Chapter 3

The connected and total double geodetic number of a graph

In this chapter we first introduce and study the connected double geodetic number $dg_c(G)$ of a graph $G$. Certain general properties of this concept are studied and bounds for the connected double geodetic number are determined. We characterize graphs which realize the bounds. The connected double geodetic number of certain classes of graphs are determined. It is shown that for a connected graph $G$ of order $n$, the $dg_c(G) = n$ if and only if every vertex of $G$ either a cut vertex or a weak extreme vertex. We proved that for any integers $n, a$ and $b$ such that $2 \leq a < b \leq n$, there exists a connected graph $G$ of order $n$ such that $dg(G) = a$ and $dg_c(G) = b$. For positive integers $r, d$ and $k \geq 4$ with $r \leq d \leq 2r$ and $k - d - 1 \geq 0$, there exists a connected graph $G$ with $rad G = r$, $diam G = d$ and $dg_c(G) = k$. Also, we introduce the concept of total double geodetic number $dg_t(G)$ of a graph $G$. Certain general properties of this concept are studied. We determinie bounds for it and characterize graphs which realize the bounds. We prove that for a connected graph $G$ with at least two vertices, $dg_t(G) \leq 2\ dg(G)$. The total double geodetic number of certain classes of graphs are determined. It is shown that for positive integers $r, d$ and $k \geq 4$ with
\( r \leq d \leq 2r \), there exists a connected graph \( G \) with \( \text{rad } G = r, \text{diam } G = d \) and \( dg_t(G) = k \). Also, for integers \( n, a, b \) such that \( 4 \leq a \leq b \leq n \), there exists a connected graph \( G \) of order \( n \), with \( dg_t(G) = a \) and \( dg_c(G) = b \). It is proved that for positive integers \( a, b \) with \( 4 \leq a \leq b \) and \( b \leq 2a \), there exists a connected graph \( G \) such that \( dg(G) = a \) and \( dg_t(G) = b \).

The connected double geodetic number of a graph

**Definition 3.1** Let \( G \) be a connected graph with at least two vertices. A connected double geodetic set of \( G \) is a double geodetic set \( S \) such that the subgraph \( G[S] \) induced by \( S \) is connected. The minimum cardinality of a connected double geodetic set of \( G \) is the connected double geodetic number of \( G \) and is denoted by \( dg_c(G) \).

**Example 3.2** For the graph \( G \) given in Figure 3.1, \( S = \{v_1, v_4, v_5, v_6\} \) is a minimum double geodetic set of \( G \) so that \( dg(G) = 4 \). Since the induced subgraph \( G[S] \) is not connected, \( S \) is not a connected double geodetic set of \( G \). It is clear that \( T = \{v_1, v_2, v_3, v_4, v_5, v_6\} \) is a minimum connected double geodetic set of \( G \) and so \( dg_c(G) = \)
Theorem 3.3  Each weak extreme vertex of a connected graph $G$ belongs to every connected double geodetic set of $G$. In particular, every end-vertex of $G$ belongs to every connected double geodetic set of $G$.

Proof. Since every connected double geodetic set is also a double geodetic set, the result follows from Proposition 2.16.  

Corollary 3.4  For the complete graph $K_n$ ($n \geq 2$), $\text{dg}_c(K_n) = n$.

Theorem 3.5  Let $G$ be a connected graph with a cut vertex $v$. Then each connected double geodetic set of $G$ contains at least one vertex from each component of $G - v$.

Proof. This follows from Theorem 2.8.

Theorem 3.6  Each cut-vertex of a connected graph $G$ belongs to every connected double geodetic set of $G$.

Proof. Let $v$ be any cut-vertex of $G$ and let $G_1, G_2, \ldots, G_r$ ($r \geq 2$) be the components of $G - \{v\}$. Let $S$ be any connected double geodetic set of $G$. Then by Theorem 3.5, $S$ contains at least one element from each $G_i(1 \leq i \leq r)$. Since $G[S]$ is connected, it follows that $v \in S$.

Corollary 3.7  For a connected graph $G$ with $k$ weak extreme vertices and $l$ cut-vertices, $\text{dg}_c(G) \geq \max \{2, k + l\}$.
Proof. This follows from Theorems 3.3 and 3.6.

