(x) \in V_1 \). Suppose that \( f \) is pairwise almost precontinuous at \( x \). Then there exists \( U \in (i, j)PO(X, x) \) such that \( f(U) \subset V_1 \). Then, we have \( x \in U \subset f^{-1}(f(U)) \subset f^{-1}(V_1) \). This shows that \( x \) is an \((i, j)\) preinterior point of \( f^{-1}(V_1) \). Therefore, we have \( x \notin (i, j)pCl(X - f^{-1}(V_1)) \) and \( x \notin (i, j)pFr(f^{-1}(V_1)) \). But this is a contradiction. This means that \( f \) is not pairwise almost precontinuous.

3.41. **Definition** [2]. A subset \( A \) of bitopological space \((X, \tau_1, \tau_2)\) is said to be \((i, j)\) \textbf{H-set} or \((i, j)\) \textbf{quasi H-closed relative to} \( X \), if for every cover \( \{U_i: i \in I\} \) of \( A \) by \( \tau_i \)-open sets of \( X \), there exists a finite subset \( I_0 \) of \( I \) such that \( A \subset \bigcup \{\tau_j-Cl(U_i): i \in I_0\} \).

3.42. **Theorem.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise almost weakly continuous and \( K \) is pairwise strongly compact relative to \( X \), then \( f(K) \) is an \((i, j)\) quasi H-closed relative to \( Y \).

**Proof.** The proof is similar to one of **Theorem 3.30.**

3.43. **Definition.** A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called \textbf{pairwise r- preopen} if the image of an \((i, j)\) preopen set in \( X \) is a \( \sigma_i \)-open in \( Y \).
3.44. Theorem. Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a pairwise almost weakly continuous bijection. If \( X \) is pairwise strongly compact and \( Y \) is pairwise Hausdorff, then \( f \) is pairwise \( r \)-preopen.

**Proof.** Suppose that \( U \) is an \((i, j)\) preopen subset of \( X \). Then \( X - U \) is an \((i, j)\) preclosed subset of the pairwise strongly compact space \( X \). This means that \( X - U \) is pairwise strongly compact relative to \( X \). By *Theorem 3.42*, \( f(X - U) \) is an \((i, j)\) quasi H-closed relative to \( Y \). Since \( f \) is bijective, we have \( f(X - U) = Y - f(U) \), where \( Y - f(U) \) is an \((i, j)\) quasi H-closed relative to \( Y \). Since \( Y \) is pairwise Hausdorff, therefore, \( Y - f(U) \) is a \( \sigma_i \)-closed. Hence \( f(U) \) is a \( \tau_i \)-open in \( Y \).

3.45. Corollary. Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a pairwise almost precontinuous bijection. If \( X \) is pairwise strongly compact and \( Y \) is pairwise Hausdorff, then \( f \) is pairwise \( r \)-preopen.

**Proof.** Since every pairwise almost precontinuous function is pairwise almost weakly continuous. Hence the proof follows from *Theorem 3.44*.

4. PAIRWISE ALMOST SEMI-CONTINUOUS FUNCTIONS :

4.1. Definition. A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be pairwise almost semi-continuous (p. a. s. c.) if for every \( x \in X \) and for every \((i, j)\) regularly open set \( G \) containing \( f(x) \) in \( Y \), there exists an \( A \in (i, j)SO(X) \) containing \( x \) such that \( f(A) \supseteq G \).

From definition 4.1. it follows that :

A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise almost semi-continuous iff for each \((i, j)\) regular-open subset \( G \) of \( Y \), \( f^{-1}(G) \in (i, j)SO(X) \).
4.2. Definition. A mapping \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be **pairwise feebly continuous (p. f. c.)** if for every \( \sigma_i \)-open subset \( A \) of \( Y \), \( f^{-1}(A) \neq \emptyset \) implies that \( \tau_i - \text{Int}[f^{-1}(A)] \neq \emptyset \).

4.3. Example. Let \( X = \) the set of all real numbers

\[
\tau_1 = \{ \emptyset, X, \text{complement of all countable subset of } X \}
\]

\[
\tau_2 = \{ \emptyset, X, \text{the sets whose complement are countable} \}
\]

\( Y = \{a, b, c\} \)

\( \sigma_1 = \{ \emptyset, Y, \{c\}, \{b\}, \{b, c\} \} \)

\( \sigma_2 = \{ \emptyset, Y, \{a\}, \{c\}, \{a, c\} \} \)

Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be define by

\[
f(x) = \begin{cases} 
a, & \text{if } x \text{ is rational} 
b, & \text{if } x \text{ is irrational} \end{cases}
\]

Then \( f \) is pairwise weakly continuous but not pairwise almost semi-continuous.

