Chapter 4

Basic Theory of Time-Frequency Methods

The need to analyze a signal in time and frequency domains simultaneously was introduced in Chapter 1 and the relevance to sonar system design was brought out in Chapter 3. But, other than Short-Time Fourier transform, time-frequency methods have been largely limited to academic research because of the complexity of the algorithms and the limitations of computing power. Since the aim of the present thesis is to come up with improved TFM based techniques for implementing sonar functions, a review of the background theory closely related to the research work carried out, is elaborated in this chapter. The topics covered include the basics of Wavelet Transform, Fractional Fourier Transform, Wigner Ville Distribution and Ambiguity Function. Each of these time-frequency methods have certain desirable features, which make them ideal for a particular application, in sonar systems. These special features are also brought out in this chapter. Implementation of Ambiguity function uses Mellin Transform and so, its basic theory is presented in the last section of this chapter.
4.1 Wavelet Transform (WT)

Wavelet Transform is a transform by which signals can be modeled as a linear combination of translations and dilations of a simple oscillatory function of finite duration called a mother wavelet \( \psi(t) \). It provides very good spectral resolution at low frequencies at the expense of temporal resolution and very good temporal resolution at high frequencies at the expense of spectral resolution.

4.1.1 Continuous Wavelet Transform (CWT)

The WT of a signal represents the signal as a linear combination of scaled and shifted versions of the mother wavelet. When the scale and shift parameters are continuous, the transform under consideration is called a Continuous Wavelet transform (CWT). Let \( f(t) \) be any square integrable function. The CWT of \( f(t) \) with respect to a wavelet \( \psi(t) \) is defined as[28,29]

\[
W(a,b) = \int_{-\infty}^{\infty} f(t) \psi^*_{a,b}(t) \, dt \tag{4.1}
\]

where \( \psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right) \)

\[
\therefore W(a,b) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{|a|}} \psi^*\left(\frac{t-b}{a}\right) \, dt \tag{4.2}
\]

4.1.1.1 Salient features of CWT

1. CWT maps a 1-D function \( f(t) \) to a 2-D time-scale plane
2. Eqn(4.1) is called the analysis or forward transform
3. Variables \( a \) and \( b \) are real and * denotes conjugation
4. \( \psi_{a,0}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t}{a}\right) \) i.e. \( \psi_{a,0}(t) \) is a time-scaled and amplitude scaled version of \( \psi(t) \). So, \( a \) is called the dilation or scaling parameter
5. For a fixed value of \( a \), \( \psi_{a,b}(t) \) is a shift of \( \psi_{a,0}(t) \) by an amount \( b \) along the time axis. So, \( b \) is called the translation or delay parameter
6. \( \psi_{1,0}(t) = \psi(t) \)
7. The term $\frac{1}{\sqrt{|a|}}$ ensures that energy stays same for all values of $a$ and $b$.

\[ \int_{-\infty}^{\infty} |\psi_{a,b}(t)|^2 dt = \int_{-\infty}^{\infty} |\psi(t)|^2 dt \] for all values of $a$ and $b$

8. If $a > 1$, $\psi(t)$ is stretched. If $0 < a < 1$, $\psi(t)$ is contracted. If $a$ is negative, $\psi(t)$ is time-reversed as well as stretched or contracted, depending on whether $|a| > 1$ or $0 < |a| < 1$.

9. $\psi(t)$ is called the mother wavelet. The set of basis functions $\psi_{a,b}(t)$ are generated from the mother wavelet by dilation and translation and are called daughter wavelets.

10. Fourier transform of mother wavelet is $\psi(t) \leftrightarrow \psi(\omega)$. Fourier transform of daughter wavelets are

\[ \psi_{a,0}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t}{a}\right) \leftrightarrow \sqrt{|a|} \psi(a\omega) \]

\[ \psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right) \leftrightarrow \sqrt{|a|} \psi(a\omega) e^{-j\omega b} \] \hspace{1cm} (4.3)

11. (a) Centre of $\psi(t)$ is given as $t_0 = \frac{\int_{-\infty}^{\infty} |\psi(t)|^2 dt}{\int_{-\infty}^{\infty} |\psi(t)|^2 dt}$ \hspace{1cm} (4.4)

Centre of $\psi_{a,b}(t) = t_0 + b$

(b) Centre of $\psi(\omega)$ is given as $\omega_0 = \frac{\int_{-\infty}^{\infty} |\psi(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |\psi(\omega)|^2 d\omega}$ \hspace{1cm} (4.5)

Centre of $\psi_{a,b}(\omega) = \omega_0 / a$

(c) RMS width of $\psi(t)$ is given as $\Delta t_{\psi} = \sqrt{\frac{\int_{-\infty}^{\infty} (t - t_0)^2 |\psi(t)|^2 dt}{\int_{-\infty}^{\infty} |\psi(t)|^2 dt}}$ \hspace{1cm} (4.6)
RMS width of $\psi_{a,b}(t) = \Delta t_{\psi}(a) = a \cdot \Delta t_{\psi}$