Corollary 3.8 For any non-trivial tree $T$ of order $n$, $d_{gc}(T) = n$.

Proof. This follows from Corollary 3.7.

Theorem 3.9 For a connected graph $G$ of order $n$, $2 \leq d_{g}(G) \leq d_{gc}(G) \leq n$.

Proof. Any double geodetic set needs at least two vertices and so $d_{g}(G) \geq 2$. Since every connected double geodetic set is also a double geodetic set, it follows that $d_{g}(G) \leq d_{gc}(G)$. Also, since $V(G)$ induces a connected double geodetic set of $G$, it is clear that $d_{gc}(G) \leq n$.

Remark 3.10 The bounds in Theorem 3.9 are sharp. For any non-trivial path $P$, $d_{g}(P) = 2$. For the complete graph $K_n$, $d_{g}(K_n) = d_{gc}(K_n)$. For any non-trivial tree $T$ of order $n$, $d_{gc}(T) = n$, by Corollary 3.8. Also, all the inequalities in Theorem 3.9 are strict. For the graph $G$ given Figure 3.1, $d_{g}(G) = 4, d_{gc}(G) = 6$ and $n = 7$ so that $2 < d_{g}(G) < d_{gc}(G) < n$.

Corollary 3.11 Let $G$ be a connected graph. If $d_{gc}(G) = 2$, then $d_{g}(G) = 2$.

Proof. This follows from Theorem 3.9.

Theorem 3.12 Let $G$ be a connected graph of order $n \geq 2$. Then $d_{gc}(G) = 2$ if and only if $G = K_2$.

Proof. If $G = K_2$, then $d_{gc}(G) = 2$. Conversely, let $d_{gc}(G) = 2$. Let $S = \{u, v\}$ be a
minimum connected double geodetic set of $G$. Then $uv$ is an edge. If $G \neq K_2$, then there exists a vertex $w$ different from $u$ and $v$, and $w$ does not lie on any $u$-$v$ geodesic so that $S$ is not a $dg_c$-set, which is a contradiction. Thus $G = K_2$.

**Theorem 3.13** Let $G$ be a connected graph of order $n$. Then $dg_c(G) = n$ if and only if every vertex of $G$ is either a cut-vertex or a weak extreme vertex.

**Proof.** Let $G$ be a connected graph with every vertex of $G$ either a cut-vertex or weak extreme vertex. Then the result follows from Theorems 3.3 and 3.6. Conversely, let $G$ be a connected graph of order $n$ with $dg_c(G) = n$. Suppose that there exists a vertex $v$ which is neither a weak extreme vertex nor a cut-vertex of $G$. We show that $S = V - \{v\}$ is a connected double geodetic set of $G$. Since $v$ is not a cut-vertex of $G$, the subgraph induced by $S$ is connected. Let $u \neq v$ be any vertex of $G$. Since $v$ is not a weak extreme vertex of $G$, we have $u, v \in I[x, y]$ for a pair of vertices $x, y \in G$ with $v \neq x$ and $v \neq y$. This shows that $S$ is a double geodetic set of $G$. Thus $S$ is a connected double geodetic set of $G$ and so $dg_c(G) \leq n - 1$, which is a contradiction. Hence every vertex of $G$ is either a cut-vertex or a weak extreme vertex.

**Theorem 3.14** If $a, b$, and $n$ are integers such that $2 \leq a < b \leq n$, then there exists a connected graph $G$ of order $n$ such that $dg(G) = a$ and $dg_c(G) = b$.

**Proof.** We prove the theorem by considering three cases.

**Case 1.** $2 \leq a < b = n$. Let $G$ be any tree of order $b$ with end vertices equal to $a$. Then by Proposition 2.16, $dg(G) = a$ and by Corollary 3.8, $dg_c(G) = n$.

