Chapter 2

MATCHING DOMINATION IN GRAPHS
Sampathkumar and Walikar [33] introduced the concept of connected domination in graphs. A dominating set $D$ is called a connected dominating set, if it induces a connected subgraph in $G$. Since a dominating set must contain at least one vertex from every component of $G$ it follows that a connected dominating set for a graph $G$ exists if and only if $G$ is connected. The minimum of the cardinalities of the connected dominating sets of $G$ is called the connected domination number of $G$ and is denoted by $\gamma_c(G)$. Sampathkumar and Walikar have obtained several interesting results on $\gamma_c(G)$. They have proved that

(i) $\gamma_c(K_p) = 1$

(ii) $\gamma_c(K_p + G) = 1$, for any graph $G$.

(iii) $\gamma_c(K_{m,n}) = \begin{cases} 1, & \text{if either } m \text{ or } n = 1 \\ 2, & \text{if } m, n \geq 1. \end{cases}$

(iv) $\gamma_c(C_p) = p - 2$

(v) For any tree $T$ of order $p$, $\gamma_c(T) = p - e$, where $e$ is the number of pendant vertices i.e. vertices of degree 1 in $T$.

They have also proved that $\gamma(G) \leq \gamma_c(G)$. 
They have also established that for any connected graph $G(p,q)$ with maximum degree $\Delta$, $\frac{P}{\Delta} \leq \gamma_c(G) \leq 2q - p$.

Later, Hedetniemi and Laskar [18] improved and extended the results obtained by Sampathkumar and Walikar. Further, they have conjectured some relations concerning connected domination.

Kulli and Sigarkanti [24] have studied the edge analogue of connected domination in graphs. They have obtained some exact bounds for the connected edge domination number. In all these results the induced subgraph $<S>$ constructed by the dominating set $S$ of the graph plays a significant role.

We have defined a new parameter called the matching dominating set and the matching domination number. These concepts of dominations are once again dependent on the induced subgraph induced by the dominating set.

We have given the necessary preliminaries in §2.1. We have obtained the matching domination number of several standard graphs.
§ 2.1

In this section we have defined all the related concepts required for defining the matching dominating set of a graph.

Definition 2.2

Let \( G \) be a simple graph with vertex set \( V \) and edge set \( E \). A matching in \( G \) is a set \( M \) of edges of \( G \) such that every vertex of \( G \) is incident to at most one edge in \( M \).

A matching \( M \) in \( G \) is called perfect matching if every vertex of \( G \) is incident to exactly one edge in \( M \).

If \( S \subseteq V \), we denote by \( G_S \), the subgraph of \( G \) obtained by removing all the vertices in \( S \) and all of the edges to which vertices in \( S \) are incident. Denoting the number of components of \( G_S \) having an odd number of vertices by \( P(S) \), Tutte [35] has obtained the following remarkable result which gives a characterization for a graph to have a perfect matching. His result is:

Theorem 2.3

The graph \( G \) has a perfect matching if and only if \( P(S) \leq |S| \), for every subset \( S \) of \( V \).
Definition 2.4

A graph $H$ is called a subgraph of a graph $G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

For any set $S$ of vertices of $G$, the induced subgraph $<S>$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $<S>$ if and only if they are adjacent in $G$.

The concept of domination in graphs was first introduced by Ore [30]. During the last three decades, the tremendous growth of research in this area has developed it, as a major area of research activity in graph theory.

The Dominating set is defined as follows

Definition 2.5

A set $S \subseteq V$ is said to be a dominating set in a graph $G$ if every vertex in $V \setminus S$ is adjacent to some vertex in $S$ and the domination number '$\gamma$' of $G$ is defined to be the minimum cardinality of all dominating sets in $G$.

We have introduced a new parameter called the matching domination set of a graph.
It is defined as follows:

**Definition 2.6**

A dominating set of a graph G is said to be matching dominating set if the induced subgraph \(<D>\) admits a perfect matching.

The cardinality of the smallest matching dominating set is called matching domination number and is denoted by \(\gamma_m\).

