Chapter 3

MATCHING DOMINATION OF PRODUCT GRAPHS
In this chapter, we discuss the matching domination of three kinds of product graphs. They are:

i. Kronecker product of two graphs

ii. Cartesian product of two graphs

iii. Lexicograph product of two graphs

We first discuss the matching domination number of Kronecker product and obtain an expression for it in an elegant manner. The Kronecker product of two graphs is defined as follows.

Definition 3.1

If $G_1$, $G_2$ are two simple graphs with their vertex sets as $V_1: \{u_1, u_2, \ldots\}$ and $V_2: \{v_1, v_2, \ldots\}$ respectively then the Kronecker product of these two graphs is defined to be a graph with its vertex set as $V_1 \times V_2$, where $V_1 \times V_2$ is the cartesian product of the sets $V_1$ and $V_2$ and two vertices $(u_i, v_j), (u_k, v_l)$ are adjacent if and only if $u_i u_k$ and $v_j v_l$ are edges in $G_1$ and $G_2$ respectively. This product graph is denoted by $G_1(k) G_2$. 
An Illustration of the product graph of $G_1 \ (k) \ G_2$ is given as follows.

**Figure 3.1**
Weichsel [40] has proved that if \( G_1, G_2 \) are connected graphs then \( G_1 \times G_2 \) is connected if and only if either \( G_1 \) or \( G_2 \) contains an odd cycle. It was further proved that if \( G_1, G_2 \) are connected graphs with no odd cycle then \( G_1 \times G_2 \) is a disconnected graph.

Sampathkumar [32] has proved that, If \( G \) is connected graph with no odd cycles then \( G \times K_2 = 2G \).

It can be easily seen that, \( \deg_{G_1 \times G_2}(u, v) = \deg_{G_1}(u) \cdot \deg_{G_2}(v) \)

**Theorem 3.2**

In \( G_1 \times G_2 \) then \( \deg (u, v) = \deg (u_i) \cdot \deg (v_j) \).

**Proof:**

Suppose \( \deg (u_i) = m \) and \( \deg (v_j) = n \), i.e., \( u_i \) is adjacent with vertices \( u_1, u_2, \ldots, u_m \) in \( G_1 \)

and \( v_j \) is adjacent with vertices \( v_1, v_2, \ldots, v_n \) in \( G_2 \).

Then in the product graph \( G_1(k) G_2 \) the vertex \( (u_i, v_j) \) is adjacent with following vertices.

\[
\begin{align*}
(u_i, v_1) & (u_1, v_j) \quad (u_1, v_n) \\
(u_2, v_1) & (u_2, v_j) \quad (u_2, v_n) \\
(u_m, v_1) & (u_m, v_j) \quad (u_m, v_n)
\end{align*}
\]
Also any other vertex \((u_k, v_l)\) in \(G_1(K) G_2\) is not adjacent with 
\((u_p, v_q)\). If \(k > m\) or \(l > n\). For, \(u_i\) is not adjacent with \(u_k\) if \(k > m\) 
and \(v_j\) is not adjacent with \(v_l\) if \(l > n\).

Hence \(\deg(u_i, v_j) = \deg(u_i) \cdot \deg(v_j)\)

Now we have the following result at once.

**Theorem 3.3**

If \(G_1\) and \(G_2\) are finite graphs without isolated vertices then
\(G_1(K) G_2\) is a finite graph without isolated vertices.

**Proof:**

Since \(G_1\) and \(G_2\) are finite graphs, if follows that \(G_1(K) G_2\) is 
also a finite graph by definition 3.1 since \(G_1, G_2\) do not have 
isolated vertices.

\(\deg_{u_i}(u_i) \neq 0\) for any \(i\) and so also \(\deg_{v_j}(v_j) \neq 0\) for any \(j\).

Thus \(\deg_{G_1, G_2}(u_i, v_j) \neq 0\) for any \(i\) and \(j\) (by Theorem 3.2).

So \(G_1(K) G_2\) do not have any isolated vertices.