**Case 2.** $2 = a < b < n$. Let $P_b : u_1, u_2, \ldots, u_b$ be a path on $b$ vertices. Add $(n-b)$ new
vertices $w_1, w_2, \ldots, w_{n-b}$ to $P_b$ and join $w_1, w_2, \ldots, w_{n-b}$ to both $u_1$ and $u_3$, thereby producing the graph $G$ of Figure 3.2. Then $G$ has order $n$ and $S = \{u_1, u_b\}$ is the unique minimum double geodetic set of $G$ and so by Proposition 2.16 $dg(G) = 2 = a$. Also, $S_1 = \{u_1, u_3, u_4, \ldots, u_b\}$ is the set of all cut-vertices and weak extreme vertices of $G$. By Theorems 3.3 and 3.6, every connected double geodetic set contains $S_1$. It is clear that $S_1$ is not a connected double geodetic set of $G$. Since $S_1 \cup \{u_2\}$ is a connected double geodetic set of $G$, we have $dg_c(G) = b$.

**Case 3.** $3 \leq a < b < n$. First assume that $b \neq a + 1$. Let $P_{b-a+2} : u_1, u_2, \ldots, u_{b-a+2}$ be a path on $(b-a+2)$ vertices. Add $(a-2+n-b)$ new vertices $v_1, w_1, w_2, \ldots, w_{a-3}, x_1, x_2, \ldots, x_{n-b}$ to $P_{b-a+2}$ and join $v_1$ to $u_2$, and join $w_1, w_2, \ldots, w_{a-3}$ to both $u_1$ and $u_3$, and join $x_1, x_2, \ldots, x_{n-b}$ to both $u_2$ and $u_4$ thereby producing the graph $G$ Figure 3.3. Then $G$ has order $n$ and $S = \{v_1, u_1, w_1, w_2, \ldots, w_{a-3}, u_{b-a+2}\}$ is the unique minimum double geodetic set of $G$ and so by Proposition 2.16 $dg(G) = a$. Also $S_1 = \{u_2, u_4, u_5, \ldots, u_{b-a+2} : v_1, u_1, w_1, w_2, \ldots, w_{a-3}\}$ is the set of all cut vertices and weak extreme vertices of $G$. By Theorems 3.3 and 3.6, every connected double
geodetic set contains $S_1$. It is clear that $S_1$ is not a connected double geodetic set of $G$. Since $S_1 \cup \{u_3\}$ is a connected double geodetic set of $G$, we have $dg_c(G) = b$.

Next, assume that $b = a + 1$. Let $V(K_{n-a}) = \{u_1, u_2, \ldots, u_{n-a}\}$ and $V(K_a) = \{v_1, v_2, \ldots, v_a\}$. Let $G = \overline{K_a} + K_{n-a}$. Then $G$ has order $n$ and $S = \{v_1, v_2, \ldots, v_a\}$ is the unique minimum double geodetic set of $G$ and so by Proposition 2.16 $dg(G) = a$.

By Theorem 3.3, every connected double geodetic set contains $S$. It is clear that $S$ is not a connected double geodetic set of $G$. Since $S \cup \{u_1\}$ is a connected double geodetic set of $G$, it follows that $dg_c(G) = a + 1 = b$.

For every connected graph $G$, $\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G$. Ostrand [16] showed that every two positive integers $a$ and $b$ with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively, of some connected graph. Now, Ostrand’s theorem can be extended so that the connected double geodetic number can also be prescribed.
Theorem 3.15  

For positive integers $r, d$ and $k \geq 4$ with $r \leq d \leq 2r$ and $k - d - 1 \geq 0$, there exists a connected graph $G$ with $\text{rad } G = r$, $\text{diam } G = d$ and $\text{dg}_c(G) = k$.

Proof. If $r = 1$, then $d = 1$ or $2$. For $d = 1$, let $G = K_k$. Then $\text{dg}_c(G) = k$. For $d = 2$, construct a graph $G$ as follows: Let $P_3 : u_1, u_2, u_3$ be a path of order 3. Add a new vertex $v_1$ to $P_3$ and join to the vertex $u_2$ and obtain the graph $H$. Also, add $(k - 4)$ new vertices $w_1, w_2, \ldots, w_{k-5}$ to $H$ and join each $w_i (1 \leq i \leq k - 4)$ to $u_1, u_2$ and $u_3$ and obtain the graph $G$ is Figure 3.4. Then $\text{rad } G = 1$ and $\text{diam } G = 2$. It is clear that $v_1, u_1, u_3, w_1, w_2, \ldots, w_{k-4}$ are the weak extreme vertices of $G$ and $u_2$ is the only cut-vertex of $G$. Hence by Theorem 2.13, $\text{dg}_c(G) = k$.