(d) RMS width of $\psi(\omega)$ is given as

$$\Delta \omega_{\psi} = \sqrt{\frac{\int (\omega - \omega_0)^2 |\psi(\omega)|^2 d\omega}{\int |\psi(\omega)|^2 d\omega}}$$

RMS width of $\psi_{a,b}(\omega) = \Delta \omega_{\psi}(a) = \Delta \omega_{\psi}/a$

12. Time-bandwidth product is a constant

ie. $\Delta t_{\psi} \cdot \Delta \omega_{\psi} = \Delta t_{\psi}(a) \cdot \Delta \omega_{\psi}(a) = \text{constant}$

13. Q-factor = center frequency/3dB bandwidth

Q-factor of $\psi(t) = \text{Q-factor of } \psi_{a,b}(t)$

4.1.1.2 Conditions for $\psi(t)$ to be a mother wavelet

1. $\int_{-\infty}^{\infty} \psi(t) \, dt = 0$ ..................................................................................... (4.8)

ie. The function integrates to zero. This symbolizes the wavy nature of wavelets

2. $\int_{-\infty}^{\infty} |\psi(t)|^2 \, dt < \infty$ ..................................................................................... (4.9)

The function is square integrable. This property implies the finite duration of wavelets

3. $\int_{-\infty}^{\infty} \frac{|\psi(\omega)|^2 \, d\omega}{|\omega|} = C \quad \text{where} \quad 0 < C < \infty$ ........................................ (4.10)

This condition called the admissibility criterion ensures that inverse CWT exists

4. The wavelet system must satisfy MRA properties of self similarity at different scales

4.1.1.3 Disadvantages of CWT

CWT is very easy to visualize and understand. But it has two major disadvantages.

- It requires analytically explicit functions to find the inner product.
- The scaling parameter $a$ and delay parameter $b$ take continuous values resulting in a redundant representation on the time-frequency plane
Both these disadvantages are overcome by the Discrete Wavelet transform. DWT requires no analytic functions and the algorithm involves only filtering and decimation and is therefore computationally very efficient.

4.1.2 Discrete Wavelet Transform using Filter Banks (DWT)

Wavelet transform can be looked at in a totally different way- as a recursive structure of filter banks[28,29]. Developed by Mallat, this algorithm is called Discrete Wavelet Transform. DWT is derived from the principles of multi-resolution analysis(MRA), wherein a function can be analyzed at different resolutions. This involves approximation of the function in a sequence of nested linear vector spaces. Given a function $x(t)$, the decomposition begins by mapping the function into a sufficiently high resolution subspace $V_j$. For this, $x(t)$ is sampled for a very large $j$ in order to get scaling coefficient $c_{j+1}$. Once the scaling function at resolution $c_{j+1}$ are got, then the scaling coefficients $c_j$ and wavelet coefficients $d_j$ at lower resolutions are got by convolving with the scaling and wavelet filters and decimating by 2 as defined in Eqns.(4.11) and (4.12). Here, $g(n)$ and $h(n)$ are the wavelet and scaling filters respectively.

$$c_j(k) = \sum_m h(m-2k)c_{j+1}(m) ..................................................(4.11)$$

$$d_j(k) = \sum_m g(m-2k)c_{j+1}(m).............................................(4.12)$$

It is found that $h(n)$ has low pass response and $g(n)$ has high pass response. So, they can be looked upon as impulse responses of an LTI system, with I/O relationship and the filter structure as in fig.4.1. Using filters $h(n)$ and $g(n)$, the signal is split into high and low frequency parts. In the next stage, the same splitting is done on the low frequency output. This decomposition is continued to the desired resolution.

$y_1(n) = h(n) * x(n)$

$y_2(n) = g(n) * x(n)$

![Fig.4.1 - DWT- Mallat's decomposition tree](image)
Mallat’s filter bank algorithm, resulting from the above decomposition tree, thus involves the computation of approximation coefficients \( c(k) \) and detailed coefficients \( d(k) \). The wavelet and scaling filters satisfy the Quadrature Mirror Filter properties [28,29] of perfect reconstruction. The coefficients at scale \( j \) are convolved with the time reversed filter coefficients \( h(n) \) and \( g(n) \) and then down sampled to get the coefficients at scale \( (j-1) \). Fig. 4.2 shows a two-stage filter bank implementation for Wavelet decomposition.

