Chapter 2

The double geodetic number of a graph

In this chapter we introduce and study the concept of double geodetic number of a graph $G$. Certain general properties of this concept are studied. We determine bounds for it and characterize graphs which realize the bounds. It is shown that for every pair $k, n$ of integers with $2 \leq k \leq n$, there exists a connected graph $G$ of order $n$ such that $dg(G) = k$. It is proved that if $G$ is a connected graph of order $n$ having a full degree vertex $v$ and $G - v$ has radius at least 3 then $dg(G) = n - 1$. It is shown that for positive integers $r, d, a$ and $b$ such that $r < d \leq 2r$ and $3 \leq a \leq b$ there exists a connected graph $G$ with $rad G = r$, $diam G = d$, $g(G) = a$ and $dg(G) = b$. Also, it is proved that for integers $n, d \geq 2$ and $l$ such that $3 \leq k \leq l \leq n$ and $n - d - l + 1 \geq 0$, there exists a graph $G$ of order $n$ diameter $d$, $g(G) = k$ and $dg(G) = l$. Also, we introduce the concept of the upper double geodetic number of a graph $G$. The upper double geodetic numbers of certain standard graphs are obtained. It is proved that for a connected graph $G$ of order $n$, $dg(G) = n$ if and only if $dg^+(G) = n$. It is also proved that $dg(G) = n - 1$ if and only if $dg^+(G) = n - 1$ for a non-complete graph $G$ of order $n$ having a vertex of degree $n - 1$. For every two positive integers $a$ and $b$, where $2 \leq a \leq b$, there exists a connected graph $G$ with $dg(G) = a$ and

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We introduce the notion of forcing double geodetic sets and the forcing double geodetic number of a graph. The general properties satisfied by these forcing subsets are discussed and the forcing double geodetic numbers for certain classes of graphs are determined.

**Double Geodetic Number of a Graph**

**Definition 2.1** Let $G$ be a connected graph with at least two vertices. A set $S$ of vertices of $G$ is called a *double geodetic set* of $G$ if for each pair of vertices $x, y$ in $G$ there exist vertices $u, v$ in $S$ such that $x, y \in I[u, v]$. The *double geodetic number* $dg(G)$ of $G$ is the minimum cardinality of a double geodetic set. Any double geodetic set of cardinality $dg(G)$ is called *$dg$-set* of $G$.

**Example 2.2** For the graph $G$ in Figure 2.1, it is clear that no 2-element or no 3-element subset of $G$ is a double geodetic set of $G$. Since $S = \{u_1, u_3, u_4, u_5\}$ is a double geodetic set, it follows that $dg(G) = 4$.

![Figure 2.1](image)

**Remark 2.3** For the graph $G$ in Figure 2.1, $S = \{u_1, u_3, u_5\}$ is a *$g$-set* of $G$ and so $g(G) = 3$. Thus the double geodetic number and the geodetic number of a graph can be different.
Theorem 2.4  For any graph $G$ of order $n$, $2 \leq g(G) \leq dg(G) \leq n$.

Proof. A geodetic set needs at least two vertices and therefore $g(G) \geq 2$. It is clear that every double geodetic set is also a geodetic set and so $g(G) \leq dg(G)$. Since the set of all vertices of $G$ is a double geodetic set of $G$, $dg(G) \leq n$. ■

Remark 2.5  The bounds in Theorem 2.4 are sharp. For the complete graph $K_n$ ($n \geq 2$), we have $dg(K_n) = n$. The set of the two end vertices of a nontrivial path $P_n$ on $n$ vertices is its unique double geodetic set so that $dg(P_n) = 2$. Thus the complete graph $K_n$ has the largest possible double geodetic number $n$ and that the nontrivial paths have the smallest double geodetic number.

Theorem 2.6  Each extreme vertex of a connected graph $G$ belongs to every double geodetic set of $G$. In particular, if the set of all end vertices of $G$ is a double geodetic set, then it is the unique $dg$-set of $G$.

Proof. Since every double geodetic set is a geodetic set, the result follows from Theorem 1.15. ■

Corollary 2.7  For a graph $G$ of order $n$ with $k$ extreme vertices, $\max\{2,k\} \leq dg(G) \leq n$.

Proof. This follows from Theorems 2.4 and 2.6. ■

Theorem 2.8  Let $G$ be a connected graph with a cut vertex $v$. Then each double geodetic set of $G$ contains at least one vertex from each component of $G - v$.

Proof. This follows from Theorem 1.16 and the fact that every double geodetic set is a geodetic set. ■

Theorem 2.9  No cut-vertex of a connected graph $G$ belongs to any $dg$-set of $G$. 

Proof. Let $S$ be any $dg$-set of $G$. Suppose that $S$ contains a cut vertex $z$ of $G$. Let $G_1, G_2, \ldots, G_r$ $(r \geq 2)$ be the components of $G - z$. Let $S_1 = S - \{z\}$. We claim that $S_1$ is a double geodetic set of $G$. Let $x, y \in V(G)$. Since $S$ is a double geodetic set, there exist $u, v \in S$ such that $x, y \in I[u,v]$. If $z \notin \{u, v\}$ then $u, v \in S_1$ and so $S_1$ is a double geodetic set of $G$, which is contradiction to the minimality of $S$. Now, assume that $z \in \{u, v\}$ say $z = u$. Assume without loss of generality that $v$ belongs to $S_1$. By Theorem 2.8, we can choose a vertex $w$ in $G_k$ $(k \neq 1)$ such that $w \in S$. Now, since $z$ is a cut vertex of $G$, it follows that $I[z,v] \subseteq I[w,v]$. Hence $x, y \in I[w,v]$ with $w, v \in S_1$. Thus $S_1$ is a double geodetic set of $G$ which is contradiction to the minimality of $S$. Thus no cut vertex belongs to any $dg$-set of $G$. 

