Chapter 1

Fréchet manifolds

In this chapter we will give some basic definitions and results that will be used in this thesis in subsequent chapters. Our discussion will be more geometric rather function analytic. Most of the definitions and other functional treatment of global analysis in the non-Banach setting, we refer to the interesting monograph [20]. Content of this chapter is influenced by [20], [15], [31], [10] and includes basic calculations of my articles [17], [18] and [19].

1.1 Fréchet space

1.1.1 Definition of Fréchet space and examples

Definition 1.1.1 (Fréchet Space). A Fréchet space is a complete Hausdorff metrizable locally convex topological vector space.

The topology on a locally convex space is metrizable if and only if it can be derived from countable semi-norms. Therefore if $E$ is a Fréchet space, we have countable semi-norms $\{\rho_n : n \in \mathbb{N}\}$ which generate topology on $E$.

Example 1.1.1. Trivial example is a Banach space.

Example 1.1.2. Loop space $L\mathbb{R} := C^\infty(S^1, \mathbb{R})$. For each $\gamma \in L\mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$, define

$$\|\gamma\|_k := \sum_{i=0}^{k} \sup_{t \in S^1} |\gamma^{(i)}(t)|.$$  

where $\gamma^{(i)}(t)$ denotes the $i$-th derivative of $\gamma$ at $t$. Each $\|\cdot\|_k$ is a semi-norm on $L\mathbb{R}$ (in-fact it is a norm). This countable collection gives the locally convex topology on $L\mathbb{R}$. 

This topology is metrizable and complete (§1.46, [29]). Therefore $L\mathbb{R}$ is a Fréchet space. It is worth mentioning here that $L\mathbb{R}$ is not normable [29].

Similarly $L\mathbb{R}^n := C^\infty(S^1, \mathbb{R}^n)$ is also a Fréchet space. Identify $S^1$ with $[0, 1]/\sim$ and elements of $I = [0, 1]$ by $t$. In this thesis we will use this identification frequently.

**Example 1.1.3.** Projective limit of Banach space: Suppose $\{E_i, \rho_{ji}\}_{i,j \in \mathbb{N}}$ be the projective system of Banach space. Let $E = \varprojlim E_i$. The projective limit topology on $E$ is given as following.

For each $n \in \mathbb{N}$, define a seminorm on $E$ by $\|(x_i)\|_n := \|x_n\|_n$. This countable collection of seminorms on $E$ makes $E$ a Fréchet space. It is worth mentioning here that the projective limit topology of countable collection is always complete [30].

In fact other way is also true. Suppose $E$ is a Fréchet space. Topology on $E$ can be realized by an increasing countable family of semi-norms [30].

$$\rho_1 \leq \rho_2 \leq \ldots.$$

$E$ can then be realized up to topological isomorphism as the projective limit of projective system $\{E_i; \rho_{ji}\}_{i,j \in \mathbb{N}}$, where $E_i$ is the completion of the quotient $E/\ker(\rho_i)$ and connecting morphism $\rho_{ji}(j \geq i)$ are given by

$$\rho_{ji}([x + \ker(\rho_j)]_j) = [x + \ker(\rho_i)]_i.$$

Here bracket denotes the corresponding equivalence classes.

In the sequel each point $x$ of $E$ will be considered as $(x_i)_{i \in \mathbb{N}}$ of its projections onto the Banach factors of the limit, with respect to the canonical mappings $f_i : E \to E_i$. If norms on each $E_i$ is denoted by $\|\cdot\|_i$. The construction allows us to consider each $f_i$ as an isometry in the sense that

$$\rho_i(x) = \|f_i(x)\|_i \quad x \in E$$

**Remark 1.1.1.** Here we fix the symbol $i$ and $j$ for the natural numbers. In the next whenever we will use $i$ and $j$ together we will mean by natural numbers $j \geq i$.

### 1.1.2 Bornology

Let $(E, \tau)$ be a locally convex topological vector space with the locally convex topology $\tau$. Bornology of $E$ is the the collection of all bounded set of $E$. The topology can vary considerably without changing the bornology. In the next section, we will see that it is not the topology but the bornology on which smoothness of a map depends.
The bornologification [30] $E_{\text{born}}$ of a locally convex space $(E, \tau)$ is the finest locally convex topology $\tau^*$ on $E$ having the same bornology.

A locally convex space is called bornological if it is stable under the bornologification. We have the following equivalent criterion for the bornological locally convex space.

1. A locally convex spaces $E$ is bornological if and only if every convex balanced set $V$ in $E$ which absorbs every bounded set $B$ (that is $B \subset tV$, for some $t > 0$) is a 0 - neighborhood.

2. A locally convex spaces $E$ is bornological if and only if each semi-norm on $E$ that is bounded on bounded sets, is continuous.

Every normed space is bornological since any set which absorbs the unit ball must contain the ball of some positive radius. It is a fact that inductive limit of a family of bornological space is bornological [30]. Thus a Fréchet space topology is bornological.

1.1.3 Dual of a Fréchet space

Let $E$ be a Fréchet space. Dual of $E$ is the set of all bounded linear map from $E \rightarrow \mathbb{R}$ with the topology of uniform convergence on bounded sets in $E$. It is a well known fact that the dual of a Fréchet is the Fréchet space if and only if Fréchet space is the Banach space.

Example 1.1.4. Dual of loop space $L\mathbb{R}^n$ defined as in Example 1.1.2 is the space of $\mathbb{R}^n$-valued distributions on the circle (§4.5 [31]).

1.2 Smooth map on Fréchet space

When one strays outside the realm of Banach spaces there are a lot of way to define the derivative. Even in Fréchet space there are three inequivalent way to define the derivative [20]. Therefore choice of the calculus is required to be fixed.