It can be easily seen that, the number of vertices \(G_1(K) G_2\) 
is the product of number of vertices in \(G_1\) and \(G_2\) and the number 
of edges in \(G_1(K) G_2\) is twice the product of the number of edges in 
\(G_1\) and \(G_2\).
Theorem 3.4

(i) \(|V_{G_1 \cup G_2}| = |V_{G_1}| + |V_{G_2}|\)

(ii) \(|E_{G_1 \cup G_2}| = 2 |E_{G_1}| + |E_{G_2}|\)

Proof:

It follows from the definition 3.1, \(|V_{G_1 \cup G_2}| = |V_{G_1}| + |V_{G_2}|\) we know that \(|E_{G_1}| = e_1 = \frac{1}{2} \sum_{i \in V_i} d(u_i)\)

and \(|E_{G_2}| = e_2 = \frac{1}{2} \sum_{j \in V_j} d(v_j)\)

Now \(|E_{G_1 \cup G_2}| = \frac{1}{2} \sum_{i,j} d(u_i, v_j)\)

\[= \frac{1}{2} \left\{ \sum_{i,j} d(u_i) \cdot d(v_j) \right\} \quad \text{(By theorem 3.2)} \]

\[= \frac{1}{2} \left\{ \sum_{i} d(u_i) \right\} \left\{ \sum_{j} d(v_j) \right\} \]

\[= \frac{1}{2} \left\{ 2e_1 \right\} \left\{ 2e_2 \right\} = 2 |E_{G_1}| |E_{G_2}| \]

Theorem 3.5

If \(G_1\) and \(G_2\) are regular graphs, then \(G_1 (K) G_2\) is also a regular graph.
Proof:

Suppose $G_1$ is a $k_1$-regular graph and $G_2$ is a $k_2$-regular graph then $\text{deg}(u_i) = k_1, \forall u_i \in V_1$ and $\text{deg}(v_j) = k_2, \forall v_j \in V_2$.

Let $(u_i, v_j)$ be any vertex in $G_1(k)G_2$ then

$$\text{deg}(u_i, v_j) = \text{deg}(u_i) \cdot \text{deg}(v_j)$$

(By Theorem 3.2)

$$= k_1 k_2$$

Thus every vertex in $G_1(k)G_2$ is of degree $k_1 k_2$ i.e. $G_1(k)G_2$ is $k_1 k_2$-regular.

Remark 3.6

However, it is to be noted that if $G_1, G_2$ are simple graphs then $G_1(k)G_2$ can never be a complete graph, for $(u_i, v_j)$ is not adjacent with $(u_i, v_k)$ for any $j \neq k$ (By definition 3.1).

Theorem 3.7

If $G_1$ or $G_2$ is a bipartite graph then $G_1(k)G_2$ is a bipartite graph.

Proof:

Suppose $G_1$ is bipartite graph with bipartition $(X,Y)$
where \( X = \{ x_1, x_2, \ldots, x_m \} \)

\( Y = \{ y_1, y_2, \ldots, y_n \} \)

Let \( V_2 = \{ v_1, v_2, \ldots, v_r \} \).

Then in \( G_1(k)G_2 \) the vertex set is

\[
\{ (x_1, v_1) (x_2, v_2) \ldots (x_m, v_r) \\
(x_2, v_1) (x_2, v_2) \ldots (x_2, v_r) \\
(x_m, v_1) (x_m, v_2) \ldots (x_m, v_r) \\
(y_1, v_1) (y_1, v_2) \ldots (y_1, v_r) \\
(y_2, v_1) (y_2, v_2) \ldots (y_2, v_r) \\
(y_n, v_1) (y_n, v_2) \ldots (y_n, v_r) \}
\]

Now, no two vertices of the form \((x_i, v_j)\) and \((x_k, v_l)\) are adjacent since \(x_i\) and \(x_k\) are not adjacent.

Similarly, no two vertices of the form \((y_i, v_j)\) and \((y_k, v_l)\) are adjacent since \(y_i, y_k\) are not adjacent.
Thus $G_{1}(k)G_{2}$ is a bipartite graph with bipartition.