4.4. Example. Let \( X = Y = \{a, b, c, d\} \)

\( \tau_1 = \{ \emptyset, X, \{a\}, \{b\}, \{a, b\} \} \)

\( \tau_2 = \{ \emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\} \} \)

\( \sigma_1 = \{ \emptyset, X, \{a\}, \{c\}, \{a, c, d\} \} \)

\( \sigma_2 = \{ \emptyset, X, \{b\}, \{b, d\} \} \).

Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be the mapping. Then \( f \) is pairwise almost semi-continuous but not pairwise weakly continuous.

4.5. Example. Let \( X = \{a, b, c, d, e\} \)

\( \tau_1 = \{ \emptyset, X, \{a\}, \{b\}, \{a, b\} \} \)

\( \tau_2 = \{ \emptyset, X, \{b\}, \{a, b\}, \{b, e\}, \{a, b, e\} \} \)
\[ Y = \{p, q, r\} \]
\[ \sigma_1 = \{\emptyset, Y, \{p\}, \{q, r\}\} \]
\[ \sigma_2 = \{\emptyset, Y, \{p\}, \{q\}, \{p, q\}, \{q, r\}\} \] be topologies on \(Y\)

Let \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be define as
\[ f(a) = p = f(b), \quad f(c) = q = f(d), \quad f(e) = r \]

Then \(f\) is pairwise feebly continuous but not pairwise almost continuous.

4.6. Example. Let \(X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{c\}\}\) and \(\tau_2 = \{\emptyset, X, \{b\}\}\). Let \(Y = \{p, q, r\}, \sigma_1 = \{\emptyset, Y, \{p, q\}\}\) and \(\sigma_2 = \{\emptyset, Y, \{r\}, \{q, r\}\}\). Let
\[ f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \] be a function define by
\[ f(a) = p, \quad f(b) = q \quad \text{and} \quad f(c) = r. \]

Then \(f\) is pairwise almost semi-continuous but not pairwise feebly continuous.

Pairwise \(\theta\)-continuous \(\Rightarrow\) pairwise weakly continuous, therefore

**Example 4.3**, shows that pairwise weakly continuous \(\not\Rightarrow\) pairwise almost semi-continuous and **Example 4.4**, proves that pairwise almost semi-continuous \(\not\Rightarrow\) pairwise weakly continuous, therefore pairwise \(\theta\)-continuous and pairwise almost semi-continuous are unrelated.

The following diagram indicates the relationship between various continuities discussed here.

\[
\begin{array}{ccc}
\text{p. s. c.} & \leftarrow & \text{p. c.} & \rightarrow & \text{p. a. c.} \\
\uparrow & & & & \downarrow \\
\text{p. f. c.} & & \text{p. a. s. c.} & & \text{p. } \theta \text{ c.} \\
& & & & \downarrow \\
& & & & \text{p. w. c.} \\
\end{array}
\]

Here, \(a = \text{almost}, c = \text{continuous}, f = \text{feebly}, s = \text{semi}, w = \text{weakly}\).
4.7. **Theorem**. Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be pairwise semi-continuous and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be pairwise almost continuous, then \( g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2) \) is pairwise almost semi-continuous.

**Proof.** Obvious.

4.8. **Remark.** Composite of two pairwise almost semi-continuous is not always pairwise almost semi-continuous is clear from the below example.

4.9. **Example.** Let \( X = X_1 = X_2 = \{a, b, c, d\} \)

\[ \tau_1 = \{\emptyset, X, \{a\}, \{a, c\}\} \]

\[ \tau_2 = \{\emptyset, X, \{b\}, \{b, d\}\} \]

\[ \sigma_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \]

\[ \sigma_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\} \]

\[ \eta_1 = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, b\}, \{a, d\}, \{a, b, d\}\} \]

\[ \eta_2 = \text{the discrete topology of } X. \]

Let

\[ f_1 : (X, \sigma_1, \sigma_2) \rightarrow (X, \tau_1, \tau_2) \]

\[ f_2 : (X, \eta_1, \eta_2) \rightarrow (X, \sigma_1, \sigma_2) \]

be the mapping

It can be seen that \( f_1 \) and \( f_2 \) are pairwise almost semi-continuous. but \( f_1 \circ f_2 : (X, \eta_1, \eta_2) \rightarrow (X, \tau_1, \tau_2) \) is not pairwise almost semi-continuous for \( \{a, c\} \in (\tau_i, \tau_j) \text{RO}(X) \) but \((f_1 \circ f_2)^{-1} \{a, c\} = \{a, c\}\) is not \((\eta_i, \eta_j)\) semi-open \((X)\).