Let $E$ be a not normable locally convex space (for example the Fréchet space as in example 1.1.2) define

$$F : E' \times E \rightarrow \mathbb{R}$$

$$(f, e) \rightarrow f(e)$$

where $E'$ is the strong dual of $E$ as define in section 1.1.3. $F$ is not continuous when $E' \times E$ is given product topology [31]. In most of the applications (at least in our
situation) we want that this map to be a smooth map. So the problem is to define the suitable notion of smoothness.

Also the definition the smoothness in Fréchet space or even in a locally convex space should agree with the definition of smooth maps in Banach space situation and fits very well in our application.

Kriegl and Michor in [20] starts by defining the smooth curve in a locally convex space. Following [20], we can define the smoothness of a curve in the same way as in Banach space (§1.2,[20]). In this way, we have the collection of smooth curves on any Fréchet space.

For the locally convex space $E$, we have the following theorems.

**Theorem 1.2.1** (Theorem 2.14(4),[20]). The curve $c : \mathbb{R} \to E$ is smooth if and only if the curves $l \circ c : \mathbb{R} \to \mathbb{R}$ are smooth for all $l \in E^*$, continuous dual of $E$.

**Theorem 1.2.2** (Corollary 2.11, [20]). A linear map $l : E \to F$ between two locally convex vector spaces is bounded if and only if it maps smooth curves in $E$ to smooth curves in $F$.

Now following [20], we give the definition of a smooth map.

**Definition 1.2.1.** [§3.11,[20]] A function $f : U(\subset E) \to F$ defined on an open subset $U$ of $E$ is smooth if it takes smooth curves in $U$ to smooth curves in $F$.

**Remark 1.2.1.** With this definition, with the product topology on $E' \times E$, the evaluation map $F$ in equation 1.2.1 (being linear and bounded) is a smooth function, but it is not continuous in the product topology.

Therefore we see that smoothness of a map does not depend upon the topology of the space but it depends upon the bornology. For the Fréchet space the bornologificaton of Fréchet space is the same as Fréchet space topology. Therefore in the case of Fréchet spaces, with the above definition of a smooth curve and a smooth map, smooth map is always a continuous map. In the case of $E' \times E$, the product topology on $E' \times E$ does not make $E' \times E$ a Fréchet space.

The main benefit of the definition of smooth map as above, is having the crucial tool as following.

**Theorem 1.2.3** (Exponential law, Theorem 3.12 [20]). Let $U \subset E$ be a open subset of Fréchet space then $C^\infty(U_1 \times U_2, F) \approx C^\infty(U_1, C^\infty(U_2, F))$.

The derivative is given explicitly by following theorem.
Theorem 1.2.4 (Theorem 3.18,[20]). Let $E$ and $F$ be Fréchet space and $U \subset E$ be open set then the differential operator

$$ d : C^\infty(U, F) \to C^\infty(U, L(E, F)) $$

$$ df(x)(v) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} $$

exists and bounded(smooth). Also the chain rule holds

$$ d(f \circ g)(x)(v) = df(g(x))dg(x)v $$

Remark 1.2.2. In above theorem space $L(E, F)$ denote the space of all bounded linear mappings from $E$ to $F$. It is closed linear subspace of $C^\infty(E, F)$. Following ($§$3.17,[20]), a mapping $f : U \to L(E, F)$ is smooth if and only if the composite mapping $U \to L(E, F) \to C^\infty(E, F)$ is smooth.

In the following, we will discuss the smoothness for particular cases of Fréchet spaces.

1.2.1 Smooth maps on loop space $L\mathbb{R}^n$

Suppose $f : U \subset L\mathbb{R}^n \to E$ be a map from open subset of $L\mathbb{R}^n$ to a Fréchet space.

Suppose we know the collection $C^\infty(\mathbb{R}, L\mathbb{R}^n)$, then by the definition 1.2.1, $f$ is smooth if and only $f \circ c$ is smooth.

Let $c : \mathbb{R} \to L\mathbb{R}^n$ be a smooth curve, define $c^\lor : \mathbb{R} \times S^1 \to \mathbb{R}^n$, by

$$ c^\lor(t, s) := c(t)(s) $$

$c^\lor$ is called the adjoint of $c$. By theorem 1.2.3, we have the following theorem for the loop space which is a particular case of the exponential law.

Theorem 1.2.5 (proposition 3.7, [31]). A curve $c : \mathbb{R} \to L\mathbb{R}^n$ is smooth if and only if its adjoint $c^\lor : \mathbb{R} \times S^1 \to \mathbb{R}^n$ is smooth.

This is an easy criterion for checking smooth curve in the loop space and which will help to determine the smoothness of a map.

In chapter 2, we will use above theorem in proving smoothness of some particular maps.
1.2.2 Smooth maps on PLB space

Suppose \( \{E_i, \phi_{ji}\}_{i,j \in \mathbb{N}} \) and \( \{F_i, \rho_{ji}\}_{i,j \in \mathbb{N}} \) be the projective system of Banach spaces and \( E = \varprojlim E_i \) and \( F = \varprojlim F_i \). \( E \) and \( F \) are Fréchet spaces.

**Definition 1.2.2.** [Projective system of mapping]. We say \( \{f_i : E_i \to F_i\}_{i \in \mathbb{N}} \) is a projective system of mapping if the following diagram commutes.

\[
\begin{array}{ccc}
E_j & \xrightarrow{f_j} & F_j \\
\downarrow{\phi_{ji}} & & \downarrow{\rho_{ji}} \\
E_i & \xrightarrow{f_i} & F_i
\end{array}
\]

We denote the canonical mapping of \( E \to E_i \) by \( e_i \) and \( F \to F_i \) by \( e'_i \).

**Definition 1.2.3.** We say \( f : E \to F \) is the projective limit of a system \( \{f_i : E_i \to F_i\}_{i \in \mathbb{N}} \) if for each \( i \), the following diagram commutes.

\[
\begin{array}{ccc}
E_i & \xrightarrow{f_i} & F_i \\
\downarrow{e_i} & & \downarrow{e'_i} \\
E & \xrightarrow{f} & F
\end{array}
\]

We define the map \( \varprojlim f_i \) as following.