$X_{\alpha_{1}\cup\alpha_{2}}$ and $Y_{\beta_{1}\cup\beta_{2}}$ where $X_{u_{i},v_{j}} = \{(x_{i}, v_{j}) / i = 1,2,\ldots,m\}$

and $Y_{u_{i},v_{j}} = \{(y_{i}, v_{j}) / j = 1,2,\ldots,n\}$

Theorem 3.8

The matching domination number of $c_{4}(k)K_{m}$ is 4

Proof:

Let $V(C_{4}) = \{u_{1}, u_{2}, u_{3}, u_{4}\}$ and $V(K_{m}) = \{v_{1}, v_{2}, \ldots, v_{m}\}$

It can be easily seen that $\{u_{1}, v_{1}\}, u_{2}, v_{2}, u_{3}, v_{3}, u_{4}, v_{4}\}$ will be a matching dominating set. The graph cannot have a matching dominating set of cardinality two, for if $\{u_{1}, v_{1}\}, u_{2}, v_{2}, u_{3}, v_{3}, u_{4}, v_{4}\}$ is a matching dominating set of $C_{4}(k)K_{m}$, it will not saturate the vertices $u_{1}, v_{1}$.

Hence the minimal matching domination number is 4 which is the same as the product of the matching domination number of $C_{4}$ and $K_{m}$.

$$= \ 2 \left\lfloor \frac{4}{4} \right\rfloor \cdot 2$$

$$= \ 4.$$
Illustration for the above theorem

\[ C_4 \times K_5 \]

Figure 3.2
In general we can prove that the matching domination number of product graph is equal to the product of matching domination numbers of the graphs.

**Theorem : 3.9**

If \( G_1, G_2 \) are two graphs without isolated vertices then
\[
\gamma_m(G_1(k)G_2) = \gamma_m(G_1) \cdot \gamma_m(G_2)
\]
where \( G_1(k)G_2 \) is the Kronecker product of graphs defined in 3.1.

**Proof :**

Let \( G_1 \) be a graph with \( p_1 \) vertices and \( G_2 \) be a graph with \( p_2 \) vertices.

Let \( D_1 = \{v_1, v_2, \ldots, v_{2r}\} \)

and \( D_2 = \{w_1, w_2, \ldots, w_{2s}\} \)

be minimal dominating sets of \( G_1 \) and \( G_2 \) respectively. Where
\[
\{(v_1, v_2); (v_3, v_4); \ldots; (v_{2r-1}, v_{2r})\} \quad \text{and} \quad \{(w_1, w_2); (w_3, w_4); \ldots; (w_{2s-1}, w_{2s})\}
\]
are pairs of adjacent vertices which constitute perfect matching in the induced subgraph \( <D_1> \) and \( <D_2> \) of \( G_1 \) and \( G_2 \) respectively.
Now consider cartesian product of $D_1$ and $D_2$

$$D_1 \times D_2 = \{(v_i, w_j) / 1 \leq i \leq 2r, 1 \leq j \leq 2s\}$$

Let $(v, w)$ be any vertex of $G_1(K)G_2$. There exists a vertex $(v_x, w_y) \in D_1 \times D_2$ such that $v$ is adjacent to $v_x$ and $w$ is adjacent to $w_y$ (By definition of 2.5). Thus $D_1 \times D_2$ is a dominating set of $G_1(K)G_2$. Deletion of any vertex in $D_1 \times D_2$ does not make $D_1 \times D_2$ a dominating set any more. For, if $(v_i, w_j)$ is deleted from $D_1 \times D_2$ where $v_i$ is adjacent to vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\}$ in $G_1$ and $w_j$ is adjacent to the vertices $\{w_{j_1}, w_{j_2}, \ldots, w_{j_s}\}$ in $G_2$, the vertices $(v_{i_1}, w_{j_1}), (v_{i_2}, w_{j_2}), \ldots, (v_{i_s}, w_{j_s})$ will not be adjacent to any of the vertices of $D_1 \times D_2$. Moreover the minimality of $D_1 \times D_2$ can also be deduced from the following observation. "Suppose $A \times B$ is a matching dominating set of $G_1(K)G_2$ then $A$ and $B$ are matching dominating sets of $G_1$ and $G_2$ respectively". Consequently it follows that $D_1 \times D_2$ is a minimal dominating set of $G_1(K)G_2$. Further, it is also a matching dominating set for the vertices $(v_1, w_2), (v_2, w_1), (v_1, w_3), (v_2, w_4), (v_1, w_4), v_2, w_3)$; ..... forms a perfect matching in the induced subgraph $\langle D_1 \times D_2 \rangle$.