Corollary 2.10 For any tree $T$, the double geodetic number $dg(T)$ equals the number of end vertices in $T$. In fact, the set of all end vertices of $T$ is the unique $dg$-set of $T$.

Proof. This follows from Theorems 2.6 and 2.9. 

Corollary 2.11 For every pair $k, n$ of integers with $2 \leq k \leq n$, there exists a connected graph $G$ of order $n$ such that $dg(G) = k$.

Proof. For $k = n$, let $G = K_n$. Then, by Theorem 2.6 $dg(G) = n = k$. Also, for each pair of integers with $2 \leq k < n$, there exists a tree of order $n$ with $k$ end vertices. Hence the result follows from Corollary 2.10.

Proposition 2.12 For a nontrivial connected graph $G$, $g(G) = 2$ if and only if $dg(G) = 2$.

Proof. If $dg(G) = 2$, then by Theorem 2.4, $g(G) = 2$. Suppose that $g(G) = 2$. Let $S = \{u, v\}$ be a $g$-set of $G$. Then it is clear that $x, y \in I[u,v]$ for any pair $x, y$ of vertices of $G$. Thus $S$ is a $dg$-set of $G$ and so $dg(G) = 2$. 

17
Corollary 2.13  

For the cycle $C_{2n}$ ($n \geq 2$), $dg(C_{2n}) = 2$.

Proof. Since $g(C_{2n}) = 2$, the result follows from Proposition 2.12.

Definition 2.14  

A vertex $v$ in a connected graph $G$ is said to be a weak extreme vertex if there exists a vertex $u$ in $G$ such that $u, v \in I[x, y]$ for a pair of vertices $x, y$ in $G$, then $v = x$ or $v = y$. Equivalently, a vertex $v$ in a connected graph is a weak extreme vertex if there exists a vertex $u$ in $G$ such that $v$ is either an initial vertex or a terminal vertex of any interval containing both $u$ and $v$.

Example 2.15  

Each extreme vertex of a graph is weak extreme. Also, for the graph $G$ in Figure 2.2, it is clear that the pair $v_2, v_5$ lies only on the $v_2 - v_5$ geodesic and so $v_2$ and $v_5$ are weak extreme vertices of $G$. Similarly, the vertices $v_4$ and $v_6$ are also weak extreme vertices of $G$. It is easily seen that $v_1$ and $v_3$ are also weak extreme vertices of $G$.

Proposition 2.16  

Every double geodetic set of a connected graph $G$ contains all the weak extreme vertices of $G$. In particular, if the set $W$ of all weak extreme vertices is a double geodetic set, then $W$ is the unique $dg$-set of $G$.

Proof. Let $S$ be a double geodetic set of $G$ and $v$ a weak extreme vertex such that $v \notin S$. Let $u$ be a vertex in $G$ such that $u \neq v$. Since $S$ is a double geodetic set of $G$, we have $u, v \in I[x, y]$ for some $x, y \in S$. Also, since $v$ is a weak extreme vertex of $G$,
we have \( v = x \) or \( v = y \). Thus \( v \in S \), which is a contradiction. 

Example 2.17 For the graph \( G \) in Figure 2.3, the set \( S = \{v_1, v_5, v_{10}\} \) of end vertices is the unique minimum geodetic set of \( G \) so that \( g(G) = 3 \). Since the pair of vertices \( v_3, v_9 \) do not lie on any geodesic of a pair vertices from \( S \), \( S \) is not a double geodetic set of \( G \). It is clear that the vertex \( v_3 \) is the only weak extreme vertex, which is not extreme. Since \( S' = \{v_1, v_3, v_5, v_{10}\} \) is a double geodetic set of \( G \), it follows from Proposition 2.16 that \( dg(G) = 4 \).

Theorem 2.18 For the cycle \( C_{2n+1} (n \geq 1) \), \( dg(C_{2n+1}) = 2n + 1 \).

Proof. Let \( v \) be a vertex of \( C_{2n+1} \) and \( u \) an eccentric vertex of \( v \). It is clear that the pair \( u, v \) of vertices lie only on the interval \( I[u, v] \) and so the vertex \( v \) is weak extreme. Hence it follows from Proposition 2.16 that the set of all vertices of \( C_{2n+1} \) is the unique double geodetic set of \( C_{2n+1} \). Thus \( dg(C_{2n+1}) = 2n + 1 \).

