Preface

This thesis embodies the work done by the author under the guidance of Dr. A. P. Santhakumaran, Professor and Head, Department of Mathematics, St. Xavier’s College (Autonomous), Palayamkottai.

The thesis consists of seven chapters.
1. Preliminaries
2. The double geodetic number of a graph
3. The connected and total double geodetic number of a graph
4. The linear geodetic number of a graph
5. The linear edge geodetic number of a graph
6. The detour and $m$-detour hull numbers of a graph
7. The vertex geodetic number of a graph.

One concept that pervades all of graph theory is that of distance and distance is used in isomorphism testing, graph operations, hamiltonicity problems, extremal problems on connectivity and diameter, and convexity in graphs. The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is defined to be the length of a shortest $u$-$v$ path in $G$. This gives rise to the concepts of convex set, geodetic set, geodetic number, hull set and hull number of a graph [1, 2, 3, 4, 5, 10, 12, 15]. A
monophonic $u-v$ path is a $u-v$ chordless path and this gives rise to the concepts of monophonic convex set, monophonic set, monophonic number, monophonic hull set and monophonic hull number of a graph [13, 17]. The detour distance $D(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is defined to be the length of a longest $u-v$ path in $G$ and this gives rise to the concepts of detour set, detour number, detour hull number, $m$-detour set and $m$-detour number of a graph [6, 29, 30].

In this thesis, we define and develop various concepts like the double geodetic number, upper double geodetic number, the forcing double geodetic number, the connected double geodetic number, the upper connected double geodetic number, the total double geodetic number, the forcing connected double geodetic number. Further, we introduce and investigate the linear geodetic number, linear edge geodetic graphs and correspondingly, the linear edge geodetic number, the detour hull number and the $m$-detour hull numbers of a graph. Finally, we introduce and investigate the vertex geodetic number of a graph. Geodetic and monophonic concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts to the problem of designing the routes for shuttles in a city network [15]. The detour concepts and colorings are widely used in the Channel Assignment Problems in radio technologies and also in special situations of molecular problems in theoretical Chemistry [11].

In Chapter 1, we collect the basic definitions and theorems, which are needed for the subsequent chapters. By a graph, we mean a finite undirected connected graph without loops or multiple edges. For basic definitions and graph theoretic
terminologies, we refer to [1, 9, 14]. Let $G = (V, E)$ be a connected graph with at least two vertices. The order and size of $G$ are denoted by $n$ and $m$ respectively. The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u$ - $v$ path in $G$. An $u$ - $v$ path of length $d(u, v)$ is called an $u$ - $v$ geodesic. The set $I[u, v]$ consists of all vertices lying on some $u$ - $v$ geodesic of $G$, while for $S \subseteq V$, $I[S] = \bigcup_{u,v \in S} I[u, v]$. The set $S$ is convex if $I[S] = S$. The convex hull $[S]$ is the smallest convex set containing $S$. A set $S \subseteq V$ is a hull set of $G$ if $[S] = V$, and a hull set of minimum cardinality is a minimum hull set of $G$. The cardinality of a minimum hull set of $G$ is the hull number $h(G)$ of $G$. A set $S \subseteq V$ is a geodetic set if $I[S] = V$, and a geodetic set of minimum cardinality is a minimum geodetic set of $G$. The cardinality of a minimum geodetic set of $G$ is the geodetic number $g(G)$ of $G$. A connected geodetic set of $G$ is a geodetic set $S$ such that the subgraph $< S >$ induced by $S$ is connected. The minimum cardinality of a connected geodetic set of $G$ is the connected geodetic number of $G$ and is denoted by $g_c(G)$. A connected geodetic set of cardinality $g_c(G)$ is called a $g_c$-set of $G$. These concepts were studied in [2, 3, 4, 12, 20]. Let $S$ be a minimum geodetic set of $G$. A subset $T$ of $S$ is called a forcing subset for $S$ if $S$ is the unique minimum geodetic set containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing geodetic number of $S$, denoted by $f(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing geodetic number of $G$, denoted by $f(G)$, is $f(G) = \{\min f(S)\}$, where the minimum is taken over all minimum geodetic sets $S$ in $G$. The forcing number of a graph was introduced and studied in [7, 21]. For any vertex $v$ in $G$, $N(v)$ denotes the set of all vertices which are adjacent to $v$. A set
$S$ of vertices is an *edge geodetic set* of a graph $G$ if each edge of $G$ lies on a geodesic of vertices in $S$, and the minimum cardinality of an edge geodetic set is the *edge geodetic number* $eg(G)$ of $G$. An edge geodetic set of cardinality $eg(G)$ is called an *$eg$-set* of $G$. The edge geodetic number of a graph was introduced and studied in [18]. Although the edge geodetic number is greater than or equal to the geodetic number for an arbitrary graph, the properties of the edge geodetic sets and results regarding edge geodetic number are quite different from that of geodetic concepts. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design. In the case of designing the route for a shuttle, although all the vertices are covered by the shuttle when considering geodetic sets, some of the edges may be left out. This drawback is rectified in the case of edge geodetic sets and hence considering edge geodetic sets is more advantageous to the real life application of routing problem. In particular, the edge geodetic sets are more useful than geodetic sets in the case of regulating and routing the goods vehicles to transport the commodities to important places [18]. This leads us to investigate further results on the edge geodetic number of a graph.