Thus $D_1 \times D_2$ is a matching dominating set of minimum cardinality.

Hence $\gamma_m [G_1(K)G_2] = \gamma_m (G_1). \gamma_m (G_2)$.
Illustration

Matching dominating Set: \{u_1, u_2\}

\[ G_1(K) G_2 \]

Matching dominating Set: \{v_1, v_2; v_4, v_5\}

Figure 3.3

Matching dominating Set: \{<u_1, v_1>, <u_1, v_2>, <u_1, v_4>, <u_1, v_6>, <u_2, v_1>, <u_2, v_2>, <u_2, v_4>, <u_2, v_6>\}

= \{u_1, u_2\} \times \{v_1, v_2, v_4, v_6\}
We next discuss a second kind of product graph known as Cartesian product of two graphs, which is defined as follows.

**Definition 3.10**

If $G_1$, $G_2$ are two simple graphs with their vertex sets $V_1$: \{u_1, u_2, \ldots\}$ and $V_2$: \{v_1, v_2 \ldots\}$ respectively then the cartesian product of these two graphs is defined to be a graph with its vertex set as $V_1 \times V_2$: \{w_1, w_2, \ldots\}$ and if $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2)$ then $w_1 w_2$ is an edge in this product graph if and only if either $u_1 = u_2$ and $v_1, v_2 \in E(G_2)$ or $u_1, u_2 \in E(G_1)$ and $v_1 = v_2$. This product graph is denoted by $G_1(C)G_2$.

Illustration follows
Figure 3.4
It can be proved that in this product also that if $G_1$, $G_2$ are finite graphs without isolated vertices then $G_1(C)G_2$ is a finite graph without isolated vertices.

For this, we first prove the following theorem

**Theorem 3.11**

If $u_i \in V_1$ and $v_j \in V_2$ then $\deg_{G_1 \times G_2}(u_i, v_j) = \deg_{G_1}(u_i) \cdot \deg_{G_2}(v_j)$

**Proof:**

By definition 3.10, $(u_i, v_j)$ is adjacent with all the vertices in $u_i \times N_{G_2}(v_j)$ and $N_{G_1}(u_i) \times v_j$

Further, $|N_{G_1}(u_i)| = \deg_{G_1} u_i$ and $|N_{G_2}(v_j)| = \deg_{G_2} v_j$

Hence $\deg_{G_1 \times G_2}(u_i, v_j) = \deg_{G_1}(u_i) \cdot \deg_{G_2}(v_j)$

We have the following result as an immediate consequence.

**Theorem 3.12**

If $G_1$, $G_2$ are simple finite graphs without isolated vertices, $G_1(C)G_2$ is a simple finite graph without isolated vertices.

**Proof:**

Then $G_1(C)G_2$ is a simple finite graph follows from the definition 3.10.
Further $G_1$, $G_2$ being graphs without isolated vertices.

$$\deg_{G_1}(u_i) \neq 0 \text{ for any } i, \quad \deg_{G_2}(v_j) \neq 0 \text{ for any } j$$

Hence from the theorem 3.11, $\deg_{G_1 \times G_2}(u_i, v_j) \neq 0$ for any $i, j$.

Thus $G_1(C)G_2$ does not have any isolated vertices.

It is interesting to see that the cartesian product of two simple graphs is not a complete graph and if $G_1$, $G_2$ are bipartite then $G_1(C)G_2$ is also a bipartite graph.

**Theorem 3.12**

The Cartesian product graph of two simple graphs is not a complete graph.

**Proof:**

If \( V_1 = \{u_1, u_2, \ldots, u_m\} \) and \( V_2 = \{v_1, v_2, \ldots, v_n\} \) then in $G_1(C)G_2$ the vertex $(u_i, v_j)$ is not adjacent with $(u_p, v_j)$ even if $u_i$ and $u_j$ are adjacent and/or $v_i$ and $v_j$ are adjacent (By definition 3.10).