4.10. **Theorem.** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be pairwise almost semi-continuous mapping. If \( A \subset X \) and \( A \) is \( \tau_i \)-open, then \( f/A : (X, \tau_{1/A}, \tau_{2/A}) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise almost semi-continuous.

4.11. **Theorem.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a mapping such that for each \( x \in X \), there exists a \( \tau_i \)-open set \( N \) containing \( x \) such that \( f/N \) is pairwise almost semi-continuous, then \( f \) is pairwise almost semi-continuous.

**Proof.** Let \( x \in X \) and let \( U \) be an \((i, j)\) regularly open set containing \( f(x) \).
Then there exists a $\tau_i$-open set $N$ containing $x$ such that $f / N$ is pairwise almost semi-continuous. It follows that there exists an $(i, j)$ semi open (in $N$) set $M$ containing $x$ such that $f(M) \subseteq U$. Since $N$ is a $\tau_i$-open, therefore $M$ is an $(i, j)$ semi open in $X$. Hence $f$ is pairwise almost semi-continuous.

4.12. Corollary. If $X = \bigcup G_\alpha$ where each $G_\alpha$ is $\tau_i$-open, then $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise almost semi-continuous iff each $G_\alpha, f / G_\alpha$ is pairwise almost semi-continuous.

4.13. Theorem. If $X = \bigcup G_\alpha$ where each $G_\alpha$ is an $(i, j)$ semi open set and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a mapping such that $f / G_\alpha$ is pairwise almost semi-continuous for each $\alpha$, then $f$ is pairwise almost semi-continuous.

Proof. Let $G$ be any $(i, j)$ regular open subset of $Y$. Then $f^{-1}(G) = \bigcup[(f / G_\alpha)^{-1}(G)]$. Since each $f / G_\alpha$ is pairwise almost semi-continuous, therefore each $(f / G_\alpha)^{-1}(G)$ is an $(i, j)$ semi open in $G_\alpha$ and hence in $X$, $G_\alpha$ being an $(i, j)$ semi open. Since every union of an $(i, j)$ semi open set is an $(i, j)$ semi-open, it follows that $f^{-1}(G)$ is an $(i, j)$ semi open. Hence $f$ is pairwise almost semi-continuous.

4.14. Lemma. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise almost semi-continuous iff for every $(i, j)$ regular closed subset $K$ of $Y$, $f^{-1}(K)$ is an $(i, j)$ semi closed in $X$.

4.15. Theorem. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise almost semi continuous iff for every subset $A$ of $X$, $f(\tau_i^{-}\text{Cl}(A)) \subseteq (f(A))_\delta$ (where $[f(A)]_\delta$ is $(i, j)$ $\delta$-closed).

Proof. Necessity. Let $A \subseteq X$, consider $[f(A)]_\delta$ which is an $(i, j)\delta$-closed in $Y$. Then $f^{-1}[f(A)]_\delta$ is an $(i, j)$ semi-closed in $X$. Since $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}[f(\tau_i^{-}\text{Cl}(A))] \subseteq f^{-1}[f(A)]_\delta$. 

therefore by the definition of an \((i, j)\) semi-closure
\[
\tau_i \text{-Cl}(A) \subset f^{-1}[f(A)]_\delta \text{ and hence }
\]
\[
f(\tau_i \text{-Cl}(A)) \subset f[f^{-1}(f(A))]_\delta = [f(A)]_\delta.
\]
**Sufficiency.** If \(C\) is an \((i, j)\) \(\delta\)-closed of \(Y\) then \([C]_\delta = C\). Taking \(A = f^{-1}(C)\),
\[
\therefore f(\tau_i \text{-Cl}(f^{-1}(C))) \subset [f(f^{-1}(C))]_\delta \subset [C]_\delta = C.
\]
Therefore \(\tau_i \text{-Cl}(f^{-1}(C)) = f^{-1}(C)\).
\[
\therefore f^{-1}(C) \text{ is an } (i, j) \text{ semi-closed. Hence } f \text{ is pairwise almost semi-continuous.}
\]

**4.16. Theorem.** A function \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is pairwise almost semi-continuous iff for every subset \(B\) of \(X\),
\[
\tau_i \text{-Cl} (f^{-1}(B)) \subset f^{-1}[B]_\delta
\]
**Proof.** Easy.