If \( \{f_i\} \) is the projective system of mappings then we see that for any \( x = (x_i) \in \varprojlim E_i \), \( f_i(x_i) \in \varprojlim F_i \). We define \( (\varprojlim f_i)(x) = (f_i(x_i)) \in F \). If \( f \) be the projective limit of system \( \{f_i\} \) then we have\( f(x) = (f_i(x_i)) \).

We denote the projective limit of system \( \{f_i\} \) by \( \varprojlim f_i \). Also \( (\varprojlim f_i)(x) = (f_i(x_i)) = f(x) \).

We are interested in knowing the criterion of checking smoothness of the map

\[ f : E \to F \] such that \( f = \varprojlim f_i \).

G. Galanis has given the following criterion.

**Theorem 1.2.6 (Lemma 1.2,[10]).** Suppose \( E = \varprojlim E_i \) and \( F = \varprojlim F_i \) and \( \{f_i : E_i \to F_i\}_{i \in \mathbb{N}} \) be a projective system of smooth mapping then the following holds.

1. \( f \) is \( C^\infty \), in the sense of J. Leslie.[22]

2. \( df(x) = \varprojlim df_i(x_i), \ x = (x_i) \in E \).
3. \( df = \lim_{\leftarrow} df_i. \)

Following [20], we already defined a smooth map between Fréchet space (definition 1.2.1). In PLB-space if a map \( f = \lim_{\leftarrow} f_i \) (as in theorem 1.2.6) is smooth in the sense of J. Leslie then it will be smooth in the sense of Kriegl and Michor too. This can be seen as following.

Let \( c : \mathbb{R} \to \lim_{\leftarrow} U_i \subset E \) be a smooth curve and \( f := \lim_{\leftarrow} f_i : \lim_{\leftarrow} U_i \to E \) be a map which is smooth in the sense of J. Leslie (i.e \( f \) satisfies theorem 1.2.6). We can identify \( c(t) \) as \( c(t) = (c_i(t)) \) where \( \phi_{ji}(c_j(t)) = c_i(t) \). \( c \) is smooth if and only each \( c_i \) is smooth (here we are using the fact that \( \pi_i : \lim_{\leftarrow} E_i \to E_i \) is a smooth map and \( c_i = \pi_i \circ c \)).

Now let \( \tilde{c} : \mathbb{R} \to \lim_{\leftarrow} U_i, \) defined by \( \tilde{c}(t) = (f \circ c)(t) \), we see that

\[
\tilde{c}(t) = \lim_{\leftarrow} (f_i \circ c_i)(t)
\]

As each \( f_i \circ c_i \) is smooth, the derivative of every order exists. Therefore by theorem 1.2.6, \( \tilde{c} \) is smooth in the sense of J. Leslie. Recall that smoothness of curves is defined by in the same way by J. Leslie and by Kriegl and Michor. This proves \( f \circ c \) is smooth curve for every smooth curve \( c \).

Therefore \( f \) defined as in theorem 1.2.6 is smooth in the sense of Kriegl and Michor too.

### 1.3 Fréchet manifolds

**Definition 1.3.1** (Fréchet manifold). A Fréchet manifold is a set \( \mathcal{M} \) together with a smooth structure represented by an atlas \( (U_\alpha, u_\alpha)_{\alpha \in A} \) such that the canonical topology on \( \mathcal{M} \) with respect to this structure is Hausdorff.

As usual, charts are bijections from open subsets of \( \mathcal{M} \) to open subsets of a fixed Fréchet space. An atlas is maximal cover of \( \mathcal{M} \) by charts, where all transition functions are defined on open subsets and are required to be smooth.

**Definition 1.3.2** (Smooth map). A map \( f : \mathcal{M} \to \mathcal{N} \) of a Fréchet manifolds is said to be smooth at \( p \in \mathcal{M} \) if it is smooth in one and hence all pair(s) of charts around \( p \) and \( f(p) \). The map is smooth if it is smooth at all points of \( \mathcal{M} \).

We have following proposition which helps in checking smooth map.

**Proposition 1.3.1.** (§27.2,[20]) \( f \) is smooth if and only if \( f \circ \gamma \) is smooth for every smooth curve \( \gamma : \mathbb{R} \to \mathcal{M} \).
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We will require Fréchet manifolds to be smoothly Hausdorff (smooth functions separate points). One can prove (§16.10, [20]) that each Fréchet manifold is smoothly paracompact, that is, each open cover admits a smooth partition of unity subordinated to it. We will discuss more about smooth partition of unity in chapter 2.

In next sections, the symbol $\mathcal{M}$ is fixed for infinite dimension manifold (Banach or Fréchet).

### 1.3.1 Loop space as Fréchet manifold

Loop space $L\mathbb{R}^n$ is a Fréchet space and we discussed the space of smooth curve $C^\infty(\mathbb{R}, L\mathbb{R}^n)$ in section 1.2.1. In this section we will give a manifold structure on the loop space $LM$, where $M$ is a finite dimension oriented manifold.

In this thesis whenever we will discuss about the loop space, we will mean by loop space over a finite dimension smooth manifold.

We start with a proposition.

**Proposition 1.3.2.** *Any oriented vector bundle over $S^1$ is trivial.*

The main point is that up to isomorphism, every real vector bundle over the circle is either trivial or the Whitney sum of a trivial bundle with the Mobius bundle. The latter is not orientable.

**Definition 1.3.3** (Local addition). A local addition on $M$ consists of a smooth map $\eta : TM \to M$ such that

1. The composition of $\eta$ with the zero section is the identity on $M$.
2. There exists an open neighborhood of $V$ of the diagonal of $M \times M$ such that $\pi \times \eta : TM \to M \times M$ is diffeomorphism onto $V$. Here $\pi : TM \to M$ is the projection map.

For any $[(p, v)] \in TM$, $\pi \times \eta([(p, v)]) = (\pi([(p, v)]), \eta([(p, v)])) = (p, \eta([(p, v)]))$. We have following proposition.