**Theorem 3.13**

If $G_1$ and $G_2$ are bipartite graphs then $G_1(C)G_2$ is a bipartite graph.
Proof:

Suppose $G_1$ is a bipartite graph with bipartition $(X_1, Y_1)$ and $G_2$, a bipartite graph with bipartition $(X_2, Y_2)$, where

$$X_1 = \{x_1, x_2, \ldots, x_r\}$$
$$Y_1 = \{y_1, y_2, \ldots, y_2\}$$
$$X_2 = \{u_1, u_2, \ldots, u_m\}$$
$$Y_2 = \{v_1, v_2, \ldots, v_n\}$$

We know that $X_1 \cup Y_1 = V_1$ and $X_2 \cup Y_2 = V_2$ and also $X_1 \cap Y_1 = \emptyset = X_2 \cap Y_2$.

Now $V_1 \times V_2 = (X_1 \cup Y_1) \times (X_2 \cup Y_2)$

$$= (X_1 \times X_2) \cup (X_1 \times Y_2) \cup (X_2 \times Y_1) \cup (X_2 \times Y_2).$$

This vertex set can partitioned as

$$X = \{X_1 \times X_2\} \cup \{Y_1 \times Y_2\} \text{ and}$$

$$Y = \{X_1 \times Y_2\} \cup \{X_2 \times Y_1\}$$

It can be easily seen that no two vertices in $X$ are adjacent in $G_1(C) G_2$ for if $t_1, t_2$ are two any vertices in $X$, then $t_1, t_2 \in \{X_1 \times X_2\} \cup \{Y_1 \times Y_2\}$

$$\Rightarrow t_1, t_2 \in \{X_2 \times X_2\} \text{ or } \{Y_1 \times Y_2\}. \]
Case (i)

If $t_1 \in \{X_1 \times X_2\}$ and $t_2 \in \{X_1 \times X_2\}$

$t_1 \in \{X_1 \times X_2\} \Rightarrow t_1 = (x_i, u_i) \text{ say and } t_2 = (x_k, u_i)$

If $x_i = x_k$; $u_i, u_i \in X_2$ and no two vertices of $X_2$ are adjacent by hypothesis; on the other hand if $u_j = u_i$, and if $x_i, x_k \in X_1$ as no two vertices of $X_1$ are adjacent (by hypothesis), $t_1, t_2$ are not adjacent.

Case (ii)

If $t_1 \in \{X_1 \times X_2\}$

$t_2 \in \{Y_1 \times Y_2\}$

From the definition of 3.10, it is evident that $t_1, t_2$ are not adjacent since no $x_i$ is equal to any $y_k$ and so also no $u_j$ is equal to any $u_i$ as $X_1 \cap Y_1 = \emptyset$; further $u_j$ and $u_i$ are in two different partitions of $G_2$. So $t_1, t_2$ are not adjacent, in this case also.

Case (iii)

$t_1 \in \{Y_1 \times Y_2\}$

$t_2 \in \{X_1 \times X_2\}$

This case is similar to the case (ii) and thus $t_1, t_2$ are not adjacent in this case also.
Case (iv)

\[ t_1 \in (Y_1 \times Y_2) \]

\[ t_2 \in (X_1 \times X_2) \]

This case is similar to case (i) discussed above and so \( t_1, t_2 \) are not adjacent in this case too.

Thus we have proved that no two vertices in \( X \) are adjacent.
Similarly it can be proved that no two vertices in \( Y \) are adjacent.

Hence the theorem

Now we prove the following result, regarding the matching domination number of the graph \( G_1(C)G_2 \).

**Theorem 3.14**

If \( G_1 \) and \( G_2 \) are any two graphs without isolated vertices

then \( \gamma_m[ G_1(C)G_2] \leq \min \{ |V_1| \cdot \gamma_{md}(G_2) \} \cup \gamma_{md}(G_1) \cdot |V_2| \}

**Proof:**

Let \( V(G_1) : \{u_1, u_2, \ldots, u_p\} = V_1 \)

\( V(G_2) : \{u_1, u_2, \ldots, u_q\} = V_2 \)

Let \( D_1 : \{u_{1'}, u_{2'}, \ldots, u_{d_x}\} \) be the matching dominating set of minimum cardinality of \( G_1 \). Let the induced sub graph \(<D_1>\) of
$G_1$ have a perfect matching in $<D_1>$ constituted by the edges
\[ u_{d_1}, u_{d_2}, u_{d_3}, u_{d_4}, \ldots, u_{d_{d_x}} \]