For vertices $x$ and $y$ in a nontrivial connected graph $G$ of order $n$, the *detour distance* $D(x, y)$ is the length of a longest $x – y$ path in $G$. An $x – y$ path of length $D(x, y)$ is an $x – y$ detour. It is known that the detour distance is a metric on the vertex set $V$ of $G$. The detour distance of a graph was studied in [8]. The interval $I_D[x, y]$ consists of $x, y$, and all vertices lying in some $x – y$ detour of $G$; while for $S \subseteq V$, $I_D[S] = \bigcup_{x, y \in S} I_D[x, y]$. A set $S \subseteq V$ is called a *detour set* if $I_D[S] = V$. The *detour number* $dn(G)$ of $G$ is the minimum cardinality of a detour set and any detour
set of cardinality \(dn(G)\) is called a \textit{minimum detour set} of \(G\). The detour number of a graph was introduced in [6] and further studied in [19].

A \textit{chord} of a path \(P : u_0, u_1, \ldots, u_n\) is an edge \(u_iu_j\), with \(j \geq i + 2\). Any chordless path connecting \(u\) and \(v\) is a \textit{u–v monophonic path} or \textit{m-path}. The \textit{monophonic interval} \(J[u, v]\) consists of all vertices lying on some \(u–v\) monophonic path of \(G\). For \(S \subseteq V\), the \textit{monophonic closure} \(J[S]\) is the set formed by the union of all monophonic intervals \(J[u, v]\) with \(u, v \in S\). A set \(S\) of vertices of \(G\) is a \textit{monophonic set} if \(J[S] = V\).

The \textit{monophonic number}, \(mn(G)\) of \(G\) is the minimum cardinality of a monophonic set in \(G\). The monophonic number of a graph was studied in [13]. The monophonic numbers of join and composition of graphs were discussed in [17].

In Chapter 2, we introduce and study the concept of double geodetic number of a graph \(G\) [23]. A set \(S\) of vertices in \(G\) is called a \textit{double geodetic set} of \(G\) if for each pair of vertices \(x, y\) in \(G\) there exist vertices \(u, v \in S\) such that \(x, y \in I[u, v]\). The \textit{double geodetic number} \(dg(G)\) is the minimum cardinality of a double geodetic set. Any double geodetic of cardinality \(dg(G)\) is called \(dg\)-set of \(G\). Certain general properties of this concept are studied. We determine bounds for it and characterize graphs which realize the bounds. It is shown that for every pair \(k, n\) of integers with \(2 \leq k \leq n\), there exists a connected graph \(G\) of order \(n\) such that \(dg(G) = k\). It is proved that if \(G\) is a connected graph of order \(n\) having a full degree vertex \(v\) and \(G − v\) has radius at least 3 then \(dg(G) = n − 1\). It is shown that for positive integers \(r, d, a\) and \(b\) such that \(r < d \leq 2r\) and \(3 \leq a \leq b\) there exists a connected graph \(G\) with \(rad \ G = r\), \(diam \ G = d\), \(g(G) = a\) and \(dg(G) = b\). Also, it is proved that for integers \(n, d \geq 2\) and \(l\) such that \(3 \leq k \leq l \leq n\) and \(n − d − l + 1 \geq 0\), there exists a
graph $G$ of order $n$ diameter $d, g(G) = k$ and $dg(G) = l$.