Now, let $r \geq 2$.

Case 1. $r = d$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$. Let $G$ be the graph given in Figure 3.5, obtained by adding the new vertices $v_1, v_2, \ldots, v_{k-r-1}$ and joining each $v_i (i \leq i \leq k - r - 1)$ with $u_1$ and $u_{2r}$ of $C_{2r}$. It is easily verified that the eccentricity of each vertex of $G$ is $r$ so that $\text{rad } G = \text{diam } G = r$. Let
$S = \{v_1, v_2, \ldots, v_{k-r-1}, u_r, u_{r+1}, u_1\}$ be the set of all weak extreme vertices of $G$. By Theorem 2.3, every connected double geodetic set of $G$ contains $S$. It is clear that $S$ is not a connected double geodetic set of $G$. Since $S_1 = S \cup \{u_2, u_3 \ldots u_{r-1}\}$ is a connected double geodetic set of $G$, we have, $dg_c(G) = k$.

**Case 2.** $r < d$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$ and let $P_{d-r+1} : v_0, v_1, \ldots, v_{d-r}$ be a path of order $d - r + 1$. Let $H$ be the graph obtained from $C_{2r}$ and $P_{d-r+1}$ by identifying $v_0$ of $P_{d-r+1}$ and $u_1$ of $C_{2r}$. Now, add $k - 6$ new vertices $w_1, w_2, \ldots, w_{k-6}$ to the graph $H$ and join $w_1$ to $u_r$, and join each vertex $w_i(2 \leq i \leq k-d-2)$ to both $u_{r+1}$ and $u_{r-1}$, thereby obtaining the graph $G$ in Figure 3.6. Then $rad G = r$ and $diam G = d$. Now, $S_1 = \{w_1, w_2, \ldots, w_{k-d-2}, u_{r+1}, u_{2r}, v_{d-r}\}$ is the set of all weak extreme vertices of $G$ and $S_2 = \{u_r, u_1, v_1, v_2, \ldots, v_{d-r-1}\}$ is the set of all cut-vertices of $G$. By Theorems 3.3 and 3.6, every connected double geodetic set contains $S_1 \cup S_2$. Although $S_1 \cup S_2$ is a double geodetic set, it is not a connected double geodetic set of $G$. It is clear that $T = S_1 \cup S_2 \cup \{u_2, u_3, \ldots, u_{r-1}\}$ is a minimum connected double geodetic set of $G$ and so $dg_c(G) = k$. 

![Figure 3.5](image-url)
The total double geodetic number of a graph

**Definition 3.16** Let $G$ be a connected graph with at least two vertices. A *total double geodetic set* of a graph $G$ is a double geodetic set $S$ such that the subgraph $G[S]$ induced by $S$ has no isolated vertices. The minimum cardinality of a total double geodetic set of $G$ is the *total double geodetic number* of $G$ and is denoted by $dg_t(G)$.

**Example 3.17** For the graph $G$ given in Figure 3.7, $S = \{v_1, v_2, v_7, v_6\}$ is the minimum double geodetic set of $G$ so that $dg(G) = 4$. Note that the subgraph induced by $S$ has isolated vertices so that $S$ is not a total double geodetic set of $G$. It is easily seen that $T = \{v_1, v_2, v_3, v_7, v_5, v_6\}$ is a minimum total double geodetic set of $G$ and so $dg_t(G) = 6$. 
It follows from Proposition 2.16 that every weak extreme vertex of a connected graph $G$ belongs to every total double geodetic set of $G$. Hence for the complete graph $K_n (n \geq 2), dg_t (K_n) = n$.