Fig. 4.2 - DWT - Two Stage decomposition tree

### 4.1.3 Types of Wavelets

There are various kinds of wavelets. Accordingly, one can choose from among smooth wavelets, compactly supported wavelets, symmetric and non-symmetric wavelets, orthogonal and biorthogonal wavelets etc. Selection of a wavelet is based on properties like smoothness, vanishing moments, symmetry, orthogonality and frequency localization. The wavelet that has been considered in this study belongs to the group of Daublets. Daublets are orthogonal wavelets with compact support. There are quite a large number of wavelets in this group viz. \( \text{db2} \), \( \text{db4} \), \( \text{db8} \) etc. The support and the smoothness of these wavelets increase as the wavelet order number increases. These wavelets have the highest number of vanishing moments among wavelets with similar properties, for a given support. For the transient detection application in Chapter 7, \( \text{db4} \) has been used in all the simulations.
4.1.4 Discrete Wavepacket Transform (DWPT)

A Wavelet basis is a member of the large collection of Wavepacket bases. According to multi-scale filtering structure, Wavepacket transform can divide the entire time-frequency plane into subtle tilings, while the classical WT can only find its finer analysis for lower-band only. The Wavepacket method is a generalization of wavelet decomposition that offers a richer range of possibilities for signal analysis. In the wave packet analysis, the details as well as the approximations can be split. If \( n \) levels of decomposition are done, the transform will yield \( 2^n \) sub-bands. Fig. 4.3 shows the Wavepacket decomposition tree for \( n=2 \).

![Wave Packet decomposition tree](image)

4.1.5 Fast Wavelet Transform using Lifting Scheme

The DWT is a very computation intensive process. When data from multiple channels have to be processed, the hardware requirement can be huge. So a faster implementation method is very much desirable. The Lifting based implementation meets this requirement. It was developed by Wim Sweldons in 1997 as a method to improve a given WT to obtain some specific properties[6]. Later, it was extended to a generic method to create the so called second generation wavelets. The theory behind the classical wavelets relies heavily on Fourier Transform, while the lifting scheme can be used to introduce wavelets without using the concept of Fourier Transform. The main feature of the lifting scheme is that all decompositions are derived in the temporal domain. This leads to a more intuitively appealing treatment, better suited to those interested in applications.
It is fruitful to view the DWT as prediction-error decomposition. The scaling coefficients at a given scale are predictors for the data at the next higher resolution or scale (j-1). The wavelet coefficients are simply the “prediction errors” between the scaling coefficients and the higher resolution data. This interpretation has led to a new framework for the DWT known as the Lifting scheme (LS).

Suppose that the low-resolution part of a signal at level \(j+1\) is given, represented by \(s_{j+1}\). This set is transformed into two other sets at level \(j\): the low-resolution part \(s_j\) and the high-resolution part \(d_j\). This is obtained first by splitting the data set \(s_{j+1}\) into two data subsets. Traditionally, this is done by separating \(s_{j+1}\) into the set of even samples and odd samples. Such a splitting is sometimes referred to as the lazy wavelet transform. Each group contains half as many samples as the original signal.

Doing just this does not improve the signal representation. The even and odd samples are interspersed. If the signal has a local correlation structure, the even and odd subsets will be highly correlated. In other words, given one of the two sets, it should be possible to predict the other one with reasonable accuracy. The even set is always used to predict the odd one. The two subsets are then recombined in several lifting steps which decorrelate the two signals.

Lifting steps usually come in pairs of a primal and a dual lifting step. A dual lifting step can be seen as a prediction; the data \(d_j\) are predicted from the data \(s_j\). When the signals are highly correlated, such a prediction will be very good, and thus we need not keep this information in both signals. We need to store only that part of \(d_j\) that differs from its prediction (the prediction error). Thus \(d_j\) is replaced by \(d_j - P(d_j)\) where \(P\) represents the prediction operator. This is the real de-correlating step. However, the new representation has lost certain basic properties, which one usually wants to keep, like for example, the mean of the signal. To restore this property, one needs a primal lifting step, whereby \(s_j\) is updated with data from the new \(d_j\). Thus \(s_j\) is replaced by \(s_j + U(d_j)\) where \(U\) represents the updating operator. These steps can be repeated by iteration on \(s_j\), creating a multilevel transform or multi-resolution decomposition. So, as the lifting stage go from level \(j+1\) to level \(j\), the steps are summarized as follows. These three steps form a lifting stage (See fig.4.4).