Also, we introduce the concept of the upper double geodetic number of a graph $G$ [24]. A double geodetic set in a connected graph $G$ is called a minimal double geodetic set if no proper subset of $S$ is a double geodetic set of $G$. The upper double geodetic number $dg^+(G)$ of $G$ is the maximum cardinality of a minimal double geodetic set of $G$. The upper double geodetic numbers of certain standard graphs are obtained. It is proved that for a connected graph $G$ of order $n, dg(G) = n$ if and only if $dg^+(G) = n$. It is also proved that $dg(G) = n - 1$ if and only if $dg^+(G) = n - 1$ for a non-complete graph $G$ of order $n$ having a vertex of degree $n - 1$. For every two positive integers $a$ and $b$, where $2 \leq a \leq b$, there exists a connected graph $G$ with $dg(G) = a$ and $dg^+(G) = b$. We introduce the notions of forcing double geodetic sets and the forcing double geodetic number of a graph. Let $S$ be a minimum double geodetic set of $G$. A subset $T$ of $S$ is called a forcing subset for $S$ if $S$ is the unique minimum double geodetic set containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing double geodetic number of $S$, denoted by $f_{dg}(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing double geodetic number of $G$, denoted by $f_{dg}(G)$, is $f_{dg}(G) = min\{f_{dg}(S)\}$, where the minimum is taken over all minimum double geodetic sets in $G$. The general properties satisfied by these forcing subsets are discussed and the forcing double geodetic numbers for certain classes of graphs are determined.

In Chapter 3, we first introduce and study the connected double geodetic number $dg_c(G)$ of a graph $G$ [27]. Let $G$ be a connected graph with at least two vertices. A connected double geodetic set of $G$ is an double geodetic set $S$ such that the subgraph
the connected double geodetic set of $G$ is connected. The minimum cardinality of a connected double geodetic set of $G$ is the connected double geodetic number of $G$ and is denoted by $dg_c(G)$. A connected double geodetic set of cardinality $dg_c(G)$ is called a $dg_c$-set of $G$. Certain general properties of this concept are studied and bounds for the connected double geodetic number are determined. We characterize graphs which realize the bounds. The connected double geodetic number of certain classes of graphs are determined. It is shown that for a connected graph $G$ of order $n$, the $dg_c(G) = n$ if and only if every vertex of $G$ either a cut vertex or a weak extreme vertex. We proved that for any integers $n, a$ and $b$ such that $2 \leq a < b \leq n$, there exists a connected graph $G$ of order $n$ such that $dg(G) = a$ and $dg_c(G) = b$. For positive integers $r, d$ and $k \geq 4$ with $r \leq d \leq 2r$ and $k - d - 1 \geq 0$, there exists a connected graph $G$ with $rad G = r$, $diam G = d$ and $dg_c(G) = k$.

Also, we introduce the concept of total double geodetic number $dg_t(G)$ of a graph $G$ [28]. A total double geodetic set of a graph $G$ is a double geodetic set $S$ such that the subgraph induced by $S$ has no isolated vertices. The minimum cardinality of a total double geodetic set of $G$ is the total double geodetic number of $G$ and is denoted by $dg_t(G)$. A total double geodetic set of cardinality $dg_t(G)$ is called a $dg_t$-set of $G$. Certain general properties of this concept are studied. We determine bounds for it and characterize graphs which realize the bounds. We prove that for a connected graph $G$ with at least two vertices, $dg_t(G) \leq 2 \cdot dg(G)$. The total double geodetic number of certain classes of graphs are determined. It is shown that for positive integers $r, d$ and $k \geq 4$ with $r \leq d \leq 2r$, there exists a connected graph $G$ with $rad G = r$, $diam G = d$ and $dg_t(G) = k$. Also, for positive integers $n, a, b$ such that
4 ≤ a ≤ b ≤ n, there exists a connected graph $G$ of order $n$, with $dg_l(G) = a$ and $dg_e(G) = b$. It is proved that for integers $a, b$ with $4 ≤ a ≤ b$ and $b ≤ 2a$, there exists a connected graph $G$ such that $dg(G) = a$ and $dg_e(G) = b$.