**Illustration:**

![Figure 2.1](image)

In this graph \(\{a, b, c, f, e, g\}\) is a matching domination set, since this is a dominating set and the induced subgraph \(\{a, b, c, e, f, g\}\) has perfect matching formed by the edges af, bc, eg. \(\{a,b,e,f\}\) is also matching dominating set. Similarly \(\{a, b, c, g\}\) is a matching dominating set where the induced subgraph of this set
admits a perfect matching given by the edges bc, ag.

However there are no matching dominating sets of lower cardinality and it follows that the matching domination number of the graph in figure 2.1 is 4. Thus a graph can have many matching dominating sets of minimal cardinality. We make the following observations as an immediate consequence.

(a) Not all dominating sets are matching domination sets. For example in figure 2.1, \( \{a,c,e\} \) is a dominating set but it is not a matching dominating set.

(b) The cardinality of matching dominating set is always even. By definition 2.6, the matching dominating set \( D \) of a graph requires the admission of a perfect matching by the induced subgraph \( <D> \). Thus it is necessary that \( D \) has even number of vertices for admitting a perfect matching.

(c) Not all dominating sets with even number of vertices are matching dominating sets. For example in figure 2.1, \( \{b,d,g,f\} \) is a dominating set containing even number of
vertices, but induced subgraph formed by these four vertices
does not have a perfect matching.

The following result is an immediate consequence of the
definition 2.2.

(d) The necessary condition for a graph $G$ to have matching
dominating set is that $G$ is a graph without isolated vertices.
The matching domination number of the graph $G$ (figure 2.1)
is 4, where as the domination number is 2 ; \{a, d\} being a
minimal dominating set. If $G$ is a graph with isolated
vertices then any dominating set should include these
isolated vertices and consequently the induced subgraph of
this set containing isolated vertices will not admit a perfect
matching.

In general, since every matching dominating set is a
dominating set with an even number of vertices, we have the
following result.

**Theorem 2.7**

If $G$ is a graph without isolated vertices, then $\gamma \leq \gamma_m$. 
Proof:

It follows from the definition 2.6 of matching dominating set and (§ 2.1(c) page 13) that the matching domination set should contain an even number of vertices, so as to allow a perfect matching in the induced subgraph, besides being a dominating set.

Thus it follows that $\gamma \leq \gamma_m$.

However it is interesting to note that there are some graphs for which $\gamma = \gamma_m$.

We call a cycle on four vertices where at each vertex of the cycle has some pendent vertices, a four cycle pendent. Consider the union of these types of graphs at nodes of the 4-cycles, we call it as union of 4-cycle-pendents. It is interesting to see that 4-cycle-pendents have the property $\gamma = \gamma_m$.

![Figure 2.2 4-Cycle-pendent](image-url)
Figure 2.3
Union of 4-Cycle-pendents
In this context, we have the following result.

We prove the result by giving example of graphs with the required conditions.

**Theorem 2.8**

(i) There exist bipartite graphs for which $\gamma \neq \gamma_m$.

(ii) There exist graphs which are not bipartite for which $\gamma = \gamma$

**Proof:**

(i) There are bipartite graphs for which $\gamma \neq \gamma_m$.

Consider the following graph which is a path on 5 vertices.

![Graph Diagram]

The above graph is a bipartite graph whose domination number is 2 with minimal dominating set $\{u_2, u_4\}$ and the matching domination number is 4. The minimal matching dominating set is $\{u_1, u_2; u_4, u_5\}$. 
Thus \( \gamma \)

(ii) Graphs which are not bipartite for \( \gamma = \gamma_1 \).

Consider the following graph.

This is not a bipartite graph as there are odd cycles \( K_3 \). The domination number is 2 and minimum dominating set is \( \{v_3, v_4\} \). The matching domination number is 2 and minimal matching domination set is \( \{v_3, v_4\} \).