**Proposition 1.3.3 ([25],[31]).** *For any finite dimension smooth manifold $M$, local addition always exits.*

Let $\eta : TM \to M$ be a local addition. Let $V \subset M \times M$ be the image of the map $\pi \times \eta : TM \to M \times M$, where $V$ is an open neighborhood of diagonal in $M$. By the definition of local addition $\pi \times \eta$ is a diffeomorphism on to $V$. 
Suppose $\alpha \in LM$, we define $(U_\alpha, \Psi_\alpha)$ as a coordinate chart around $\alpha$, where $U_\alpha$ is defined as following:

$$U_\alpha := \{ \beta \in LM : (\alpha, \beta) \in LV \} \subset LM$$

The pre-image of $\{\alpha\} \times U_\alpha$ under $\pi \times \eta^L$ is naturally identified with $\Gamma^s_1(\alpha^*TM)$.

$\Psi_\alpha : \Gamma^s_1(\alpha^*TM) \rightarrow U_\alpha$ is defined as following. Let $\beta \in \Gamma^s_1(\alpha^*TM)$ and $\tilde{\beta}$ be the corresponding loop in $TM$, so $\beta(t) = (t, \tilde{\beta}(t))$. Then we have $(\pi \times \eta)^L(\tilde{\beta}) = (\alpha, \eta^L(\tilde{\beta}))$. Now define

$$\Psi_\alpha := \eta^L(\tilde{\beta})$$

This gives a chart for loop space around $\alpha \in LM$.

For the full discussion we refer to various article [20],[25] etc.

**Proposition 1.3.4 ([31]).** With the atlas consisting charts as above and with the manifold topology on the loop space $LM$, $LM$ is Hausdorff, regular, second countable and paracompact.

Thus the loop space $LM$ is a Fréchet manifold.

### 1.3.2 PLB manifold

We say $M$ is the projective limit of Banach manifolds (PLB-Manifold) modeled on a PLB-space $E = \varprojlim E_i$ if we have followings.

1. There is a projective system of Banach manifolds $\{M_i, \phi_{ji}\}_{i,j \in \mathbb{N}}$ such that $M = \varprojlim M_i$.

2. For each $p \in M$, we have $p = (p_i)$. $p_i \in M_i$, and there is a chart $(U_i, \psi_i)$ of $p_i \in M_i$ such that

   a) $\phi_{ji}(U_j) \subset U_i$, $j \geq i$.
   
   b) Let $\{E_i, \rho_{ji}\}_{i,j \in \mathbb{N}}$ be a projective systems of Banach spaces, where each $\rho_{ji}$ is inclusion map and the diagram

   $\begin{array}{ccc}
   U_j & \xrightarrow{\psi_j} & \psi_j(U_j) \\
   \downarrow{\phi_{ji}} & & \downarrow{\rho_{ji}} \\
   U_i & \xrightarrow{\psi_i} & \psi_i(U_i)
   \end{array}$

   commutes.
(c) \( \lim \psi_i(U_i) \) is open in \( E \) and \( \lim U_i \) is open in \( M \) with the inverse limit topology. The space \( M \) satisfying the above properties has a natural Fréchet manifold structure. The differential structure on \( M \) is determined by the co-ordinate map \( \psi : U = \lim U_i \to \psi(U) = \lim \psi_i(U_i) \). Therefore a smooth structure on these type of manifolds is completely determined by the smooth structure on the sequence. George Galanis in [10, 11, 12, 13] has studied this type of manifolds and smooth structure on it.

G. Galanis, in a series of articles [10, 11, 12, 13] discussed various properties of PLB-manifolds. H. Omri [27] discussed the ILB-manifolds (similar to PLB manifolds) and ILB-normal manifolds (here ILB stands for inverse limit of Banach). A PLB-manifold is the projective limit of Banach manifolds in general not only modeled over ILB space in contrary Omri-strong ILB-manifolds.

In view of discussion in section 1.2.2, we see that the calculus on PLB-manifolds agrees with the Kriegl and Michor calculus.

In particular we have the following proposition.

**Proposition 1.3.5.** Let \( E \) be a Fréchet space and \( E = \lim E_i \), \( E_i \)'s are Banach spaces. The co-ordinate map \( \psi_i \) as defined above, \( \psi_i : U = \lim U_i \to \psi(U) = \lim \psi_i(U_i) \) is smooth in the sense of Kriegl and Michor [20].

**Proof.** We have to check that \( c : \mathbb{R} \to \lim U_i \) is smooth if and only if \( \tilde{c} : \mathbb{R} \to \lim E_i \) defined by

\[
\tilde{c}(t) := (\lim \psi_i) \circ c(t) = \lim (\psi_i \circ c_i)(t)
\]

is smooth. But this follows by the remark after theorem 1.2.6. \( \square \)

Summarizing above, PLB manifolds are Fréchet manifolds and fit with Kriegl and Michor calculus.

### 1.3.3 Tangent bundle

Let \( p \in E \) be a point in a Fréchet space \( E \). The kinematic tangent space \( T_p E \) of \( E \) at \( p \) is the set of all pairs \((p, X)\) with \( X \in E \). Equivalently, \( T_p E \) is the set of equivalence classes of smooth curves through \( p \), where \( \gamma_1 \sim \gamma_2 \) if both have the same derivative at \( p \). Each tangent vector \( X \in T_p E \) yields a continuous (hence bounded) derivation \( X : C^\infty(E \supset \{p\}, \mathbb{R}) \to \mathbb{R} \) on the germs of smooth functions at \( p \).

In the general Fréchet space, it is not true that each such derivation comes from a tangent vector. However if \( E \) is a nuclear Fréchet reflexive space (§28.7, [20]), \( T_p E \) does coincide with the set derivations on the stalk \( C^\infty(E \supset \{p\}, \mathbb{R}) \).
If Fréchet manifold is not modeled over nuclear Fréchet space then there are two notion of vector which may not agree. We call the space of tangent vector at a point \( p \) as Kinematic tangent space. We call the set of all derivation at a point as Operational tangent space.