1. Splitting (lazy wavelet transform) \(s_{j+1} \rightarrow\) odd samples \(d_j\) and even odd samples \(s_j\)
Review of Basic Theory

2. Prediction (dual lifting) \( d_j \leftarrow d_j - P(d_j) \).................................(4.13)

3. Update (primal lifting) \( s_j \Rightarrow s_j + U(d_j) \).................................(4.14)

The lifting scheme has a number of advantages:

a) All calculations can be performed in place resulting in memory savings

b) Computations are reduced since the sub expressions can be reused

![Lifting Scheme-Forward Transform](image)

Fig.4.4 – Lifting Scheme-Forward Transform

Iteration of the lifting stage on the output \( s(n) \) creates the complete set of DWT scaling and wavelet coefficients. By first factoring a classical wavelet filter into lifting steps, the computational complexity of the corresponding DWT can be reduced. The lifting steps can be easily implemented with ladder type structures, which is different from the direct finite impulse response (FIR) implementations of Mallat’s algorithm. Hence, this implementation will require lesser hardware resources while achieving higher utilization.

4.1.6 Desirable Features of Wavelet Transform for Transient Detection

The broadband nature and relatively short duration of transient signals demand a transform with variable time and frequency resolutions. This is an inherent feature of Wavelet transform. According to multi-scale filtering structure, Wavepacket transform can divide all the time-frequency plane into subtle tilings, while the classical WT can only find its finer
analysis for lower-band only. Hence Discrete Wave packet transform will be more competent to handle wide-band and high-frequency narrow band signals like transients. The results of applying DWT to the analysis of transients are elaborated in Chapter 7.

4.2 Fractional Fourier Transform (FrFT)

Chirps are signals which exhibit a change in instantaneous frequency with time (either linear or non-linear) and are of particular interest in sonar, radars, acoustic communications, seismic surveying, ultrasonic applications, etc. The potential of FrFT lies in its ability of FrFT to process chirp signals better than the conventional Fourier Transform. The transform absorbs the chirp parameters in its kernel by a parameter $\alpha$.

Namias introduced Fractional Fourier Transform[75] in the field of quantum mechanics for solving some classes of differential equations efficiently. Later, Ozaktas et al[76] came up with the discrete implementation of FrFT. Since then, a number of applications of FrFT have been developed, mostly in the field of optics. However, it remains relatively unknown in acoustics.

Little need to be said of the importance and ubiquity of the ordinary Fourier transform in many diverse areas of science and engineering. As a generalization of the ordinary Fourier transform, the FrFT is only richer in theory and more flexible in applications, but not more costly in applications. Therefore, the transform is likely to have something to offer in every area in which Fourier transforms and related concepts are used. The FrFT is basically a time-frequency distribution. It provides us with an additional degree of freedom(order of the transform), which in most cases results in significant gains over the classical Fourier transform. With the development of FrFT and related concepts, we see that the ordinary frequency domain is merely a special case of a continuum of fractional Fourier domains. So in every area in which Fourier transforms and frequency domain concepts are used, there exists the potential for improvement by using the FrFT.

4.2.1 Linear Chirp Signal

A linear chirp signal, its phase and its instantaneous frequency are given by the following equations. Two parameters completely define a chirp namely the start frequency $f_0$ and slope $a$ of the chirp.
**Review of Basic Theory**

**chirp signal**  \[ \Rightarrow e^{i(at^2 + f_0 t + c)} \] .................................(4.18)

**phase**  \[ \Rightarrow at^2 + f_0 t + c \] ......................................................(4.19)

**instantaneous frequency**  \[ \Rightarrow 2at + f_0 \] .............................................(4.20)

where

- \( f_0 \): start frequency of chirp,
- \( c \): initial phase,
- \( 2a \): chirp rate or slope

**4.2.2 Overview of FrFT**

FrFT is defined [87,88] with the help of transformation kernel \( K_\alpha \),

where

- \( \alpha \) \{0 ≤ \( \alpha \) ≤ 1.0\} defines the transform order.

\( X_\alpha (y) \) is the fractional transform of order \( \alpha \)

\[ X_\alpha(y) = F_\alpha[f(x)] = \int_{-\infty}^{\infty} K_\alpha(x,y) f(x)dx \] .................................(4.15)

\[ K_\alpha = \sqrt{\frac{1 - j \cot \phi}{2\pi}} e^{\frac{1}{2} j \phi \cot \phi} e^{-jxy \csc \phi + \frac{1}{2} jx^2 \cot \phi} \] .................................(4.16)

\[ : X_\alpha(y) = \sqrt{\frac{1 - j \cot \phi}{2\pi}} e^{\frac{1}{2} j \phi \cot \phi} \int_{-\infty}^{\infty} f(x) e^{-jxy \csc \phi + \frac{1}{2} jx^2 \cot \phi} dx \] .................................(4.17)

where \( \phi = \alpha \frac{\pi}{2} \)

FrFT computation can be interpreted as a sequence of steps viz. a multiplication by a chirp in one domain followed by a Fourier transform, then multiplication by a chirp in the transform domain and finally a complex scaling. So, chirps form the basis functions of FrFT.