Theorem 2.19 For the wheel \( W_n (n \geq 5) \), \( dg(W_n) = \begin{cases} 2 & \text{if } n = 5, \\ n - 1 & \text{if } n \geq 6. \end{cases} \)

Proof. Since \( g(W_5) = 2 \), it follows from Proposition 2.12 that \( dg(W_5) = 2 \). Let
\[ W_n = K_1 + C_{n-1} \ (n \geq 6) \] with \( x \) the vertex of \( K_1 \). Let \( v \) be any vertex of \( C_{n-1} \). First we prove that \( v \) is a weak extreme vertex of \( W_n \). Let \( v' \) be an eccentric vertex of \( v \) in \( W_n \). Then \( v \neq x \) and \( v, v' \) lie only on \( I[v, v'] \) so that \( v \) is a weak extreme vertex of \( W_n \). Hence it follows from Proposition 2.16 that \( dg(W_n) \geq n - 1 \). It is clear that the set of all vertices of \( C_{n-1} \) is a double geodetic set of \( W_n \) and so \( dg(W_n) = n - 1 \). 

**Theorem 2.20**  
For the complete bipartite graph \( G = K_{m,n} \) \((m, n \geq 2)\), \( dg(G) = \min\{m, n\} \).

**Proof.** Let \( X \) and \( Y \) be the partite sets of \( K_{m,n} \). Let \( S \) be a double geodetic set of \( K_{m,n} \). We claim that \( X \subseteq S \) or \( Y \subseteq S \). Otherwise, there exist vertices \( x, y \) such that \( x \in X, y \in Y \) and \( x, y \not\in S \). Now, since the pair of vertices \( x, y \) lie only on the intervals \( I[x,y], I[x,t] \) and \( I[s,y] \) for some \( t \in X \) and \( s \in Y \), it follows that \( x \in S \) or \( y \in S \), which is a contradiction to \( x, y \not\in S \). Thus \( X \subseteq S \) or \( Y \subseteq S \). Also it is clear that both \( X \) and \( Y \) are double geodetic sets of \( K_{m,n} \) and so the result follows.

**Theorem 2.21**  
Let \( G \) be a connected graph of order \( n \geq 2 \). If \( G \) has exactly one vertex \( v \) of degree \( n - 1 \), then \( dg(G) \leq n - 1 \).

**Proof.** Let \( v \) be the vertex of degree \( n - 1 \). Let \( S = V(G) - \{v\} \). We claim that \( S \) is double geodetic set of \( G \). Let \( x, y \in V(G) \). If \( x, y \in S \), then it is clear that \( x, y \in I[x,y] \). So assume that \( x \in S \) and \( y = v \). Since \( v \) is the only vertex of degree \( n - 1 \), there exists a vertex \( x' \neq v \) which is non adjacent to \( x \). Hence \( x, y \in I[x,x'] \) and it follows that \( S \) is a double geodetic set of \( G \) and so \( dg(G) \leq |S| = n - 1 \).

**Remark 2.22**  
For the graph given in Figure 2.1 \( dg(G) = 4 \) so that the bound in Theorem 2.21 is sharp. For the wheel \( W_5, dg(W_5) = 2 \) (see Theorem 2.19) and so the inequality in Theorem 2.21 can also be strict.
The following theorem gives a necessary condition for a graph $G$ of order $n$ to have $dg(G) = n - 1$.

**Theorem 2.23**  Let $G$ be a connected graph such that $G$ has a full degree vertex $v$ and $G - v$ has radius at least 3. Then $dg(G) = n - 1$.

**Proof.** Let $u \neq v$ be a vertex of $G$ and $u'$ an eccentric vertex of $u$. Since $G - v$ has radius at least 3, we have $d(u, u') \geq 3$. Now, since $diam G = 2$, it follows that $P : u, v, u'$ is the unique $u-u'$ geodesic containing the vertices $u$ and $u'$ in $G$. Hence $u$ is a weak extreme vertex of $G$. Thus, all the vertices of $G$ except $v$ are weak extreme vertices. Since $G - v$ has radius at least 3, it follows that $v$ is not a weak extreme vertex of $G$. Now, $V(G) - \{v\}$ is a double geodetic set of $G$ and hence by Proposition 2.16, $dg(G) = n - 1$.

We leave the following problem as an open question.

**Problem 2.24**  Characterize graphs $G$ of order $n$ for which

(i) $dg(G) = n - 1$,

(ii) $dg(G) = n$.

**The Double Geodetic Number and Diameter of a Graph**

If $G$ is connected graph of order $n$ and diameter $d$, it is proved in [5] that $g(G) \leq n - d + 1$. However, the same is not true for the double geodetic number of a graph. For the graph $G$ given in Figure 2.4, $n = 6, d = 3$ and $dg(G) = 4$ so that $dg(G) = n - d + 1$. Similarly, for the graph $G$ given in Figure 2.5, $dg(G) = 5$ so that $dg(G) > n - d + 1$ and for the graph $G$ given in Figure 2.6, $dg(G) = 3$ so that $dg(G) < n - d + 1$. 

21
A caterpillar is a tree, the removal of whose end-vertices leaves a path.

**Theorem 2.25** For every nontrivial tree $T$ of order $n$ with diameter $d$, $dg(G) = n - d + 1$ if and only if $T$ is a caterpillar.

**Proof.** Let $T$ be any nontrivial tree. Let $d = d(u, v)$ and let $P : u = v_0, v_1, v_2, \ldots, v_{d-1}, v_d = v$ be a diametral path. Let $k$ be the number of end vertices of $T$ and $l$ the number of internal vertices of $T$ other than $v_1, v_2, \ldots, v_{d-1}$. Then $d - 1 + l + k = n$. 