In Chapter 4, we introduce and study the linear geodetic number of a graph $G$ [25]. For a connected graph $G$ of order $n$, an ordered set $S = \{u_1, u_2, \ldots, u_k\}$ of vertices in $G$ is a linear geodetic set of $G$ if for each vertex $x$ in $G$, there exists an index $i, 1 ≤ i < k$ such that $x$ lies on a $u_i - u_{i+1}$ geodesic on $G$, and a linear geodetic set of minimum cardinality is the linear geodetic number $g_l(G)$. The linear geodetic numbers of certain standard graphs are obtained. It is shown that if $G$ is a graph of order $n$ and diameter $d$, then $g_l(G) ≤ n - d + 1$ and this bound is sharp. For positive integers $r, d$ and $k ≥ 2$ with $r < d ≤ 2r$, there exists a connected graph $G$ with $\text{rad } G = r$, $\text{diam } G = d$ and $g_l(G) = k$. Also, for integers $n, d$ and $k$ with $2 ≤ d < n$, $2 ≤ k ≤ n - d + 1$, there exists a connected graph $G$ of order $n$, diameter $d$ and $g_l(G) = k$. We characterize connected graphs $G$ of order $n$ with $g_l(G) = n$ and $g_l(G) = n - 1$. It is shown that for each pair $a, b$ of integers with $3 ≤ a ≤ b$, there is a connected graph $G$ with $g(G) = a$ and $g_l(G) = b$. We also discuss how the linear geodetic number of a graph is affected by adding a pendant edge to the graph. For each pair $a, b$ of integers with $4 ≤ a ≤ b + 1$, there is a connected graph $G$ with $g_l(G') = a$ and $g_l(G) = b$, where $G'$ is a graph obtained from $G$ by adding a pendant edge.

In Chapter 5, we introduce and study a new class of graphs called linear edge geodetic graphs, and correspondingly the linear edge geodetic number of such graphs [26]. For a connected graph $G$ of order $n$, an ordered set $S = \{v_1, v_2, \ldots, v_k\}$ of vertices
in $G$ is a linear edge geodetic set of $G$ if for each edge $e = xy$ in $G$, there exists an index $i, 1 \leq i < k$ such that $e$ lies on a $v_i - v_{i+1}$ geodesic in $G$, and a linear edge geodetic set of minimum cardinality is the minimum linear edge geodetic set is the linear edge geodetic number $\text{leg}(G)$ of $G$. A graph $G$ is called a linear edge geodetic graph if it has a linear edge geodetic set. The linear geodetic numbers of a certain standard graphs are obtained. It is shown that for a connected linear edge geodetic graph $G$ of order $n$, $2 \leq g(G) \leq \text{eg}(G) \leq \text{leg}(G) \leq n$ and $2 \leq g(G) \leq g_l(G) \leq \text{leg}(G) \leq n$.

For every nontrivial tree $T$ of order $n$, $\text{leg}(G) = n - d + 1$ if and only if $T$ is a caterpillar. It is shown that for a connected graph $G$, if there exist vertices $x, y$ and $z$ with $N[x] = N[y] = N[z]$, then $G$ is not a linear edge geodetic graph. For positive integers $r, d$ and $k \geq 2$ with $r < d \leq 2r$, there exists a connected graph $G$ with $\text{rad} G = r, \text{diam} G = d$ and $\text{leg}(G) = k$. It is shown that for each pair $a, b$ of integers with $3 \leq a \leq b$, there is a connected graph $G$ with $\text{eg}(G) = a$ and $\text{leg}(G) = b$. Also, it is proved that the complete bipartite graph $G = K_{r,s}$ ($2 \leq r \leq s$) is a linear edge geodetic graph.