Let \( M \) be the Fréchet manifold with a smooth atlas \( (M \supset U_{\alpha} \rightarrow u_{\alpha})_{\alpha \in A} \). We define the kinematic tangent bundle \( T_{\mathcal{M}} \) of a Fréchet manifold \( \mathcal{M} \) to be the quotient of the disjoint union \( \bigcup_{\alpha} \{\alpha\} \times U_{\alpha} \times E \) by the equivalence relation

\[
(\alpha, p, X) (\beta, q, Y) \iff p = q, \quad d(u_{\alpha\beta})(u_{\beta}(p))(Y) = X,
\]

where the \( u_{\alpha} : M \supset U_{\alpha} \rightarrow E \) denote the charts, and \( u_{\alpha\beta} = u_{\alpha} \circ u_{\beta}^{-1} \) transition function of the manifold.

We will denote the kinematic tangent bundle of a Fréchet manifold \( \mathcal{M} \) by \( T_{\mathcal{M}} \). Kriegl and Michor defined the operational tangent bundle \( D_{\mathcal{M}} \) in \((\S 28.12, [20])\). In our application (see section 1.4.4) we will need only kinematic tangent bundle therefore in next, by tangent bundle we will mean by kinematic tangent bundle.

**Tangent bundle of the loop space**

\( \Gamma_{S^1}(\gamma^*(TM)) \) seems trivially the tangent space at \( \gamma \) because chart map at \( \gamma \) identifies it. But this is not the case. There are many possibility of chart maps because of various choices of local addition. Hence we do not have a canonical choice for the tangent space.

Andrew Stacey in \([31]\) showed that the tangent bundle \( TLM \) of the loop space \( LM \) has the structure of a bundle of \( L\mathbb{R} \)-modules \((\S 4.1, [31])\). Using this fact further he proved in \([31]\) that \( TLM \) and \( LTM \) are diffeomorphic, covering the identity on \( LM \) \((\S 4.1,[31])\). This proves that the tangent space (kinematic tangent space) at the point \( \gamma \in LM \)

\[
T_{\gamma}LM \approx \Gamma_{S^1}(\gamma^*(TM)) \equiv L\mathbb{R}^n.
\]

**Tangent bundle of a PLB manifold**

We will follow \([14],[1]\) for the following discussion on the tangent bundle of a PLB manifold. If \( \{M_i, \phi_{ji}\} \) be a projective system of Banach manifolds and \( M = \varprojlim M_i \) is the PLB manifold for this projective system. For \( p \in M = \varprojlim M_i \), we have \( p = (p_i) \). We observe that \( \{T_p M_j, T_p \phi_{ji}\} \) is a projective system of Banach spaces \( T_p \phi_{ji} \) is the usual Banach space derivative of \( \phi_{ji} \) at point \( p_j \). The identification \( T_p M \approx \varprojlim T_p M_i \) is given by the mapping \( h := \varprojlim T_p \phi_i \), where \( \phi_i \) are the canonical projection of \( M \). We refer to \([14]\) or \([1]\) for the proof.
Galanis proved in [14] that \( \{ TM_i, T\phi_{ji} \} \) is a projective system of Banach manifolds and \( TM \) is a PLB manifold and

\[
TM \simeq \varprojlim TM_i \text{ by } g = \varprojlim T\phi_i.
\]

Remark 1.3.1. The strong dual of a Fréchet space need not be metrizable. If we consider the strong dual of \( T_pM \) as the cotangent space, we would drop out of the Fréchet space category. In order to avoid this we will consider tensors not as sections of a certain bundle, but simply as smooth, fiberwise multilinear maps \( A : TM \times_M \cdots \times_M TM \to E \) with \( \pi \circ A = \text{id} \) and \( \pi : E \to M \) a vector bundle over \( M \).

1.4 Differential geometry on Fréchet manifold

1.4.1 Vector fields

As we discussed in section 1.3.3, for Fréchet spaces in general there are two types of tangent space, the kinematic tangent space and the operational tangent space. Therefore vector fields on a general Fréchet manifold \( M \) are of two types.

A kinematic vector field \( X \) on \( M \) is a smooth section of the kinematic tangent bundle \( TM \to M \). The space of all kinematic vector fields will be denoted by \( \mathfrak{X}(M) \).

By an operational vector field we mean a bounded derivation of the sheaf \( C^\infty(\cdot, \mathbb{R}) \). That is for an open set \( U \subset M \) we are given bounded derivations \( X_U : C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}) \) commuting with the restriction mappings. This can be identified with the smooth sections of the operational tangent bundle (§32.2,[20]). Denote the space of all operational vector fields by \( \text{Der}(C^\infty(M, \mathbb{R})) \).

For a reflexive nuclear Fréchet space both notion of vector field agrees ([20]). But in general we have the following proposition:

**Proposition 1.4.1** (Lemma 32.3,[20]). There is a natural embedding of convenient vector spaces

\[
\mathfrak{X}(M) \to \text{Der}(C^\infty(M, \mathbb{R}))
\]

For our purpose (for example while working on 2 form on a Fréchet manifold) we need only kinematic vector field. For example in section 1.4.4 we will see that for defining the differential form we need only kinematic vector fields.
Vector field (kinematic) on the loop space $LM$

A vector field $X$ over $LM$ is defined as a smooth section of $TLM$. We denote the collection of all vector fields on $LM$ by $\mathfrak{X}(LM)$. There are vector field on the loop space which arise from the base manifold and there are other type of vector fields which do not arise in this way.