22
By Theorem 2.6, \( dg(T) = k = n - d + 1 - l \). Hence \( dg(T) = n - d + 1 \) if and only if \( l = 0 \), if and only if all the internal vertices of \( T \) lie on the diametrical path \( P \), if and only if \( T \) is caterpillar.

**Corollary 2.26** For a non trivial tree \( T \) of order \( n \) with diameter \( d \), the following are equivalent:

(i) \( g(T) = n - d + 1 \),

(ii) \( dg(T) = n - d + 1 \),

(iii) \( T \) is a caterpillar.

**Proof.** This follows from Theorem 2.25 and the fact that \( g(T) = dg(T) \) for any tree \( T \).

For every connected graph \( G \), \( rad G \leq diam G \leq 2 rad G \). Ostrand [16] showed that every two positive integers \( a \) and \( b \) with \( a \leq b \leq 2a \) are realizable as the radius and diameter respectively, of some connected graph. Now, Ostrand’s theorem can be extended so that the geodetic number and double geodetic number can also be prescribed, when \( r < d \leq 2r \).

**Theorem 2.27** For positive integers \( r, d, a \) and \( b \) such that \( r < d \leq 2r \) and \( 3 \leq a \leq b \), there exists a connected graph \( G \) with \( rad G = r \), \( diam G = d \), \( g(G) = a \) and \( dg(G) = b \).

**Proof.** Case 1. \( r = 1 \). Then \( d = 2 \). Construct a graph \( G \) as follows: Let \( P_3 = u_1, u_2, u_3 \) be a path of order 3. Add \( a - 2 \) new vertices \( v_1, v_2, \ldots, v_{a-2} \) to \( P_2 \) and join each \( v_i \) (\( 1 \leq i \leq a - 2 \)) to the vertex \( u_2 \) and obtain the graph \( H \). Also, add \( (b-a) \) new vertices \( w_1, w_2, \ldots, w_{b-a} \) to \( H \) and join each \( w_i \) (\( 1 \leq i \leq b-a \)) to \( u_1, u_2 \) and \( u_3 \) and obtain the graph \( G \) in Figure 2.7. Then \( G \) has radius 1 and diameter 2. It is clear that \( S_1 = \{v_1, v_2, \ldots, v_{a-2}, u_1, u_3\} \) is a minimum geodetic set of \( G \) and so by Theorem 1.15, \( g(G) = a \). It is clear that \( S_2 = \{v_1, v_2, \ldots, v_{a-2}, u_1, u_3, w_1, w_2, \ldots, w_{b-a}\} \) is the
set of all weak extreme vertices of $G$ and since $S_2$ is a double geodetic set of $G$, it follows from Proposition 2.16 that $dg(G) = b$.

\[\text{Figure 2.7}\]

Case 2. $r \geq 2$. Construct a graph $G$ as follows: Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let $H$ be the graph obtained from $C_{2r}$ and $P_{d-r+1}$ by identifying $v_1$ in $C_{2r}$ and $u_0$ in $P_{d-r+1}$. First, assume that $b > a$. Add $a - 2$ new vertices $w_1, w_2, \ldots, w_{a-2}$ to $H$ and join each vertex $w_i$ ($1 \leq i \leq a - 2$) to the vertex $v_r$ and add $b - a - 1$ new vertices $x_1, x_2, \ldots, x_{b-a-1}$ to $H$ and join each $x_i$ ($1 \leq i \leq b - a - 1$) to both $v_{r+1}$ and $v_{r-1}$ and obtain the graph $G$ of Figure 2.8. Then $G$ has radius $r$ and diameter $d$. Let $S = \{w_1, w_2, \ldots, w_{a-2}, u_{d-r}\}$ be the set of all extreme vertices of $G$. Since the vertices $v_i$ ($r + 1 \leq i \leq 2r$) and $x_i$ ($1 \leq i \leq b - a - 1$) do not lie on a geodesic joining any pair of vertices $S$, we see that $S$ is not a geodetic set of $G$. Since $T = S \cup \{v_{r+1}\}$ is a geodetic set of $G$, it follows from Theorem 1.15 that $g(G) = a$. It is clear that the vertex $x_i$ ($1 \leq i \leq b - a - 1$) is either an initial vertex or a terminal vertex of any geodesic containing the vertices $v_i$ and $w_1$ and so each $x_i$ is weak extreme. Similarly, $v_{r+1}$ is either an initial vertex or a terminal vertex of any geodesic containing the vertices $v_{r+1}$ and $u_1$. Similarly, $v_{2r}$ is either an initial vertex or a terminal vertex of
any geodesic containing the vertices \( v_2r \) and \( w_1 \). Hence \( x_1, x_2, \ldots, x_{b-a-1}, v_{r+1}, v_{2r} \) are weak extreme vertices. Let \( S' = \{ w_1, w_2, \ldots, w_{a-2}, u_{d-r}, x_1, x_2, \ldots, x_{b-a-1}, v_{r+1}, v_{2r} \} \).

It is easily verified that \( S' \) is the set of all weak extreme vertices of \( G \). Since \( S' \) is a double geodetic set of \( G \), it follows from Proposition 2.16 that \( dg(G) = b \).