**4.17. Theorem.** \((X_1, \tau_1, \tau_1^1, \tau_2, \tau_2^1, \tau_2^2)\) and \((X_2, \tau_2, \tau_2^1, \tau_2^2)\) be bitopological spaces \(X = (X_1 \times X_2, \mathcal{I}_1, \mathcal{I}_2)\) be the bitopological product. Let \(A_1 \in (i, j) \text{SO}(X_1)\) and \(A_2 \in (i, j) \text{SO}(X_2)\). Then \(A_1 \times A_2 \in (i, j) \text{SO}(X_1 \times X_2)\).

**4.18. Lemma.** Let \((X_1, \tau_1^{(1)}, \tau_1^{(2)})\) and \((X_2, \tau_2^{(1)}, \tau_2^{(2)})\) be bitopological spaces. Let \(\tau_{16}^{(1)}, \tau_{26}^{(1)}\) and \(\tau_{16}^{(2)}, \tau_{26}^{(2)}\) denote the topologies generated by \((i, j)\) regularly open sets of \(X_1\) and \(X_2\) respectively. If \(\mathcal{I}_1, \mathcal{I}_2\) denotes the product topology of \(X_1 \times X_2\) and \(\mathcal{I}_{16}, \mathcal{I}_{26}\) denotes the topology generated by \((i, j)\) regularly open sets of \(X_1 \times X_2\), then
\[
\tau_{16}^{(1)} \times \tau_{16}^{(2)} = \mathcal{I}_{16} \text{ and } \tau_{26}^{(1)} \times \tau_{26}^{(2)} = \mathcal{I}_{26}
\]
**Proof.** It is sufficient to show that
\[
(i) \ G^{(1)} \text{ and } G^{(2)} \text{ are } (i, j) \text{ regularly open subsets of } X_1 \text{ and } X_2 \text{ respectively, then } G^{(1)} \times G^{(2)} \text{ is an } (i, j) \text{ regularly open in } X_1 \times X_2.
\]
\[
(ii) \text{ If } G \text{ is any } (i, j) \text{ regularly open subset of } X_1 \times X_2. \text{ Then } G \text{ can be}
\]
expressed as \( \bigcup \alpha \in I G^{(1)}_\alpha \times G^{(2)}_\alpha \), where \( G^{(1)}_\alpha \) and \( G^{(2)}_\alpha \) are an \((i, j)\) regularly open subsets of \( X_1 \) and \( X_2 \), respectively.

(i) Holds because
\[
\tau^{(1)}_i \text{-Int}(\tau^{(1)}_j \text{-Cl}(G^{(1)})) \times (\tau^{(2)}_i \text{-Int}(\tau^{(2)}_j \text{-Cl}(G^{(2)}))) = \exists \text{-Int}(\exists_j \text{-Cl}(G^{(1)} \times G^{(2)})).
\]

Proof of (ii) Let \((x_1, x_2)\) be any point of \( G = \exists \text{-Int}(\exists_j \text{-Cl}(G)) \). Since \( G \) is a \( \tau_i \)-open therefore there exists \( \tau^{(1)}_i \)-open set \( O^{(1)} \) and \( \tau^{(2)}_i \)-open set \( O^{(2)} \) in \( X_1 \) and \( X_2 \) containing \( x_1 \) and \( x_2 \) respectively, such that
\[
O^{(1)} \times O^{(2)} \subset G = \exists \text{-Int}(\exists_j \text{-Cl}(G)).
\]

\[
\tau^{(1)}_i \text{-Int}(\tau^{(1)}_j \text{-Cl}(O^{(1)})) \times \tau^{(2)}_i \text{-Int}(\tau^{(2)}_j \text{-Cl}(O^{(2)})) \subset G
\]

Now \( \tau^{(1)}_i \text{-Int}(\tau^{(1)}_j \text{-Cl}(O^{(1)})) = G^{(1)} \) and \( \tau^{(2)}_i \text{-Int}(\tau^{(2)}_j \text{-Cl}(O^{(2)})) = G^{(2)} \) are an \((i, j)\) regularly open subset of \( X_1 \) and \( X_2 \) respectively. Hence (ii) follows.