In Chapter 6, we introduce and study the monophonic detour hull number of a graph $G$ [29]. The monophonic detour distance $d_m$ was introduced and studied in [22]. A set $S$ of vertices is a $m$-detour convex set if $I_{dm}[S] = S$. The $m$-detour convex hull $[S]_{dm}$ is the smallest detour convex set containing $S$. A set $S$ of vertices of $G$ is a $m$-detour hull set if $[S]_{dm} = V$ and a $m$-detour hull set of minimum cardinality is the $m$-detour hull number $\text{mdh}(G)$ of $G$. Some general properties satisfied by this concept are studied. For a connected graph $G$ of order $n$ with $m$-detour diameter $d_m$, it is shown that $k \leq \text{mdh}(G) \leq n - d_m + 1$, where $k$ is the number of $m$-detour
extreme vertices of $G$. Graphs $G$ for which $md_h(G) = n$, and $md_h(G) = n - 1$
are characterized. It is proved that for each pair of integers $2 \leq a \leq b$, there is a
connected graph $G$ with $md_h(G) = a$ and $md(G) = b$. Also, it is proved that for each
triple $a, b$ and $k$ of positive integers with $a < b$ and $k \geq 3$, there is a connected graph
$G$ with $rad_m(G) = a$, $diam_m(G) = b$ and $md_h(G) = k$. Finally, we investigate how
the $m$-detour hull numbers of a graph are affected by adding a pendant edge. If $G'$
is the graph obtained from $G$ by adding a pendant edge $uv$ at a vertex $v$ of $G$, then
it is proved that $md_h(G) \leq md_h(G') \leq md_h(G) + 1$.

In chapter 7, we define and develop the concept of vertex geodetic number of a
graph $G$ [31]. Let $G$ be a connected graph and $x$ a vertex in $G$. For a vertex $y$ in $G$,
$I[y]_x$ denotes the set of all vertices that lie on some $x-y$ geodesic of $G$, while $S \subseteq V(G)$,
$I[S]_x = \bigcup_{y \in S} I[y]_x$. A set of vertices $S$ is an $x$-geodetic set of $G$ if $I[S]_x = V(G)$ and
an $x$-geodetic set of minimum cardinality is a minimum $x$-geodetic set of $G$. The
cardinality of a minimum $x$-geodetic set of $G$ is the $x$-geodetic number $g_x(G)$ of $G$.
Any $x$-geodetic set of cardinality $g_x(G)$ is the $g_x$-set of $G$. Some general properties
satisfied by this concept are studied. It is proved that for any vertex $x$ in a connected
graph $G$, $g_x$-set is unique. If $T$ is a tree with number of pendent vertices $l$, then
it is proved that $g_x(T) = l - 1$ or $l$ according as $x$ is a pendent or non-pendent
vertex. For any vertex $x$ in $G$, it is shown that $g(G) \leq g_x(G) + 1$. Further, we
investigate the vertex geodetic number on Cartesian product of graphs. It is shown
that if $S$ is an $(x, y)$-geodetic set of $G\Box H$, then $\pi_G(S)$ is an $x$-geodetic set of $G$ and
$\pi_H(S)$ is an $y$-geodetic set of $H$. For connected graphs $G$ and $H$, it is shown that
$max\{g_x(G), g_y(H)\} \leq g_{(x,y)}(G\Box H)$. Also, it is proved that if $G$ and $H$ are connected
graphs such that $S \subseteq V(G)$ and $T \subseteq V(H)$, then $S$ is an $x$-geodetic set of $G$ and $T$ is a $y$-geodetic set of $H$ if and only if $S \times T$ is an $(x, y)$-geodetic set of $G \square H$. For integers $m, n \geq 2$, we prove that $g_{(x,y)}(K_m \square K_n) = (m - 1)(n - 1)$.