Similarly let $D_2 : \{ v_{d_1}, v_{d_2}, \ldots, v_{d_{d_x}} \}$ be the matching dominating set of minimum cardinality of $G_2$. Let the induced subgraph $<D_2>$ of $G_n$ have a perfect matching in $<D_2>$ constituted by the edges
\[ v_{d_1}, v_{d_2}, v_{d_3}, v_{d_4}, \ldots, v_{d_{d_x}} \]

Then it can be easily seen that the sets
\[
D : \{ (u_i, v_{d_i}), (u_j, v_{d_j}), \ldots, (u_{d_x}, v_{d_{d_x}}) \}
\]

and the set
\[
D' : \{ (u_{d_1}, v_i), (u_{d_2}, v_i), \ldots, (u_{d_x}, v_i) \}
\]
will be both matching dominating sets of $G_1(C) \cup G_2$. Hence, it follows that the minimal matching dominating set will be $\leq \min \{ |D|, |D'| \}$.

\[ |D| = p.2s = |V_1| \cdot \gamma_m (G_2); \]

\[ |D'| = 2r.q = \gamma_m (G_1). |V_2| \]

Hence $\gamma_m (G_1(C) \cup G_2) \leq \min \{ 2ps, 2qr \}

\leq \min \{ \gamma_m (G_1), |V_2|, |V_1| \cdot \gamma_m (G_2) \}$

Illustration follows:
Figure 3.5
matching dominating Set $D = \{u_1, u_2\} \times |V_2|,$

$= \{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_1, v_4),$

$(u_1, v_5), (u_1, v_6), (u_2, v_1), (u_2, v_2),$

$(u_2, v_3), (u_2, v_4), (u_2, v_5), (u_2, v_6)\},$

$|D| = 12$

matching dominating Set $D' = |V_1| \times \{v_1, v_2, v_3, v_4, v_5\}$

$= \{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_1, v_4), (u_1, v_5);$

$(u_2, v_1), (u_2, v_2), (u_2, v_3), (u_2, v_4), (u_2, v_5);$

$(u_3, v_1), (u_3, v_2), (u_3, v_3), (u_3, v_4), (u_3, v_5);$

$(u_4, v_1), (u_4, v_2), (u_4, v_3), (u_4, v_4), (u_4, v_5)\},$

$|D'| = 16$

However, \{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_1, v_4), (u_1, v_5), (u_2, v_1),$

$(u_2, v_2), (u_3, v_1), (u_3, v_2), (u_3, v_3), (u_3, v_4), (u_3, v_5)\}\ is a matching dominating set of $G_1 \ (C) \ G_2$

with cardinality 10.

Hence, $\gamma_m \{G_1 \ (C) \ G_2\} \leq \min \{D, D'\}$

Next we define another kind of product graph called Lexicograph product of two graphs.

**Definition 3.15**

If $G_1, G_2$ are simple graphs with their vertex sets as $V_1: \{u_1, u_2, \ldots\}$ and $V_2: \{v_1, v_2, \ldots\}$ respectively, then the Lexicograph product is a graph with its vertex set as $V_1 \times V_2 = \{w_1, w_2, \ldots\}$ and if $w_1 = (u_1, v_1), w_2 = (u_2, v_2)$ then $w_1 w_2$ is an edge in this product graph if and only if either (i) $u_1, u_2 \in E (G_1)$ or (ii) $u_1 = u_2$ and $v_1, v_2 \in E (G_2)$. This product graph is called Lexicograph product graph and is denoted by $G_1 \ (L) \ G_2$. 
Illustration

$G_1$ (L) $G_2$

Figure 3.6
In this product graph also, it can be proved that if $G_1$ and $G_2$ are simple finite graphs without isolated vertices then $G_1 \times G_2$ is also finite graph without isolated vertices.

To establish this, we first obtain an expression for $N_{(u_i, u_j)} (u_1, v_j)$. Where $N_G(u)$ denotes the neighbourhood set of $u$ in the graph $G$.