Next, assume that \( a = b \). Add \( (a - 1) \) new vertices \( y_1, y_2, \ldots, y_{a-1} \) to \( H \) and join each \( y_i (1 \leq i \leq a-1) \) to both \( v_r \) and \( v_{r+2} \) in \( H \), and obtain a graph \( G' \). Then \( G' \) has radius \( r \) and diameter \( d \). Let \( S_1 = \{ y_1, y_2, \ldots, y_{a-1}, u_{d-r} \} \) be the set of all weak extreme vertices of \( G' \). Since \( S_1 \) is a geodetic set as well as a double geodetic set of \( G' \), it follows from Theorem 1.15 and Proposition 2.16 that \( g(G') = dg(G') = a \). Thus the proof is complete.

**Theorem 2.28** If \( n, d \geq 2 \) and \( l \) are integers such that \( 3 \leq k \leq l \leq n \) and \( n - d - l + 1 \geq 0 \), then there exists a graph \( G \) of order \( n \) diameter \( d \) with \( g(G) = k \) and \( dg(G) = l \).

**Proof.** Let \( P_{d+1} = u_0, u_1, \ldots, u_d \) be a path of length \( d \). Add \( k - 2 \) new vertices \( v_1, v_2, \ldots, v_{k-2} \) to \( P_d \) and join each \( v_i \) \((1 \leq i \leq k-2)\) to \( u_1 \), thereby producing a tree.
Let $H$ be the graph obtained from $T$ by adding $(l - k)$ new vertices $w_1, w_2, \ldots, w_{l-k}$ to $T$ and joining each $w_i$ ($1 \leq i \leq l-k$) to both $u_0$ and $u_2$. Now, let $G$ be the graph in Figure 2.9 obtained from $H$ by adding $n - d - l + 1$ new vertices $x_1, x_2, \ldots, x_{n-d-l+1}$ to $H$ and joining each $x_i$ ($1 \leq i \leq n - d - l + 1$) to both $u_1$ and $u_3$. Then $G$ has order $n$ and diameter $d$. Let $S = \{u_d, v_1, v_2, \ldots, v_{k-2}\}$ be the set of all extreme vertices of $G$. It is not a geodetic set of $G$. Since $S' = S \cup \{u_0\}$ is a geodetic set of $G$ it follows from Theorem 1.15 that $g(G) = k$. Now, it is clear that each vertex $w_i$ ($1 \leq i \leq l-k$) is an end of any geodesic containing the vertex $w_i$ and $v_1$ and so each $w_i$ is weak extreme. Similarly, $u_0$ is an end of any geodesic containing the vertices $u_0$ and $u_d$. Now, it is easily verified that $S_1 = \{u_d, v_1, v_2, \ldots, v_{k-2}, w_1, w_2, \ldots, w_{l-k}, u_0\}$ is the set of all weak extreme vertices of $G$. Since $S_1$ is double geodetic set of $G$, it follows from Proposition 2.16 that $dg(G) = l$. 

![Figure 2.9](image-url)
The Upper Double Geodetic Number of a Graph

Definition 2.29  A double geodetic set in a connected graph $G$ is called a minimal double geodetic set if no proper subset of $S$ is a double geodetic set of $G$. The upper double geodetic number $dg^+(G)$ of $G$ is the maximum cardinality of a minimal double geodetic set of $G$.

Example 2.30  For the graph $G$ in Figure 2.10, $S = \{v_2, v_4\}$ is a double geodetic set of $G$ so that $dg(G) = 2$. The set $S' = \{v_1, v_3, v_5\}$ is a double geodetic set of $G$ and it is clear that no proper subset of $S'$ is a double geodetic set of $G$ and so $S'$ is a minimal double geodetic set of $G$. It is easily verified that no 4-element subset is a minimal double geodetic set and so $dg^+(G) = 3$.

![Figure 2.10](image)

Remark 2.31  Every minimum double geodetic set of $G$ is a minimal double geodetic set of $G$ and the converse is not true. For the graph $G$ given in Figure 2.10, $S' = \{v_1, v_3, v_5\}$ is a minimal double geodetic set but not a minimum double geodetic set of $G$.

Theorem 2.32  For a connected graph $G$ of order $n$, $2 \leq dg(G) \leq dg^+(G) \leq n$.

Proof. Any double geodetic set needs at least two vertices and so $dg(G) \geq 2$. Since
every minimal double geodetic set is double geodetic set, \( dg(G) \leq dg^+(G) \). Thus

\[ 2 \leq dg(G) \leq dg^+(G) \leq n. \]

**Remark 2.33** The bounds in Theorem 2.32 are sharp. For any non-trivial path \( P, dg(P) = 2 \). It follows from Proposition 2.16 that \( dg(T) = dg^+(T) \) for any tree \( T \) and \( dg^+(K_n) = n, n(\geq 2) \). Also, all the inequalities in the theorem are strict. For the complete bipartite graph \( = K_{r,s} (3 \leq r < s) \), \( dg(G) = r, dg^+(G) = s \) and \( n = r + s \). (See Theorems 2.20 and 2.42 (iii))

**Theorem 2.34** For a connected graph \( G \), \( dg(G) = n \) if and only if \( dg^+(G) = n \).

**Proof.** Let \( dg^+(G) = n \). Then the vertex set \( V \) is the unique minimal double geodetic set of \( G \). Since no proper subset of \( V \) is an double geodetic set, it is clear that \( V \) is also the unique minimum double geodetic set of \( G \) and so \( dg(G) = n \). The converse follows from Theorem 2.32. \[ \]

For the complete graph \( G = K_n \), it is clear that \( dg(G) = n \). Hence we have the following corollary.