Following Biswas and Chatterjee [3], we define a special vector field on the loop space $LM$ as following:

**Definition 1.4.1.** A vector field $\xi$ on $LM$ is said to be the vector field on $LM$ associated to a vector field $K$ on $M$, if

$$ev_t(\xi(\gamma)) = (K)_{\gamma(t)}, \forall \gamma \in LM, t \in [0, 1]$$

(1.4.1)

where for each $t \in [0, 1]$, $ev_t : LM \to M$, defined by $ev_t(\gamma) = \gamma(t)$, is a smooth map.

Let $\mathfrak{X}'(LM)$ is a collection of all vector fields $\xi$ on $LM$ such that there is a vector field $K$ of $M$ and which satisfies 1.4.1.

Many physicists ([3],[6],[7]) use the above collection $\mathfrak{X}'(LM)$ as a definition of vector field on the loop space. But below we will see that, in the manifold structure on $LM$ given as in section 1.3.1, $\mathfrak{X}'(LM)$ is not same as $\mathfrak{X}(LM)$.

**Example 1.4.1.** Define $X(\gamma) = (\gamma, \gamma')$ from $LR^n \to TLR^n = LR^n \times LR^n$. This is a bounded linear map and hence smooth. Thus this is an example of a vector field on $LR^n$.

For this vector field, it is trivial to see that $X \notin \mathfrak{X}'(LR^n)$. Therefore $\mathfrak{X}'(LM) \subset \mathfrak{X}(LM)$.

### 1.4.2 Flow of a vector field

In this section we will discuss the existence of the flow of a kinematic vector field.

Let $c : J \to M$ be a smooth curve in a manifold $M$ defined on an interval $J$. It will be called an integral curve or flow line of a kinematic vector field $X \in \mathfrak{X}(M)$ if $c'(t) = X(c(t))$ holds for all $t \in J$. For a Fréchet manifold, the flow line of a vector field may not exist (page 330, [20]).

Let $X \in \mathfrak{X}(M)$ be a kinematic vector field. A local flow $F$ for $X$ is smooth mapping $F : \mathcal{U} \subset M \times \mathbb{R} \to M$ defined on an open neighborhood $\mathcal{U}$ of $M \times \{0\}$ such that

1. $\mathcal{U} \cap (\{x\} \times \mathbb{R})$ is a connected open interval.
2. If $F(s, x)$ exists then $F(t + s, x)$ exists if and only if $F(t, F(s, x))$ exits and we have the equality.

3. For each $x \in U$, $\gamma_x(t) := F(t, x)$ is the integral curve of $X$ passing through a point $x$ at $t = 0$.

Suppose $Y : U \to E$ be a vector field on an open subset $U$ of a Fréchet space $E$. For ensuring existence of the flow of a vector field on a Fréchet space we can go either in direction set by H. Omri as in [27] or we can demand for some kind of tame condition as in [15]. For a general Fréchet space, the flow of a vector field may not exist for example see [20].

The situation become simpler while working on some special vector fields on $PLB$-manifolds. In rest of this section we will use the notation of example 1.1.3.

**Definition 1.4.2** (Projective $\mu$-Lipschitz map). Let $E = \varprojlim E_i$ be a Fréchet space. A mapping $\phi : E \to E$ is called projective $\mu$-Lipschitz ($\mu$, a positive real number) if there are $\phi_i : E_i \to E_i$ such that $\phi = \varprojlim \phi_i$ and for every $i$, $\phi_i$ is a $\mu$ Lipschitz map on each $E_i$.

We have a theorem below which will be helpful in determining existence of the flow.

**Theorem 1.4.2.** Let $E = \varprojlim E_i$ be a Fréchet space and suppose $X : E \to E$ is a projective $\mu$ Lipschitz map such that each $X_i$ is smooth map on $E_i$ and $X = \varprojlim X_i$. Suppose for each $i$,

$$M := \sup \{ \rho_i(X(x)) : i \in \mathbb{N}, \ x \in E \} < +\infty$$

Then there is a unique $C^\infty$ curve $x(t)$ defined on $\mathbb{R}$ such that

$$x'(t) = X(x(t)), \ x(0) = x_0.$$  \hspace{1cm} (1.4.2)

Above theorem is a version of theorem proved by G. Galanis (Theorem 3, [13]). The proof below is motivated from [13].

**Proof.** $\{X_i\}_{i \in \mathbb{N}}$ be a family of smooth map realizing $X$. Equation 1.4.2 gives a system of ordinary differential equations on the Banach spaces $E_i$ defined by

$$x_i'(t) = X_i(x_i(t)), \ x_i(0) = x_i^0$$  \hspace{1cm} (1.4.3)

where $x_0 = (x_i^0)$. It is given that each $X_i$ is $\mu$-Lipschitz and $\|X_i(x_i)\|_i < M$. Therefore by (§4.1, [23]), a unique smooth solution can be defined for each equation 1.4.3 on $\mathbb{R}$. 

These solution are related. For any \( j \geq i \), we have
\[
(\rho_{ji} \circ x_j)'(t) = \rho_{ji}(x_j'(t)) = \rho_{ji}(X_j(x_j(t))) = X_i(x_i(t)).
\]
We refer to (theorem 3, [13]) for the above calculation.

Both \( \rho_{ji} \circ x_j \) and \( x_i \) emanate from the same initial point. Therefore they coincide, as a result, the mapping \( x = \lim_{\leftarrow} x_i \) is defined on the \( \mathbb{R} \). Following (theorem 3, [14]), we see that \( x \) is unique desired solution of the given differential equation 1.4.2. Each \( x_i \) are smooth curve, therefore \( x := \lim_{\leftarrow} x_i \) is a smooth curve (see 1.2.2).

**Theorem 1.4.3.** Let \( X : E \to E \) be a smooth vector field on \( E \) as in theorem 1.4.2. Then for every \( y \in E \) there is a unique integral curve \( t \to x^y(t) \in E \) defined on \( \mathbb{R} \) such that \( x^y(0) = y \). Also the map \( F : \mathbb{R} \times E \to E \) defined by
\[
F(t, p) := x_p(t)
\]
is a smooth map.

Proof given below is taken from [23], [13] and [5].