**Theorem 3.16**

$$N_{G_1} (u_i, v_j) = \{ N_{G_1} (u_i) \times V_2 \} \cup \{ (u_i) \times N_{G_2} (v_j) \}$$

**Proof:**

Suppose $N_{G_1} (u_i) = \{ u_1, u_2, \ldots, u_r \}$,

$$V_1 = \{ u_1, u_2, \ldots, u_m \};$$

and $N_{G_2} (v_j) = \{ v_1, v_2, \ldots, v_s \}$

$$V_2 = \{ v_1, v_2, \ldots, v_m \}.$$

Then vertex $(u_i, v_j)$ is adjacent to the vertices

$$\{ (u_1, v_1), (u_1, v_2), \ldots, (u_1, v_s) \}$$

$$\{ (u_2, v_1), (u_2, v_2), \ldots, (u_2, v_s) \}$$

$$\{ (u_3, v_1), (u_3, v_2), \ldots, (u_3, v_s) \}$$

$$\{ (u_r, v_1), (u_r, v_2), \ldots, (u_r, v_s) \}$$
The vertex \((u_i, v_i)\) is adjacent with any vertex \((u_m, v)\), if \(u_m\in N_{\alpha}(u_i)\) and \(v \in V_2\) (By definition 3.15). Thus \((u_i, v_j)\) is adjacent with all the vertices of the set \(N_{\alpha}(u_j) \times V_2\).

Also \((u_i, v_j)\) is adjacent with all the vertices of the set \(\{u_i\} \times N_{\alpha}(v_j)\), for if \((u_i, v_j)\) is any element in the set \(\{u_i\} \times N_{\alpha}(v_j)\) where \(v_i \in N_{\alpha}(v_j)\), then \((u_i, v_j)\) is adjacent with \((u_i, v_i)\) since \(v_j, v_i\) are adjacent. Thus

\[
\{N_{\alpha}(u_i) \times V_2\} = \{u_i\} \times N_{\alpha}(v_j) \subset N(u_i, v_j) \quad .... \quad (1)
\]

Conversely, if \((u_x, v_y) \in N_{\alpha}(u_i, v_j) = (u_i, v_j)\) is adjacent with \((u_x, v_y)\) (By definition 3.15), this is possible only if \(u_i\) is adjacent with \(u_x\) i.e. \(u_x \in N_{\alpha}(u_i)\) or if \(u_i = u_x\) and \(v_j\) is adjacent with \(v_y\) i.e. \(v_j \in N_{\alpha}(v_j)\).

\[
\{u_i, v_j\} \in N_{\alpha}(u_i) \times V_2
\]

or \((u_x, v_i) \in \{u_i\} \times N_{\alpha}(v_j)\).

Thus \((u_x, v_j) \in \{\{u_i\} \times N_{\alpha}(v_j)\}\)

Hence

\[
N_{\alpha}(u_i, v_j) \subset \{N_{\alpha}(u_i) \times V_2\} \cup \{\{u_i\} \times N_{\alpha}(v_j)\} \quad .... \quad (2)
\]

From (1) and (2) the theorem follows.
To obtain an expression for the matching domination number of $G_1 (L) G_2$. We require the following result also

**Theorem 3.17**

$$\deg_{G_1} (u, v) = |N_{\alpha_1} (u, v)| \leq |V_2| \cdot |N_{\alpha_2} (v)|$$

**Proof:**

From the theorem 3.16

$$N_{\alpha_1} (u, v) \leq \{ N_{\alpha_1} (u) \times V_2 \} \cup \{ \{ u \} \times N_{\alpha_2} (v) \}$$

The two Cartesian product sets on the RHS are disjoint sets; for, any element in the Cartesian product $\{ u \} \times N_{\alpha_2} (v)$ is of the form $(u, v_s)$, where as any element in the Cartesian product

$$N_{\alpha_1} (u) \times V_2$$

is of the form $(v, v)$; $v \in V_2$. None of the elements $(u, v_s)$ can be the element $(v, v)$ since $v \neq \emptyset$ i.e. $u \neq u$.