**Corollary 2.35** For the complete graph \( G = K_n \) \((n \geq 2)\), \( dg^+(G) = n \).

However, a non-complete graph \( G \) of order \( n \) can have \( dg(G) = dg^+(G) = n \). For the graph \( G \) given in Figure 2.2, all the vertices are weak extreme and so it follows from Proposition 2.16 that \( dg(G) = dg^+(G) = 6 \).

**Theorem 2.36** If \( G \) is a connected graph of order \( n \) with \( dg(G) = n - 1 \), then \( dg^+(G) = n - 1 \).

**Proof.** Since \( dg(G) = n - 1 \), it follows from Theorem 2.32 that \( dg^+(G) = n \) or \( dg^+(G) = n - 1 \). It follows from Theorem 2.34 that \( dg^+(G) = n - 1 \). \[ \]
A vertex in a graph $G$ of order $n$ is called a full degree vertex if its degree is $n - 1$.

**Theorem 2.37**  
Let $G$ be a non-complete connected graph. Then a full degree vertex does not belong to any minimal double geodetic set of $G$.

**Proof.** Let $S$ be a minimal double geodetic set of $G$ containing a full degree vertex $v_0$. Let $S' = S - \{v_0\}$. We claim that $S'$ is a double geodetic set of $G$. Let $u, v \in V$.

**Case 1.** $u, v \in S$.

If $v_0 \neq u, v$, then $u, v \in S'$ and so $S'$ is a double geodetic set of $G$. So assume that $u = v_0$. If $v$ is not a full degree vertex, then there exists $v' \neq v$ such that $v$ and $v'$ are non-adjacent and so $u, v \in I[v, v']$ with $v, v' \in S'$. Now, if $v$ is a full degree vertex, then since the subgraph induce by $S$ is not complete, there exist non-adjacent vertices $v', v''$ in $S$ such that $u, v \in I[v', v'']$. Thus $S'$ is a double geodetic set of $G$, which is a contradiction to $S$ a minimal double geodetic set.

**Case 2.** $u \notin S$ or $v \notin S$.

Since $S$ is a double geodetic set, there exist $x, y \in S$ such that $u, v \in I[x, y]$. Since $v_0$ is a full degree vertex, it follows that $x \neq v_0$ and $y \neq v_0$. Thus $x, y \in S'$ and so $S'$ is a double geodetic set of $G$, which is again contradiction to $S$ a minimal double geodetic set of $G$. Thus the proof is complete.

**Theorem 2.38**  
Let $G$ be a non-complete graph of order $n$ with a full degree vertex $v$. Then $dg^+(G) = n - 1$ if and only if $dg(G) = n - 1$.

**Proof.** Let $dg(G) = n - 1$. Then by Theorem 2.36, $dg^+(G) = n - 1$. Let $dg^+(G) = n - 1$. Let $S$ be a minimal double geodetic set of cardinality $n - 1$. By Theorem 2.37, $v \notin S$. Suppose that $dg(G) \leq n - 2$. Let $S'$ be a minimum double geodetic set of $G$. Then it follows from Theorem 2.37 that $v \notin S'$ and $S' \subseteq S$, which is a contradiction to $S$ a minimal double geodetic set of $G$. Hence $dg(G) = n - 1$. 

29
Theorem 2.39  Let $G$ be a connected graph with a cutvertex $v$. Then every minimal double geodetic set of $G$ contains at least one vertex from each component of $G - v$.

Proof. This follows from Theorem 2.8.

Theorem 2.40  No cutvertex of a connected graph $G$ belongs to any minimal double geodetic set of $G$.

Proof. The proof is similar to that of Theorem 2.9.

Theorem 2.41  For any tree $T$ with $k$ end-vertices $dg(T) = dg^+(T) = k$.

Proof. This follows from Corollary 2.10 and Theorem 2.40.

Theorem 2.42  For the complete bipartite graph $G = K_{m,n}$,

(i) $dg^+(G) = 2$ if $m = n = 1$.

(ii) $dg^+(G) = n$ if $m = 1, n \geq 2$.

(iii) $dg^+(G) = \max\{m, n\}$ if $m, n \geq 2$.

Proof. (i) and (ii) follow from Theorem 2.41. (iii) The proof is exactly similar to that of Theorem 2.20. Since the upper double geodetic number $dg^+(G)$ is the maximum cardinality of a minimal double geodetic set, it follows that $dg^+(G) = \max\{m, n\}$.

Theorem 2.43  For any positive integers $2 \leq a \leq b$, there exists a connected graph $G$ such that $dg(G) = a$ and $dg^+(G) = b$.

Proof. If $a = b$, let $G = K_{1,a}$. By Theorem 2.41, $dg(G) = dg^+(G) = a$. If $a < b$, let $G = K_{a,b}$. It follows from Theorems 2.20 and 2.42 that $dg(G) = a$ and $dg^+(G) = b$.
The forcing double geodetic number of a graph

**Definition 2.44** Let $G$ be a connected graph and $S$ a minimum double geodetic set of $G$. A subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique minimum double geodetic set containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing double geodetic number of $S$, denoted by $f_{dg}(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing double geodetic number of $G$, denoted by $f_{dg}(G)$, is $f_{dg}(G) = \min\{f_{dg}(S)\}$, where the minimum is taken over all minimum double geodetic sets $S$ in $G$.