**Theorem 3.18**  For a connected graph $G$ of order $n$, $2 \leq dg(G) \leq dg_t(G) \leq dg_c(G) \leq n$.

**Proof.** Any double geodetic set needs at least two vertices and so $dg(G) \geq 2$. Since every total double geodetic set is a double geodetic set, we have $dg(G) \leq dg_t(G)$, and every connected double geodetic is a total double geodetic set of $G$ so that $dg_t(G) \leq dg_c(G)$. Also, since $G$ is connected, it is clear that $dg_c(G) \leq n$. Thus $2 \leq dg(G) \leq dg_t(G) \leq dg_c(G) \leq n$. $\blacksquare$

A vertex $v$ of a connected graph $G$ is called a *support* of $G$ if it is adjacent to an endvertex of $G$.

**Theorem 3.19**  Every total double geodetic set of a connected graph $G$ contains all the weak extreme vertices and the support vertices of $G$. In particular, if the set
$S$ of all weak extreme vertices and support vertices is a total double geodetic set, then $S$ is the unique $dg_t$-set of $G$.

**Proof.** This follows from Proposition 2.16. ■

For a nontrivial tree $T$, the set $S$ of all endvertices and all support vertices forms a total double geodetic set of $T$ so that $dg_t(T) = |S|$.

**Theorem 3.20** For a nontrivial tree $T$ of order $n$, $dg_t(T) = n$ if and only if $T$ is a caterpillar with all its cutvertices are support vertices.

**Proof.** Let $dg_t(T) = n$. By Theorem 3.19, all the vertices of $T$ are end vertices and support vertices. Hence it follows that $T$ is a caterpillar. The converse part is clear. ■

**Theorem 3.21** For the complete bipartite graph $G = K_{m,n}$,

$$dg_t(G) = \begin{cases} 
2 & \text{if } m = n = 1 \\
 n + 1 & \text{if } m = 1, n \geq 2 \\
\min\{m, n\} + 1 & \text{if } m, n \geq 2. 
\end{cases}$$

**Proof.** The first two parts are clear as $G$ is a tree. For $m, n \geq 2$, let $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ be the partite sets of $G$. Let $S$ be a double geodetic set of $G$. We claim that $X \subseteq S$ or $Y \subseteq S$. Otherwise, there exist vertices $x, y$ such that $x \in X, y \in Y$ and $x, y \notin S$. Now, since the pair of vertices $x, y$ lie only on the intervals $I[x, y], I[x, t]$ or $I[s, y]$ for some $t \in X$ and $s \in Y$, it follow that $x \in S$ or $y \in S$, which is a contradiction. Hence $X \subseteq S$ or $Y \subseteq S$. Also it is clear that both...
X and Y are double geodetic sets of $K_{m,n}$ and so $dg(G) = \min\{m, n\}$. Assume that $m \leq n$. Then $S = X \cup \{y\}$, where $y \in Y$, is a total double geodetic set of $G$ and so $dg_t(G) = m + 1 = \min\{m, n\} + 1$.

**Theorem 3.22** For an even cycle $G = C_{2n}$, $dg_t(G) = \begin{cases} 3 & \text{if } n = 2 \\ 4 & \text{if } n \geq 3. \end{cases}$

**Proof.** For $n = 2$, it is clear that any set of three vertices is a minimum total double geodetic set of $G$ so that $dg_t(G) = 3$. For $n \geq 3$, let $G$ be the cycle $C_{2n} : v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}, v_1$. Since $S = \{v_1, v_{n+1}\}$ is a double geodetic set, it is easily seen that $S_1 = S \cup \{v_2, v_n\}$ is a minimum total double geodetic set of $G$ so that $dg_t(G) = 4$.

**Theorem 3.23** For the odd cycle $G = C_{2n+1}(n \geq 1)$, $dg_t(G) = 2n + 1$.

**Proof.** This follows from Theorem 2.18.

**Theorem 3.24** For any connected graph $G$, $dg_t(G) = 2$ if and only if $G = K_2$.

**Proof.** Let $dg_t(G) = 2$ and let $S = \{u, v\}$ be a total double geodetic set of $G$. Then $uv$ is an edge. It is clear that a vertex different from $u$ and $v$ cannot lie on a $u$-$v$ geodesic and so $G = K_2$. The converse is clear.