There are various other definitions of the FrFT. Of all these, the definition given above is particularly desirable because of its many properties and the relation to the classical Fourier transform. It is also interesting to note that this definition of the FrFT reduces to the classical FT when the order of the transformation \( \alpha = 1 \). The variable \( x \) and \( y \) emphasize the generality of the transform, rather than assuming time and frequency for the domains. For \( \alpha = 1 \) and -1, the transform corresponds to ordinary forward and inverse Fourier Transforms respectively where \( x \) and \( y \) represent frequency and time respectively.
4.2.3 Transform Optimization

The FrFT parameter $\alpha$ is used to tune the transform to provide an optimal response to a given linear chirp signal. When the axis of rotation is matched to the chirp rate of the signal, the magnitude response of FrFT reaches its maximum. This procedure is known as transform optimization. The corresponding $\alpha$ is called the optimum $\alpha$.

\[ \phi' = 2\alpha t + f_0 \]

**Fig.4.5- Relationship of Chirp rate and FrFT order**

Fig.4.5 shows the time-frequency plot of a chirp. There are two methods to describe chirp rate. The first is the quadratic phase parameter $a$ in the algebraic definition of the linear chirp given in Eqn.(4.18). It is also given as the optimum $\alpha$ parameter in the FrFT definition in Eqn.(4.17). The relationship between the two is given as [87, 88]

\[ \alpha_{opt} = \frac{2\phi}{\pi} = \frac{2}{\pi} \tan^{-1}\left(\frac{1}{2a}\right) \]

(4.21)

\[ \alpha_{opt} = \frac{2}{\pi} \tan^{-1}\left(\frac{f_s^2}{N} \frac{\pi}{2a}\right) \]

(4.22)

The true relationship is dependent on the digital sampling scheme used and is given in Eqn.(4.22) where $f_s$ is the sampling frequency and $N$ is the number of samples in the chirp signal. This relation is used to calculate the optimal order for a sampled linear chirp signal with known chirp rate $a$. Conversely, it can be used to estimate chirp rate, given the FrFT
order $\alpha$. The optimum FrFT order cannot be found analytically in general. So, a one-dimensional search for $\alpha$ is necessary to find the optimum order, with which the chirp focuses well. i.e. On a given block of data, FrFT is done for different values of $\alpha$, and we select the one that yields the maximum peak value. We can scan values of $\alpha [-1,1]$ using a finer spacing to get a good estimate.

### 4.2.4 Properties of FrFT

1. **Linearity** $F_{\alpha}[c_1 f(t) + c_2 g(t)] = c_1 F_{\alpha} f(t) + c_2 F_{\alpha} g(t)$

2. **Identity** $F_{0}[f(t)] = f(t)$

3. FrFT reduces to Fourier transform when $\alpha=1$ i.e. $F_{1} f(t) = F(f)$

4. **Additivity** $F_{\alpha,n}[f(t)] = F_{\alpha} F_{n} f(t)$. Successive application of FrFT is equivalent to a single transform whose order is equal to the sum of the individual orders.

5. Rotation by $2\Pi$- FrFT of order $\alpha=4$ corresponds to successive application of Fourier transform 4 times and therefore acts as identity operator

6. Inverse FrFT is done by taking $\alpha=-1$

7. FrFT is both associative and commutative

8. **Time shift** – FrFT of a time shifted signal is a shifted version of FrFT of original signal modulated by a chirp function

   \[ x(t) \Leftrightarrow F_{\alpha}(y) \]

   \[ x(t-\tau) \Leftrightarrow e^{\frac{-1}{2}j\tau^2 \sin \phi \cos \phi - j\tau \sin \phi} F_{\alpha} (y - \tau \cos \phi) \]

9. **Modulation** in time domain results in corresponding modulation with a chirp and shift in FrFT domain

   \[ x(t) \Leftrightarrow F_{\alpha}(y) \]

   \[ x(t)e^{j\omega t} \Leftrightarrow e^{\frac{-1}{2}j\omega^2 \sin \phi \cos \phi + j\omega \sin \phi} F_{\alpha} (y - \omega \sin \phi) \]

10. **Time inversion** results in a corresponding inversion of FrFT

    \[ x(t) \Leftrightarrow F_{\alpha}(y) \quad x(-t) \Leftrightarrow F_{\alpha}(-y) \]

    If signal is an even function, its FrFT is also an even function

    \[ x(t) = x(-t) \Leftrightarrow F_{\alpha}(y) = F_{\alpha}(-y) \]

    If the signal is an odd function, then its FrFT is also odd.
\[ x(t) = -x(-t) \quad \Leftrightarrow \quad F_\alpha(y) = -F_\alpha(-y) \]

11. Scaling of axis – A compression of the time axis usually results in an expansion of the fractional axis with varying multiplying factors. Similarly, an expansion of time axis results in a compression of fractional axis.