Hence

$$\deg (u_i, v) = |N_{\alpha_1} (u_i, v)|$$

$$= |N_{\alpha_1} (u_i) \times V_2| \cdot |N_{\alpha_2} (v)|$$

**Corollary 3.18**

$$\deg_{G_2} (u_i, v_j) = 0 \text{ if and only if } \deg_{\alpha_1} (u_i) = 0 \text{ and } \deg_{\alpha_2} (v_j) = 0$$

**Proof:**

If $\deg (v_i, v_j) = 0$;
by the previous theorem, \( |N_{v_1}(u_i) \times V_x| \cdot |N_{v_2}(v_j)| = 0 \)

\[ \Rightarrow |N_{v_1}(u_i) \times V_x| = 0 \text{ and } |N_{v_2}(v_j)| = 0 \]

\[ \Rightarrow N_{v_1}(u_i) = 0 \text{ and } N_{v_2}(v_j) = 0 \]

\[ \Rightarrow \deg_{v_1}(u_i) = 0 \text{ and } \deg_{v_2}(v_j) = 0 \]

Conversely, if \( \deg_{v_1}(u_i) = 0 \) and \( \deg_{v_2}(v_j) = 0 \), by retracing the above steps, we get \( \deg_{G_i}(u_i, v_j) = 0 \).

Now the following result is an immediate consequence

**Theorem 3.19**

If \( G_1, G_2 \) are simple finite graphs without isolated vertices then \( G_1 (L) G_2 \) is a finite graph without isolated vertices.

**Proof:**

Since \( G_1 (L) G_2 \) is a finite graph follows by the definition 3.15.

Further \( G_1, G_2 \) are graphs without isolated vertices

i.e. \( \deg_{v_1}(u_i) \neq 0 \) for any \( i \),

\( \deg_{v_2}(v_j) \neq 0 \) for any \( j \)

Hence from corollary 3.18, \( \deg_{G_1(L)G_2}(u_i, v_j) \neq 0 \) for any \( i, j \)

It can also be established that, \( G_1(L)G_2 \) is a complete graph if and only if \( G_1 \) and \( G_2 \) are complete graphs.
We now obtain an expression for the matching domination number for the Lexicograph product of two graphs.

It is interesting to see that in this type of product graphs the matching domination number of Lexicograph product graph $G_1$ and $G_2$ is same as the matching domination number of the graph $G_1$.

**Theorem 3.20**

If $G_1$, $G_2$ are any two graphs without isolated vertices then

$$\gamma_m [G_1(L) G_2] = \gamma_m (G_1)$$

**Proof:**

Let $D_1$, $D_2$ be the matching dominating sets of minimum cardinality of $G_1$ and $G_2$ respectively.

Let $D_1 = \{ u_1, u_2, \ldots, u_{t_1} \}$

$D_2 = \{ v_1, v_2, \ldots, v_{t_2} \}$

Consider the set $D = \{ (u_1, v_1), (u_2, v_2), \ldots, (u_{t_1}, v_{t_1}) \}$

(if $r < s$) or consider the set

$D = \{ (u_1, v_1), (u_2, v_2), \ldots, (u_{t_1}, v_{t_1}), (u_{t_1} + 1, v_1), (u_{t_1} + 2, v_1), \ldots, (u_{t_1} + r, v_1) \}$

($if r > s$)

$D$ will be a matching dominating set. Further $D$ is of minimum cardinality for if we remove any of the vertices in $D$, $D$ is not a matching dominating set any more in view of (1), and from the definition of Lexicograph product.

Thus $D$ is a matching dominating set of minimum cardinality.

Hence, $\gamma_m (G_1(L)G_2) = \gamma_m (G_1)$.

Illustrations follows
Figure 3.7
matching dominating
set: \{u_1, u_2\}
\(\gamma_m(G_1) = 2\)

matching dominating
set: \{v_1, v_2, v_3, \ldots, v_n\}
\(\gamma_m(G_2) = 4\)

matching dominating
set: \{(u_1, v_1), (u_2, v_2)\}
\(\gamma_m[G_1 \cup G_2] = 2\)
Figure 3.8
matching dominating
set: \( \{ v_1, v_2, v_3, v_4 \} \)
\( \gamma_m(G_1) = 2 \)

matching dominating
set: \( \{ u_1, u_2 \} \)
\( \gamma_m(G_2 (L) G_1) = 4 \)