**Theorem 3.25** Let $G$ be a connected graph with at least two vertices. Then $dg_t(G) \leq 2dg(G)$.

**Proof.** Let $S = \{v_1, v_2, \ldots, v_k\}$ be a minimum double geodetic set of $G$. Let $u_i \in$
\(N(v_i)\) for \(i = 1, 2, \ldots, k\) and \(T = \{u_1, u_2, \ldots, u_k\}\). Then \(S \cup T\) is a total double geodetic set of \(G\) so that \(dg_t(G) \leq |S \cup T| \leq 2k = 2dg(G)\).

In view of Theorem 3.25, we have the following realization results.

**Theorem 3.26**  For integers \(a, b\) with \(4 \leq a \leq b \leq 2a\), there exists a connected graph \(G\) such that \(dg(G) = a\) and \(dg_t(G) = b\).

**Proof.** Case 1. For \(a = b\), the complete graph \(K_a\) has the desired properties.

Case 2. \(a < b\). Let \(b = a + p\), where \(1 \leq p \leq a\). For \(p = 1\), the star \(K_{1,a}\) has the desired properties. Now, let \(p \geq 2\). For each integer \(i\) with \(1 \leq i \leq p - 1\), let \(F_i\) be a copy of the cycle \(C_4\) with vertex set \(V(F_i) = \{x_i, y_i, z_i, w_i\}\). Let \(H\) be a graph formed by identifying all these \(F_i\) at \(x_i\), and let \(x\) be the identified vertices. Let \(P_b : u_1, u_2, \ldots, u_b\) be a path of order \(b\). Let \(H'\) be the graph obtained by identifying the vertex \(u_1\) of \(P_b\) with the vertex \(x\) of \(H\). Let \(G\) be the graph obtained from \(H'\) by adding the new vertices \(v_1, v_2, \ldots, v_{a-p}\), and joining \(v_1\) with \(u_{b-1}\), and each \(v_i(2 \leq i \leq a - p)\) with both \(u_{b-2}\) and \(u_b\). The graph \(G\) is shown in Figure 3.8. Let \(S = \{v_1, v_2, \ldots, v_{a-p}, u_b, z_1, z_2, \ldots, z_{p-1}\}\) be the set of all weak extreme vertices of \(G\).

Since \(S\) is a double geodetic set of \(G\), it follows from Proposition 2.16 that \(dg(G) = a\).

By Theorem 3.19, every total double geodetic set of \(G\) contains \(S\). It is clear that \(S_1 = S \cup \{y_1, y_2, \ldots, y_{p-1}, u_{b-1}\}\) is a minimum total double geodetic set of \(G\) so that \(dg_t(G) = b\). 

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Now, Ostrand’s theorem [16] can be extended so that the total double geodetic number can be prescribed.

**Theorem 3.27** For positive integers \( r, d \) and \( k \geq 4 \) with \( r \leq d \leq 2r \), there exists a connected graph \( G \) with \( \text{rad} \, G = r, \text{diam} \, G = d \) and \( d_{gt}(G) = k \).

**Proof.** If \( r = 1 \), then \( d = 1 \) or 2. For \( d = 1 \), let \( G = K_k \). Then \( d_{gt}(G) = k \). For \( d = 2 \), let \( G = K_{1,k-1} \). Then \( d_{gt}(G) = k \). Now, let \( r \geq 2 \). We construct a graph \( G \) with the desired properties as follows.

Case 1. \( r = d \). Let \( C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1 \) be a cycle of order \( 2r \). Let \( G \) be the graph given in Figure 3.9, obtained by adding the new vertices \( v_1, v_2, \ldots, v_{k-3} \) and joining each \( v_i (i \leq i \leq k-3) \) with \( u_1 \) and \( u_{2r} \) of \( C_{2r} \). It is easily verified that the eccentricity of each vertex of \( G \) is \( r \) so that \( \text{rad} \, G = \text{diam} \, G = r \). Let \( S = \{v_1, v_2, \ldots, v_{k-3}, u_{r+1}\} \).
be the set of all weak extreme vertices of $G$. By proposition 2.16, every total double geodetic set of $G$ contains $S$. It is clear that for any $x \notin S, S \cup \{x\}$ is not a total double geodetic set of $G$. Since $S_1 = S \cup \{u_1, u_r\}$ is a total double geodetic set of $G$, we have, $dg_t(G) = k$.