12. Parseval’s Theorem – The energy preservation property holds good for FrFT just as for FFT

\[ \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(y)|^2 \, dy \]

### 4.2.5 Discrete Implementation of FrFT

A number of discrete implementations have been put forward. The most satisfactory ones, consistent with the important properties of index additivity, unitarity and reduction to DFT for unit order, are those implementations based on the discrete Hermite-Gaussian functions. To date, there is no fast algorithm for the exact computation of the discrete FrFT. However, a fast \( O(N \log N) \) algorithm has been proposed, which calculates an approximation to the discrete samples of the FrFT with sufficient accuracy for many applications[77].

### 4.2.6 Desirable Features of FrFT for Active and Intercept Sonar Processing

Chirps are not compact in the time or frequency domain. But, since chirps form the basis functions in FrFT, there exists an order for which it is compact in the FrFT domain. So, FrFT will improve solutions to problems where chirps signals are involved. Hence, FrFT is the ideal transform for processing chirp signals in active and intercept sonars. However, the algorithms for these two applications will be different. In the case of active sonar, the transmitted chirp signal is known a priori, and hence calculation of the optimum transform order \( \alpha \) is straightforward. However, in the case of intercept sonar, the received waveform is unknown. So, to apply FrFT, a search algorithm has to be implemented to find the optimum transform order. For the problem of multiple chirps overlapping in time and frequency, an extraction algorithm will be required. All these additional challenges have been addressed successfully in the new technique, developed in the present thesis work and are detailed in Chapter 6.
4.3 Wigner Ville Distribution (WVD)

WVD belongs to the class of quadratic TFMs, which also called energy distributions. In contrast with the linear TFMs which decompose the signal on elementary components, the purpose of the energy distributions is to distribute the energy of the signal over the two variables, time and frequency. The Wigner distribution was originally developed in the area of quantum mechanics, back in 1932 and was introduced by French scientist Ville 15 years later. It is now commonly known in the SP community as Wigner Ville Distribution and is defined as

\[ WVD_x(t, f) = \int s(t + \frac{\tau}{2}) s^*(t - \frac{\tau}{2}) e^{-j2\pi f \tau} d\tau \] ................................. (4.23)

This distribution satisfies a large number of desirable mathematical properties, as summarized in the next sub-section. In particular, the WVD is always real-valued, it preserves time and frequency shifts and satisfies the marginal properties. WVD possesses many useful properties and also has better resolution than STFT spectrogram. But it has one major drawback, the so called cross-term interference.

4.3.1 Properties of WVD

Main aim of any TFM is that it should bring out the signal's frequency changes over time. This is the most difficult part to satisfy. In addition, there are a number of additional desirable properties. Given below are the main properties of WVD.

1. Energy Conservation – Energy of a signal can be deduced from the squared modulus of either the signal or its Fourier transform.

\[ E_x = \int |x(t)|^2 \, dt = \int |X(f)|^2 \, df \] ................................. (4.24)

By integrating the WVD of \( x(t) \) all over the time and frequency plain will give the energy of \( x(t) \).

\[ E_x = \int_\infty^{\infty} \int_\infty^{\infty} W_x(t, f) \, dt \, df \] ................................. (4.25)

2. Marginal Properties
\[
\int_{-\infty}^{\infty} W_x(t,f) \, dt = |X(f)|^2 \quad \text{(4.26)}
\]
\[
\int_{-\infty}^{\infty} W_x(t,f) \, df = |x(t)|^2 \quad \text{(4.27)}
\]
Integration along time axis yields the total power spectrum. This is called the frequency marginal condition. Conversely, the integration along the frequency axis gives the instantaneous energy of the signal.

3. WVD is real-valued
\[
W_x(t,f) = W_x^*(t,f) \quad \text{(4.28)}
\]

4. WVD is time-shift invariant and frequency modulation invariant.
\[
y(t) = x(t-t_0) \Rightarrow W_y(t,f) = W_x(t-t_0,f) \quad \text{(4.29)}
\]
\[
y(t) = x(t) e^{j2\pi ft} \Rightarrow W_y(t,f) = W_x(t_0, f-f_0) \quad \text{(4.29)}
\]

5. Dilation covariance - WVD preserves dilations
\[
y(t) = \sqrt{k} x(kt); \quad k > 0 \Rightarrow W_y(t,f) = W_x(kt, \frac{f}{k}) \quad \text{(4.30)}
\]

6. Compatibility with filterings - If a signal \(y(t)\) is generated by convolving \(x(t)\) with filter \(h(t)\), WVD of \(y(t)\) is the time convolution of WVD of \(x(t)\) and WVD of \(h(t)\)
\[
y(t) = \int h(t-s) x(s) \, ds \Rightarrow W_y(t,f) = \int W_x(t-s,f) W(s,f) \, ds \quad \text{(4.31)}
\]