4.19. Theorem. Let \( f_i : X_i \rightarrow Y_i \) be pairwise almost semi-continuous for \( i = 1, 2 \). Let \( f : X_1 \times X_2 \rightarrow Y_1 \times Y_2 \) be defined as \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \). Then \( f : X_1 \times X_2 \rightarrow Y_1 \times Y_2 \) is pairwise almost semi-continuous.

Proof. Let \( G \) be any \((i, j)\) regular open set in \( Y_1 \times Y_2 \), therefore \( \tau_i \text{-Int}(\tau_j \text{-Cl}(G)) = G \)

By the above Lemma 4.18,
\[
G = \bigcup \alpha G^{(1)}_\alpha \times G^{(2)}_\alpha \), where \( G^{(1)}_\alpha \) and \( G^{(2)}_\alpha \) are an \((i, j)\) regularly open sets in \( Y_1 \) and \( Y_2 \)

respectively.
\[
f^{-1}(G) = \bigcup \alpha f^{-1}(G^{(1)}_\alpha \times G^{(2)}_\alpha)
\]
\[
= \bigcup \alpha [f^{-1}(G^{(1)}_\alpha) \times f^{-1}(G^{(2)}_\alpha)]
\]
\[
= \bigcup \alpha [(i, j) \text{ SO set in } (X_1 \times X_2)]
\]
\[
= (i, j) \text{ SO set in } X_1 \times X_2.
\]
4.20. **Theorem.** Let $h : X \rightarrow X_1 \times X_2$ be pairwise almost semi-continuous, where $X, X_1$ and $X_2$ are bitopological spaces. If $f_i : X \rightarrow X_i$ be defined for each $x \in X$ as $f_i(x) = x_i$, where $h(x) = (x_1, x_2)$, then $f_i : X \rightarrow X_i$ is pairwise almost semi-continuous.

**Proof.** Let $G$ be an $(i, j)$ regular-open $(X_1)$. Then $G \times X_2$ is $(i, j)$ regular-open $(X_1 \times X_2)$. As $h$ is pairwise almost semi-continuous

$h^{-1}(G \times X_2) = f_1^{-1}(G) \cap f_2^{-1}(X_2)$

$= f^{-1}(G)$ is an $(i, j)$ semi-open $(X)$. Hence $f_1$ is pairwise almost semi-continuous. Similarly $f_2$ is pairwise almost semi-continuous.

4.21. **Definition.** A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise **pre-semi $\delta$-open** (p.p.s.$\delta$.o) mapping if the image of every $(i, j)$ semi-open subset of $X$ is an $(i, j)$ $\delta$-open subset of $Y$.

4.22. **Definition.** A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise **pre-semi $\delta$-closed** (p.p.s.$\delta$.c) if the image of every $(i, j)$ semi-closed subset of $X$ is an $(i, j)$ $\delta$-closed subset of $Y$.

4.23. **Remark.** Every p.p.s.$\delta$.o mapping is a pairwise open (pairwise closed) mapping but converse is not true.

4.24. **Definition** [3]. A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise **almost open** (p.a.o) if the image of every $(i, j)$ regularly open subset of $X$ is a $\sigma_i$-open subset of $Y$.

4.25. **Remark.** Obviously, every p.p.s.$\delta$.o mapping is pairwise almost open but not converse is not true.

4.26. **Definition.** A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise **pre-semi-open** (p.p.s.o) iff for $A \in (i, j)\text{SO}(X)$, $f(A) \in (i, j)\text{SO}(Y)$. 
4.27. Remark. Every p.p.s.δ.o mapping is a p.p.s.o. mapping but not converse is not true.

Following is the implication diagram of three types of mappings considered above.

\[
\begin{array}{c}
\text{p. p. s. } \delta \text{. o.} \\
\text{p. o. } \rightarrow \text{ p. a. o. } \quad \text{p. p. s. o}
\end{array}
\]


4.29. Remark. The restriction of a p. p. s. o. mapping may fail to be p. p. s. o. mapping.

4.30. Remark. The restriction of p. p. s. δ. o mapping may fail to be a p. p. s. δ. o. mapping.
REFERENCES