**Proof.** Theorem 1.4.2, implies that for every \( y \in E \), there exists a unique curve
\[
x^y : \mathbb{R} \to E \text{ such that } x^y(0) = y.
\]
In fact \( x^y = \lim_{\leftarrow} x^y_i \) (\( \lim_{\leftarrow} x^y_i \) is well defined as we saw in the theorem 1.4.2, where \( x^y_i : \mathbb{R} \to E_i \) be the integral curves passing through \( y_i \) of corresponding vector fields \( X_i \)).

For each \( i \) define \( F_i : \mathbb{R} \times E_i \to E_i \) such that \( F_i(t, p_i) = x^p_i(t) \) and \( F : \mathbb{R} \times E \to E \) such that \( F(t, p) = x^p(t) \).

\( \{F_i\} \) makes projective system of map with projective limit \( F \). That is we have \( F = \lim_{\leftarrow} F_i \). Authors in [23] showed that each \( F_i \) is smooth map. This proves that \( F \) is a smooth map.

\[\square\]

### 1.4.3 Lie Bracket

Let \( X, Y \in \mathfrak{X}(\mathcal{M}) \) where \( \mathfrak{X}(\mathcal{M}) \) is the collection of kinematic vector field. Define a map
\[
f \to X(Y(f)) - Y(X(f)).
\]
This is a bounded derivation of sheaf \( C^\infty(\cdot, \mathbb{R}) \). We denote it by \([X, Y]\).
In general Fréchet space where two notions of vector fields (kinematic and operational) do not agree, it is not obvious that \([X,Y]\) is a kinematic vector field. Though by definition \([X,Y]\) is an operational vector field.

We mention here a theorem in (§32.8 [20]). Theorem states that the bounded derivation \([X,Y] \in \mathfrak{X}(\mathcal{M})\) whenever \(X,Y \in \mathfrak{X}(\mathcal{M})\). We call this map as Lie bracket of \(X\) and \(Y\).

### 1.4.4 Differential form and de Rham cohomology

Space of \(k\)-differential forms on \(\mathcal{M}\) is the closed linear subspace of \(C^\infty(\mathcal{T}\mathcal{M} \times_\mathcal{M} \ldots \times_\mathcal{M} T\mathcal{M}, \mathbb{R})\) consisting of all fiber-wise \(k\)-linear alternating smooth functions in the vector bundle structure \(T\mathcal{M} \oplus \ldots \oplus T\mathcal{M}\). We denote this space by \(\Omega^k(\mathcal{M})\).

For example, a 2-form \(\omega \in \Omega^2(\mathcal{M})\) is a fiberwise bilinear alternating smooth function in the vector bundle structure \(T\mathcal{M} \oplus T\mathcal{M}\). \(T\mathcal{M} \oplus T\mathcal{M}\) has a vector bundle structure (§29.4,[20]) as in the case of finite dimension manifold. Therefore for each \(p \in \mathcal{M}\), \(\omega_p\) (as a bilinear map) is a bounded map (being smooth map, we refer section 1.2).

As we discussed that there are two types of tangent bundle, kinematic tangent bundle \((\mathcal{T}\mathcal{M})\) and operational tangent bundle \((\mathcal{D}\mathcal{M})\). In view of these two types of tangent bundles there are many ways to define differential forms which agree with the usual differential form in the finite dimensional case.

Kriegl and Michor (§33, [20]) showed that there are 12 classes of possible differential form. But out of these there is only one (defined above) which satisfies all the useful identities as in the finite dimensional case.

With this definition of the differential form all the important mappings are defined in usual way and smooth:

\[
\begin{align*}
\text{d} : \Omega^k(\mathcal{M}) & \to \Omega^{k+1}(\mathcal{M})(\text{exterior derivative, §33.12 [20]}). \\
\text{i} : \mathfrak{X}(\mathcal{M}) \times \Omega^k(\mathcal{M}) & \to \Omega^{k-1}(\mathcal{M})(\text{insertion operator, §33.10 [20]}). \\
\mathcal{L} : \mathfrak{X}(\mathcal{M}) \times \Omega^k(\mathcal{M}) & \to \Omega^k(\mathcal{M})(\text{Lie derivative, §33.17 [20]}). \\
f^* : \Omega^k(\mathcal{M}) & \to \Omega^k(\mathcal{N})(\text{pull back operator, §33.9 [20]})
\end{align*}
\]

Following (§34,[20]), for a Fréchet manifold \(\mathcal{M}\) consider a graded algebra

\[
\Omega(\mathcal{M}) = \bigoplus_{k=0}^{\infty} \Omega^k(\mathcal{M})
\]

\[
d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^{deg(\phi)} \phi \wedge d\psi.
\]

Define

\[
H^k_{\text{DR}}(\mathcal{M}) = \frac{\{\omega \in \Omega^k(\mathcal{M}) : d\omega = 0\}}{\{d\phi : \phi \in \Omega^{k-1}(\mathcal{M})\}}
\]

(1.4.4)
$H^k_{DR}(\mathcal{M})$ is called the $k$-th de-Rham cohomology of $\mathcal{M}$.

1.5 Symplectic geometry on a Fréchet manifold

1.5.1 Weak symplectic structure

**Definition 1.5.1.** A 2 form $\sigma \in \Omega^2(\mathcal{M})$ is called a weak symplectic form on $\mathcal{M}$ if it is closed ($d\sigma = 0$) and if the associated vector bundle homomorphism $\sigma^b : T\mathcal{M} \to T^*\mathcal{M}$ is injective.

**Definition 1.5.2.** A 2 form $\sigma$ on a Banach manifold is called a strong symplectic if it is closed ($d\sigma = 0$) and its associated vector bundle homomorphism $\sigma^b : T\mathcal{M} \to T^*\mathcal{M}$ is invertible with smooth inverse.