7. Wide-sense support conservation - If a signal has a compact support in time (respectively in frequency), then its WVD has the same compact support in time (respectively in frequency).
\[
x(t) = 0, \quad |t| > T \Rightarrow W_x(t,f) = 0, \quad |t| > T \quad \text{(4.32)}
\]
\[
X(f) = 0, |f| > B \Rightarrow W_x(t,f) = 0, |f| > B \quad \text{(4.32)}
\]

8. Instantaneous Frequency property - IF of a signal can be recovered from the WVD as its first order moment in frequency

9. Group delay property - GD of a signal can be recovered from the WVD as its first order moment in time
4.3.2 Cross-term Interference

WVD possesses many useful properties and also has better resolution than STFT spectrogram. But one main deficiency of the WVD is the so-called cross-term interference. When a signal has more than one component or contains noise, its WVD is not just sum of their respective WVDs. In addition, cross-terms also appear. For N individual components, the total number of cross-terms is N(N-1)/2. Because the cross-term usually oscillates and its magnitude is twice as large as that of auto-terms, it often obscures the useful time-dependent spectral patterns. For $s(t)=s_1(t)+s_2(t)$, the WVD is

$$WVD_s(t,f) = WVD_{s1}(t,f) + WVD_{s2}(t,f) + 2\text{Re}\{WVD_{s1,s2}(t,f)\} \quad (4.34)$$

In simple signals with two or three components, we can identify the cross-term interferences. But for real life signals containing many components and noise, the pattern of cross-terms, which usually overlap with auto-terms, will be more complicated. Consequently, the desired spectrum could be deceiving and confusing. It is these undesired terms that prevents the application of WVD, even though the WVD possesses many useful properties for signal analysis. How to reduce the cross-term interference without destroying the useful properties has been a topic of many studies.

4.3.3 Psuedo WVD (PWVD)

One method to reduce cross-terms is to apply a low pass filter $H(t,f)$ to the WVD\[^{15}\] as shown in Eqn.(4.35) ie.2D convolution of WVD of the analyzed signal $s(t)$ and 2D filter $H(t,f)$. Low pass filter performs a smoothing operation and hence the name smoothed WVD. Low pass filtering suppresses cross-terms. But it reduces the resolution. So, a trade-off needs to be made between the resolution and the degree of smoothing.

$$PWVD(t,f) = \iint WVD_s(x,y) H(t-x,w-y) \, dx \, dy \quad (4.35)$$

4.3.4 Desirable Features of WVD for Echo Characterization

Among all the TFMs, the Wigner Ville Distribution, is the most efficient representation, in giving the best resolution in both time and frequency and is independent of any analysis width. However, it is the least used one, mainly because of the problem of cross-terms. If the cross-term problem can be rectified by a suitable denoising technique, WVD is the ideal time-frequency method for echo characterization in sonars. A new technique
combining FrFT and WVD has been developed in this thesis work with very promising results.

4.4 Ambiguity Function

Ambiguity function is a TFM, having relevance wherever matched filtering is used, like radars and sonars. Basically, ambiguity function has two roles. The first one is in the evaluation of active sonar waveforms. Second, it is used in the matched filtering based detection processing of active sonars. These two functions of ambiguity function are explained in the following sections.

4.4.1 Detection in Active Sonars

Active sonar involves the transmission of an acoustic signal which, when reflected from a target, provides the sonar receiver with a basis for detection and estimation of its range and radial velocity. For the sake of continuity, detection in active sonars is repeated in this chapter also. The relation between the transmitted signal, echo, range and radial velocity are derived as follows [12]

\[
x(t) \quad \text{transmitted signal} \\
y(t) \quad \text{received signal} \\
R_0 \quad \text{initial range} \\
R \quad \text{range at time } t \\
v \quad \text{radial velocity} \\
R = R_0 + vt \\
y(t) = s(t - 2R_0/c) \quad \text{[without signal attenuation ]} \\
= s[(t-2R_0 + vt)/c \quad s[(1-2v/c)t-2R_0/c] = s[(1-\delta)t-\tau] \\
\]

where \( \delta = 2v/c \) - time scaling or Doppler parameter \( \tau = 2R_0/c \) - delay parameter

Therefore, the estimates of range and velocity can be obtained as a linear function of delay and Doppler (\( \delta \) and \( \tau \)) measurements. In modern sonars, \( \delta \) and \( \tau \) measurements are made by cross correlating overlapping segments of the incoming signal with a set of stored references. Each of the references is a replica of the transmitted signal that has been artificially time compressed. Enough of these references are employed to cover a range of expected target velocities. When detection is achieved, the elapsed time since transmission
provides the delay estimate. The Doppler parameter of the reference which results in maximum correlation is taken as the Doppler estimate. The optimum detector for a known signal in the background of white Gaussian noise is the correlation receiver, also called matched filtering. The range and radial velocity can be obtained by passing the received signal through an array of matched filters where each filter in the array is matched to a different target velocity. A sufficient number of filters are employed to span the range of probable target velocities. The output of each filter is then passed through a simple threshold detector. The output of the threshold detector peaks with a delay, which provides the range estimates. The estimated velocity is inferred from the filter of best match. The process is illustrated in fig.4.6.