**Example 2.45** For the graph $G$ given in Figure 2.11, $S_1 = \{v_3, v_6\}$ is the unique minimum double geodetic set of $G$ so that $f_{dg}(G) = 0$ and for the graph $G$ given in Figure 2.12, $S_1 = \{v_2, v_3, v_4, v_5\}$ and $S_2 = \{v_1, v_3, v_5, v_6\}$ are the only two minimum double geodetic sets of $G$. It is clear that $f_{dg}(S_1) = f_{dg}(S_2) = 1$ and so $f_{dg}(G) = 1$.

![Figure 2.11](image)
The next theorem follows immediately from the definition of the double geodetic number and the forcing double geodetic number of a connected graph $G$.

**Theorem 2.46** For every connected graph $G$ of order $n$, $0 \leq f_{dg}(G) \leq dg(G) \leq n$.

**Remark 2.47** The bounds in Theorem 2.46 are strict. For the graph $G$ given in Figure 2.11, $f_{dg}(G) = 0$. For the graph $G$ given in Figure 2.12, $V(G) = 6, dg(G) = 4$ and $f_{dg}(G) = 1$. Thus $0 < f_{dg}(G) < dg(G) < n$.

**Theorem 2.48** Let $G$ be a connected graph. Then

(i) $f_{dg}(G) = 0$ if and only if $G$ has a unique minimum double geodetic set.

(ii) $f_{dg}(G) = 1$ if and only if $G$ has at least two minimum double geodetic sets, one of which is the unique minimum double geodetic set containing one of its elements.

(iii) $f_{dg}(G) = dg(G)$ if and only if no minimum double geodetic set of $G$ is the unique minimum double geodetic set containing any of its proper subsets.

**Proof.** (i) Let $f_{dg}(G) = 0$. Then by definition, $f_{dg}(S) = 0$ for some double geodetic set $S$ of $G$ so that the empty set $\emptyset$ is the minimum forcing subset for $S$. Since the empty set $\emptyset$ is a subset of every set, it follows that $S$ is the unique minimum double geodetic set of $G$. The converse is clear.
(ii) Let $f_{dg}(G) = 1$. Then by Theorem 2.48 (i), $G$ has at least two double geodetic sets. Also, since $f_{dg}(G) = 1$, there is a singleton subset $T$ of an double geodetic set $S$ of $G$ such that $T$ is not a subset of any other double geodetic set of $G$. Thus $S$ is the unique double geodetic set containing one of its elements. The converse is clear.

(iii) Let $f_{dg}(G) = dg(G)$. Then $f_{dg}(S) = dg(G)$ for every double geodetic set $S$ in $G$. Also, $dg(G) \geq 2$ and hence $f_{dg}(G) \geq 2$. Then by Theorem 2.48(i), $G$ has at least two double geodetic sets and so the empty set $\emptyset$ is not a forcing subset for double geodetic set of $G$. Since $f_{dg}(G) = dg(G)$, no proper subset of $S$ is a forcing subset of $S$. Thus no double geodetic set of $G$ is the unique double geodetic set containing any of its proper subsets. Conversely, the data implies that $G$ contains more than one double geodetic sets and no subset of any minimum double geodetic set $S$ other than $S$ is a forcing subset for $S$. Hence it follows that $f_{dg}(G) = dg(G)$. }

\textbf{Definition 2.49} A vertex $v$ of a connected graph $G$ is said to be an \textit{double geodetic vertex} of $G$ if $v$ belongs to every minimum double geodetic set of $G$.

\textbf{Example 2.50} For the graph $G$ given in Figure 2.12, $S_1 = \{v_2, v_3, v_4, v_5\}$ and $S_2 = \{v_1, v_3, v_5, v_6\}$ are the only $dg$-sets of $G$. It is clear that $v_3$ and $v_5$ are the double geodetic vertices of $G$.

\textbf{Theorem 2.51} Let $G$ be a connected graph and let $\mathcal{J}$ be the set of relative complements of the minimum forcing subsets in their respective minimum double geodetic sets in $G$. Then $\cap_{F \in \mathcal{J}} F$ is the set of double geodetic vertices of $G$.

\textbf{Proof}. Let $W$ be the set of all double geodetic vertices of $G$. We are to show that $W = \cap_{F \in \mathcal{J}} F$. Let $v \in W$. Then $v$ is an double geodetic vertex of $G$ that belongs to every minimum double geodetic set $S$ of $G$. Let $T \subseteq S$ be any minimum forcing subset for any minimum double geodetic set $S$ of $G$. We claim that $v \notin T$, then $T' = T - \{v\}$ is a proper subset of $T$ such that $S$ is the unique minimum double geodetic set containing $T'$ so that $T'$ is a forcing subset for $S$ with $|T'| < |T|$, which
is a contradiction to $T$ is a minimum forcing subset for $S$. Thus $v \notin T$ and so $v \in F$, where $F$ is the relative complement of $T$ in $S$. Hence $v \in \cap_{F \in \mathcal{J}} F$ so that $W \subseteq \cap_{F \in \mathcal{J}} F$.