\[ \text{Figure 3.9} \]

Case 2. $r < d$. First, assume that $k \geq 7$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$ and let $P_{d-r+1} : v_0, v_1, \ldots, v_{d-r}$ be a path of order $d - r + 1$. Let $H$ be the graph obtained from $C_{2r}$ and $P_{d-r+1}$ by identifying $v_0$ of $P_{d-r+1}$ and $u_1$ of $C_{2r}$. Now, add $k - 6$ new vertices $w_1, w_2, \ldots, w_{k-6}$ to the graph $H$ and join $w_1$ to $u_r$, and join each vertex $w_i (2 \leq i \leq k - 6)$ to both $u_{r+1}$ and $u_{r-1}$, thereby obtaining the graph $G$ in Figure 3.10. Then $\text{rad } G = r$ and $\text{diam } G = d$. Let $S_1 = \{w_1, w_2, \ldots, w_{k-6}, u_{r+1}, u_2r, v_{d-r}, u_r, v_{d-r-1}\}$ be set of all weak extreme vertices and support vertices of $G$. By Theorem 3.19, every total double geodetic set contains $S_1$. It is clear that $S_1$ is not a total double geodetic set of $G$. Since $S_2 = S_1 \cup \{u_1\}$ is a total double geodetic set of $G$, we have $dg_t(G) = k$. 

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Now, for \( k = 4, 5, 6 \), let \( G_1 \) be the graph obtained from \( H \) by adding \((k - 4)\) new vertices \( y_1, y_2, \ldots, y_{k-4} \) and joining each \( y_i (1 \leq i \leq k - 4) \) to both \( u_r \) and \( u_{r+2} \). Then \( \text{rad} \ G_1 = r \) and \( \text{diam} \ G_1 = d \). Let \( T = \{ y_1, y_2, \ldots, y_{k-4}, u_{r+1}, v_{d-r-1}, v_{d-r} \} \) be the set of all weak extreme vertices and support vertices of \( G_1 \). By Theorem 3.19, every total double geodetic set contains \( T \). It is clear that \( T_1 = T \cup \{ u_r \} \) is a minimum total double geodetic set of \( G_1 \) so that \( \text{dg}_t(G) = k \).

**Theorem 3.28** If \( n, a, b \) are integers such that \( 4 \leq a \leq b \leq n \), then there exists a connected graph \( G \) of order \( n \) with \( \text{dg}_t(G) = a \) and \( \text{dg}_c(G) = b \).

**Proof.** We prove this theorem by considering four cases.

Case 1. \( a = b = n \). Let \( G = K_n \). Then \( \text{dg}_t(G) = \text{dg}_c(G) = n \).

Case 2. \( a < b < n \). Let \( P_{b-a+4} : u_1, u_2, \ldots, u_{b-a+4} \) be a path of order \( b - a + 4 \). Add \( n + a - b - 4 \) new vertices \( v_1, v_2, \ldots, v_{n-b}, w_1, w_2, \ldots, w_{a-4} \) to \( P_{b-a+4} \) and
join \( w_1 \) to \( u_2 \), and join \( w_2, w_3, \ldots, w_{a-4} \) with \( u_1 \) and \( u_3 \) and also join \( v_1, v_2, \ldots, v_{n-b} \) with both \( u_2 \) and \( u_4 \) to get the graph \( G \) of order \( n \) given in Figure 3.11. Let \( S = \{w_1, w_2, \ldots, w_{a-4}, u_1, u_2, u_{b-a+3}, u_{b-a+4}\} \) be the set of all weak extreme vertices and support vertices of \( G \). Since \( S \) is a total double geodetic set of \( G \), by Theorem 3.19 \( dg_t(G) = a \). Let \( S_1 = \{w_1, w_2, \ldots, w_{a-4}, u_1, u_{b-a+4}, u_2, u_4, u_5, \ldots, u_{b-a+3}\} \) be the set of all weak extreme vertices and cutvertices. By Theorems 3.3 and 3.6, every connected double geodetic set of \( G \) contains \( S_1 \). It is clear \( S_1 \) is not a connected double geodetic set of \( G \). Since \( T = S_1 \cup \{u_3\} \) is a minimum connected double geodetic set of \( G \), we have \( dg_c G = b \).