Thus \( \gamma \)
§ 2.2

Matching domination number of some standard graphs:

Theorem 2.9:

\[ \gamma_m(K_p) = 2 \]

Proof:

We know that for any +ve integer \( p \), \( \gamma(K_p) = 1 \), as every vertex in \( K_p \) is adjacent to every other vertex in \( K_p \). Further \( \gamma_m(K_p) = 2 \), as any pair of vertices will constitute a matching dominating set.

Theorem 2.10:

\[ \gamma_m(K_{s,t}) = 2 \]

Proof:

\( K_{s,t} \) is a complete bipartite graph with bipartition \((X,Y)\). Any pair of vertices \((u,v)\), \( u \in X, v \in Y \) will constitute a dominating set as well as matching dominating set.
Thus $\gamma(K_{n,k}) = \gamma_m(K_{n,k}) = 2$

**Theorem 2.11**

For any positive integer $n, \gamma(K_{i,n}) = 1, \gamma_m(K_{i,n}) = 2$

**Proof:**

If $K_{i,n}$ is a bipartite graph with bipartition $(X,Y)$, $u \in X$ will be the only vertex in $X$ and is adjacent to every vertex in $Y$.

Hence $\{u\}$ is minimal dominating set.

$\gamma(K_{i,n}) = 1$.

For any $v \in Y$, $\{u,v\}$ will form a minimal matching dominating set

$\therefore \gamma_m(K_{i,n}) = 2$

**Theorem 2.12**

For any positive integer $n, \gamma(W_n) = 1, \gamma_m(W_n) = 2$ where $W_n$ is a wheel on $n$ vertices.

**Proof:**

![Figure 2.4](image)
\{1\} is minimal dominating set and \{1,v\}, \forall v \in \{2, 3, 4, 5, 6\}
will constitute a minimal matching dominating set.

**Theorem 2.13**

For any positive integer \( n \), 
\[
\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil, \quad \gamma_m(P_n) = 2 \left\lceil \frac{n}{4} \right\rceil
\]

where \( \lceil x \rceil \) denotes least positive integer \( \geq x \).

**Proof:**

Let \( P_n \) be a path on \( n \) vertices. We can label vertices of \( P_n \) as
\{1, 2, 3, ............., n\}. Divide them into subpaths of length not exceeding 4. We have the subpaths \{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}, ........... \{......n......\}.

We will have four cases.

Case (i)

When \( n = 4t \), then \{2, 3, 6, 7, 10, 11, ........ n-2, n-1\} is minimal matching dominating set and \{2, 3\}, \{6, 7\}, \{10, 11\}, ......... \{n-2, n-1\} forms a perfect matching of the induced graph of the minimal dominating set.
Case (ii)

When \( n = 4t + 1 \), then \( \{2, 3, 6, 7, \ldots, n-3, n-2, n-1, n\} \) is minimal matching dominating set and \( \{2, 3\}, \{6, 7\}, \ldots, \{n-3, n-2\}, \{n-1, n\} \) forms a perfect matching of the induced graph of the minimal dominating set.

Case (iii)

When \( n = 4t + 2 \), then \( \{2, 3, 6, 7, \ldots, n-4, n-3, n-1, n\} \) is minimal matching dominating set and \( \{2, 3\}, \{6, 7\}, \ldots, \{n-4, n-3\}, \{n-1, n\} \) forms a perfect matching of the induced graph of the matching dominating set.

Case (iv)

When \( n = 4t + 3 \) then \( \{2, 3, 6, 7, 10, 11, \ldots, n-5, n-4, n-1, n\} \) is a minimal matching dominating set and \( \{2, 3\}, \{6, 7\}, \ldots, \{n-5, n-4\}, \{n-1, n\} \) forms a perfect matching of the induced graph of the matching dominating set.

The cardinality in each case is \( 2\lceil \frac{n}{4} \rceil \).

The following result follows similarly,

**Theorem 2.14**

For any positive integer \( n \), \( \gamma(C_n) = \lceil \frac{n}{3} \rceil \), \( \gamma_m(C_n) = 2\lceil \frac{n}{4} \rceil \).