4.4.2 Evaluation of Active Waveforms

The ambiguity function $|\chi(\tau, \delta)|^2$ is a 2-D function of correlator output power against range $\tau$ and Doppler frequency shift $\delta$. Let us consider an illustration of $|\chi(\tau, \delta)|^2$ versus $\tau$ and $\delta$. For a hypothetical signal, the resulting surface may appear as in fig.4.7(a). The detection threshold can be visualized as a plane parallel to the $\tau$ and $\delta$ axes, the presence of a target being indicated if a correlation point exceeds this value (fig.4.7b). The intersection of this threshold with $|\chi(\tau, \delta)|^2$ defines a contour within which a target cannot be located unambiguously (with a single pulse), since all $\tau, \delta$ combinations enclosed by the contour give rise to detections. This contour sketched for the hypothetical signal is called the ambiguity contour for the waveform s(t). Two-dimensional plot of an ambiguity contour $\tau$ versus $\delta$ is called an ambiguity diagram (fig. 4.7c).
The ambiguity diagram indicates, for a given waveform, the accuracy with which range and velocity can be measured. So, the resolution obtainable with a given waveform is defined as the height and width of the ambiguity diagram for that waveform, measured at zero range and zero velocity \([1]\), as shown in fig. 4.8.
So, performance of any active waveform can be got from its ambiguity function. Lot of work is going on in the design of new waveforms, with specific capabilities like reverberation resistance and so on. To evaluate them, a correct picture of their ambiguity functions need to be generated, which is only possible with the WB AF definition. The Ambiguity Functions using the narrow band assumption, though easy to calculate, may not give a true picture of the waveform’s capabilities. This difficulty has been resolved using Mellin transform, addressed in the section to follow. A fast implementation of WB AF using Mellin transform is elaborated in Chapter 9.

4.5 Mellin Transform

One of the main properties of Fourier transform is that it allows one to compare translated functions and to remove the translation factor. That is the case because the energy density spectrum, the absolute square of the Fourier transform, is insensitive to translation. The importance of this is that if we have two functions at different locations, the energy spectrum will tell us the inherent differences between the two, irrespective of the translation factor. If the two functions are the same, then absolute square of the two transforms will be the same. That is, if we have a function $x(t)$ and a translated version $x_{t_0}(t)=x(t+t_0)$, then their respective Fourier transform $X(f)$ and $X_{t_0}(f)$ are related by
\[ X_{\nu}(f) = e^{j2\pi \nu t} X(f) \] .................. (4.37)

Hence \[ |X_{\nu}(f)|^2 = |X(f)|^2 \]

Now, instead of translating the function, one attempts to magnify it. This requires a transform that will remove the magnification factor so that the inherent differences can be compared. In other words one must use a transform that is insensitive to scaling or magnification – the answer is Mellin transform. A brief mathematical treatment of Mellin transform is given here [4,5,6]. Given a function \( x(t) \) which is assumed to have energy for \( t > 0 \), the continuous Mellin transform is given by Eqn.(4.38)

\[ M_x(\beta) = \int_0^\infty x(t) t^{p-1} dt \quad t > 0 \quad \text{where} \quad p = \sigma - j\beta \] ........................ (4.38)

Converting the variable \( t \) into an exponential function \( e^z \), the above equation is written as

\[ t = e^z \quad \therefore dt = e^z \, dz \]

\[ M_x(\beta) = \int_{-\infty}^{\infty} x(e^z) (e^z)^{\sigma-j\beta-1} e^z \, dz \] .......................... (4.39)

\[ \therefore M_x(\beta) = \int_{-\infty}^{\infty} x(e^z) e^{-j\beta z} e^{\sigma z} \, dz = \int_{-\infty}^{\infty} \tilde{x}(z)e^{-j\beta z} e^{\sigma z} \, dz = FT[e^{\sigma z} \tilde{x}(z)] \]

This equation indicates that the Mellin transform is equivalent to the Fourier transform after the logarithmic conversion of the time variable \( t \).

### 4.6 Conclusion

In this chapter, the background theory of four TFMs has been elaborated. Certain desirable features of these TFMs, which make them ideal for sonar applications, are also brought out in this chapter. A perspective of some sonar functions have been given in the previous chapter. The objective of this research work has been to improve the performance of these sonar functions using the time-frequency methods elaborated in this chapter.

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