Case 3. \( a = b < n \). Let \( P_3 : u_1, u_2, u_3 \) be a path of order 3. Add the new vertices \( v_1, v_2, \ldots, v_{n-a} \) and join each \( v_i (1 \leq i \leq n-a) \) with \( u_1 \) and \( u_3 \). Also, add new vertices \( w_1, w_2, \ldots, w_{a-3} \) and join each \( w_i (1 \leq i \leq a-3) \) with \( u_1 \), thereby obtaining the graph \( G \) of order \( n \) given in Figure 3.12.
Let $S_2 = \{w_1, w_2, \ldots, w_{a-3}, u_1, u_3\}$ be the set of all weak extreme vertices and support vertices of $G$. By Theorems 3.3 and 3.19, every total double geodetic set as well as every connected double geodetic set of $G$ contains $S_2$. Since $S_2$ is neither a total double geodetic set nor a connected double geodetic set of $G$ and $S_2 \cup \{u_2\}$ is a connected double geodetic set of $G$, we have $dg_c(G) = dg_t(G) = a = b$.

Case 4. $a < b = n$. Let $P_{b-a+4} : u_1, u_2, \ldots, u_{b-a+4}$ be a path of order $b-a+4$. Add $a-4$ new vertices $v_1, v_2, \ldots, v_{a-4}$ to $P_{b-a+4}$ and join $v_1$ to $u_{b-a+3}$ and join $v_2, v_3, \ldots, v_{a-4}$ with both $u_{b-a+2}$ and $u_{b-a+4}$, thereby producing the graph $G$ of order $n$ given in Figure 3.13. Let $S_3 = \{u_1, u_2, v_1, v_2, \ldots, v_{a-4}, u_{b-a+3}, u_{b-a+4}\}$ be the set of all weak extreme vertices and support vertices of $G$. Since $S_3$ is a total double geodetic set of $G$, it follows from Theorem 3.19 that $dg_t(G) = a$. Since each vertex of $G$ is either a weak extreme vertex or a cutvertex of $G$, it follows from Theorems 3.3 and 3.6 that $dg_c(G) = n = b$. 

Figure 3.12

Let $S_2 = \{w_1, w_2, \ldots, w_{a-3}, u_1, u_3\}$ be the set of all weak extreme vertices and support vertices of $G$. By Theorems 3.3 and 3.19, every total double geodetic set as well as every connected double geodetic set of $G$ contains $S_2$. Since $S_2$ is neither a total double geodetic set nor a connected double geodetic set of $G$ and $S_2 \cup \{u_2\}$ is a connected double geodetic set of $G$, we have $dg_c(G) = dg_t(G) = a = b$.

Case 4. $a < b = n$. Let $P_{b-a+4} : u_1, u_2, \ldots, u_{b-a+4}$ be a path of order $b-a+4$. Add $a-4$ new vertices $v_1, v_2, \ldots, v_{a-4}$ to $P_{b-a+4}$ and join $v_1$ to $u_{b-a+3}$ and join $v_2, v_3, \ldots, v_{a-4}$ with both $u_{b-a+2}$ and $u_{b-a+4}$, thereby producing the graph $G$ of order $n$ given in Figure 3.13. Let $S_3 = \{u_1, u_2, v_1, v_2, \ldots, v_{a-4}, u_{b-a+3}, u_{b-a+4}\}$ be the set of all weak extreme vertices and support vertices of $G$. Since $S_3$ is a total double geodetic set of $G$, it follows from Theorem 3.19 that $dg_t(G) = a$. Since each vertex of $G$ is either a weak extreme vertex or a cutvertex of $G$, it follows from Theorems 3.3 and 3.6 that $dg_c(G) = n = b$.
Figure 3.13