Conversely, let $v \in \cap_{F \in \mathcal{J}} F$. Then $v$ belongs to the relative complement of $T$ in $S$ for every $T$ and every $S$ such that $T \subseteq S$, where $T$ is a minimum forcing subset for $S$. Since $F$ is the relative complement of $T$ in $S$, we have $F \subseteq S$ and thus $v \in S$ for every $S$, which implies that $v$ is an double geodetic vertex of $G$. Thus $v \in W$ and so $\cap_{F \in \mathcal{J}} F \subseteq W$. Hence $W = \cap_{F \in \mathcal{J}} F$.

**Corollary 2.52** Let $G$ be a connected graph and $S$ an minimum double geodetic set of $G$. Then no double geodetic vertex of $G$ belongs to any minimum forcing set of $S$.

**Proof.** The proof is contained in the proof of the first part of Theorem 2.51.

**Theorem 2.53** Let $G$ be a connected graph and $W$ be the set of all double geodetic vertices of $G$. Then $f_{dg}(G) \leq dg(G) - W$.

**Proof.** Let $S$ be any minimum double geodetic set of $G$. Then $dg(G) = |S|$, $W \subseteq S$ and $S$ is the unique minimum double geodetic set containing $S - W$. Thus $f_{dg}(G) \leq |S - W| = |S| - |W| = dg(G) - W$.

**Corollary 2.54** If $G$ is a connected graph with $k$ weak extreme vertices, then $f_{dg}(G) \leq dg(G) - k$.

**Proof.** This follows from Proposition 2.16 and Theorem 2.53.

**Theorem 2.55** For any even cycle $G = C_n(n \geq 4)$, a set $S \subseteq V(G)$ is an minimum double geodetic set if and only if $S$ consists of two antipodal vertices.

**Proof.** If $S$ consists of two antipodal vertices, then it is clear that $S$ is an minimum double geodetic set of $C_n$. Conversely, let $S$ be any minimum double geodetic set of
$C_n$. Then $dg(C_n) = |S|$. Let $S'$ be any set of two antipodal vertices of $C_n$. Then as in the first part of this Theorem, $S'$ is an minimum double geodetic set of $C_n$. Hence $|S'| = |S|$. Thus $S$ consists of two vertices, say $S = \{u, v\}$. If $u$ and $v$ are not antipodal, then any part of vertices that is not on the $u$-$v$ geodesic. Thus $S$ is not an minimum double geodetic set, which is contradiction. □

**Theorem 2.56** For any cycle $C_n (n \geq 4)$, $f_{dg}(C_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$

**Proof.** If $n$ is even, then by Theorem 2.55, every $dg$-set of $C_n$ consists of pair of antipodal vertices. Hence $C_n$ has $\frac{n}{2}$ $dg$-sets and it is clear that each singleton set is the minimum forcing set for exactly on $dg$-set of $C_n$. Hence it follows from Theorem 2.48 (ii) that $f_{dg}(C_n) = 1$.

Let $n$ be odd. Let the cycle $C : v_1, v_2, \ldots, v_{2n+1}, v_1$. By Theorem 2.18 $S = \{v_1, v_2, \ldots, v_{2n+1}\}$ is the unique minimum double geodetic set of $C_n$. Now, it follows from Theorem 2.48 (i) that $f_{dg}(C_n) = 0$. □

**Theorem 2.57** For any complete graph $G = K_n (n \geq 2)$ or any non-trivial tree $G = T, f_{dg}(G) = 0$.

**Proof.** For $G = K_n$, it follows from Proposition 2.16 that the set of all vertices of $G$ is the unique minimum double geodetic set. Now, it follows from Theorem 2.48 (i) that $f_{dg}(G) = 0$. If $G$ is a non-trivial tree, then by Corollary 2.10, set of all end-vertices of $G$ is the unique minimum double geodetic set of $G$ and so $f_{dg}(G) = 0$ by Theorem 2.48 (i). □

**Theorem 2.58** For the complete graph $G = K_{m,n} (m, n \geq 2)$,

$$f_{dg}(G) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$
Proof. Without loss of generality $m < n$. Let $X = \{u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$ be a partition of $K_{m,n}$. Let $S$ be a double geodetic set of $K_{m,n}$. We claim that either $X \subseteq S$ or $Y \subseteq S$. If not, there exist $x \in X$ and $y \in Y$ such that $x, y \not\in S$. Now, since the pair of vertices $x, y$ lies only on the $x$-$y$ geodsic, $x$-$t$ geodesic with $t \in X$ such that $t \neq x$ and $s$-$y$ geodesic with $s \in Y$ such that $s \neq y$, it follows that either $x \in S$ or $y \in S$, which is a contradiction. Thus either $X \subseteq S$ or $Y \subseteq S$. Also, it is clear that both $X$ and $Y$ are double geodetic sets of $K_{m,n}$. It follows that $X$ is the unique minimum double geodetic set of $G$. Hence it follows from Theorem 2.48 (i) that $f_{dg}(G) = 0$.

Now, let $m = n$. Then as in the first part of this Theorem, both $X$ and $Y$ are the double geodetic sets of $K_{m,n}$. Hence it follows from Theorem 2.48 (ii) that $f_{dg}(G) = 1$. \qed