In the case of strong symplectic Banach manifold, the vector bundle $T\mathcal{M}$ has reflexive fibers $T_x\mathcal{M}$. For a Fréchet manifold with the strong topology $T^*_x\mathcal{M}$ is not topologically isomorphic to $T_x\mathcal{M}$. Hence for our situation that is for the case of the loop space and for the case of PLB manifold, there is only weak symplectic structure. There are no strong symplectic structure on these.

We will discuss a symplectic structure on the loop space in chapter 2. In chapter 3, we will discuss a weak symplectic structure on the PLB manifold.

1.5.2 Symplectic cohomology defined by Kriegl and Michor

By the symplectic cohomology, we mean the definition given by Kriegl and Michor in [20]. We will follow notations and definitions of section 48 of [20]. For sake of completeness, below we will define the required terms.

Let $(\mathcal{M}, \sigma)$ be a weak symplectic Fréchet manifold. Let $T^*_x\mathcal{M}$ denotes the real linear subspace $T^*_x\mathcal{M} = \sigma^b(T_x\mathcal{M}) \subset T^*_x(\mathcal{M})$. These vector space fit to form a sub bundle of $T^*\mathcal{M}$ and $\sigma^b : T\mathcal{M} \to T^*\mathcal{M}$ is bundle isomorphism ($\S 48.4,[20]$). Define $C^\infty_\sigma(\mathcal{M}, \mathbb{R}) \subset C^\infty(\mathcal{M}, \mathbb{R})$ to be the linear subspace consisting of all smooth functions $f : \mathcal{M} \to \mathbb{R}$ such that $df : \mathcal{M} \to T^*\mathcal{M}$ factors to a smooth mapping $\mathcal{M} \to T^*\mathcal{M}$.

In other words, $f \in C^\infty_\sigma(\mathcal{M}, \mathbb{R})$ if there exists a smooth $\sigma$-gradient $grad^\sigma f \in \mathfrak{X}(\mathcal{M})$ such that for given $p \in \mathcal{M}$ and $Y \in T_p\mathcal{M}$ we have $df_p(Y) = \sigma_p(grad^\sigma f|_p, Y)$. Detailed description of these spaces and analysis of weak symplectic manifold is given in ($\S 48,[20]$).

Let $C^\infty(L^k_{alt}(T\mathcal{M}, \mathbb{R})^\sigma)$ be the space of smooth sections of a vector bundle with fiber $L^k_{alt}(T_x\mathcal{M}, \mathbb{R})^\sigma_x$ consisting of all bounded skew symmetric forms $\omega$ with $\omega(., X_2, ..., X_k) \in \mathbb{R}$.
$T_x^\sigma \mathcal{M}$. Let

$$\Omega^k_\sigma(\mathcal{M}) := \{ \omega \in C^\infty(L^k_{alt}(T\mathcal{M}, \mathbb{R})^\sigma) : d\omega \in C^\infty(L^{k+1}_{alt}(T\mathcal{M}, \mathbb{R})^\sigma) \}.$$ 

d^2 = 0 and the wedge product of $\sigma$-dual forms is again a $\sigma$-dual form (see page 527 of [20]). We have a graded differential subalgebra $(\Omega_\sigma(\mathcal{M}), d)$, whose cohomology is called the symplectic cohomology and will be denoted by $H^k_\sigma(\mathcal{M})$. We mention that this definition of the symplectic cohomology is not same as the symplectic cohomology defined by Floer.

### 1.5.3 Darboux chart

Let $E$ be a Fréchet space. We have $TE = E \times E$. Let $\mathcal{F}$ be a bounded, skew symmetric, non singular, bilinear map $\mathcal{F} : E \times E \to \mathbb{R}$. $\mathcal{F}$ defines a 2-form $\omega$ on $E$ by the following:

$$\omega_x : T_x E \times T_x E \to \mathbb{R}$$

$$\omega_x(v, w) := \mathcal{F}(\tilde{v}, \tilde{w})$$

where $T_x E$ is identified with $E$ and $\tilde{v}$ and $\tilde{w}$ corresponds to $v, w \in T_x E$.

**Example 1.5.1.** Let $E = L\mathbb{R}^n$, then we have $TE = L\mathbb{R}^n \times L\mathbb{R}^n$. Define $\mathcal{F} : L\mathbb{R}^n \times L\mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{F}(X, Y) := \int_0^1 \langle X'(t), Y(t) \rangle dt$$

Then $\mathcal{F}$ is a skew symmetric bilinear map. $\mathcal{F}$ defines a symplectic structure on $L\mathbb{R}^n$ as following: For $\gamma \in L\mathbb{R}^n$ and $X, Y \in T_\gamma L\mathbb{R}^n$, define:

$$\omega_\gamma(X, Y) := \int_0^1 \langle \tilde{X}'(t), \tilde{Y}(t) \rangle dt.$$ 

Now we will proceed to define a Darboux chart:

For a general infinite dimensional smooth manifold $\mathcal{M}$, we have the following definition:

By a Darboux chart around $p \in \mathcal{M}$, we mean a coordinate chart $\{ (\mathcal{U}, \Phi), \Phi : \mathcal{U} \to E \}$ around $p$ such that there exists a bounded alternating bilinear map $\mathcal{F}$ on $E$ for which, for $v_1, v_2 \in T_q \mathcal{M}$ and for every $q \in \mathcal{U}$

$$\sigma_q(v_1, v_2) = \mathcal{F}(d\Phi_q(v_1), d\Phi_q(v_2))$$

This chart around $p$ is called a Darboux chart around $p$. We say that $(\mathcal{M}, \sigma)$ admits Darboux chart if for every $p \in \mathcal{M}$ there is a Darboux chart around $p \in \mathcal{M}$. 

In the case of finite dimensional manifold $M$, Darboux theorem states that every point in $M$ has a coordinate neighborhood $N$ with coordinate functions $(x_1, ..., x_n, y_1, ..., y_n)$ such that $\sigma = \sum_{i=1}^{n} dx_i \wedge dy_i$ on $N$. 