CHAPTER 4

ANALYSIS OF AN $M^X/G/1$ FEEDBACK RETRIAL QUEUE WITH NON-PERSISTENT CUSTOMERS AND MULTIPLE VACATIONS WITH N-Policy

4.1 INTRODUCTION

The retrial queueing model that takes into account the impatience of customers was introduced by Cohen (1957). The author studied an $M/M/C$ retrial queue in which the orbital customers leave the system with a certain degree of intensity. An $M/M/1$ model with impatient subscribers was considered by Falin (1997). Yang et al (1990) presented an $M/G/1$ model with impatient customers and obtained expressions for the moments of the queue length in terms of server utilization. Krishnamoorthy et al (2005) presented an analysis of an $M/G/1$ retrial queue with non-persistent customers with an orbital search. Senthilkumar & Arumuganathan (2009) presented performance analysis of an $M/G/1$ retrial queue with non-persistent calls, two phases of heterogeneous service and different vacation policies.

Queuing systems with vacation time have been found to be useful in modeling systems in which the server has additional tasks. Gautam Choudhury & Madhuchanda Paul (2004) discussed a batch arrival queue with an additional service channel under N-policy.

Most papers on retrial queues have analyzed the system without taking customer feedback into consideration. A more practical retrial queue with feedback of the customers occurs in many real world situations. Choi & Kulkarni (1992) studied an $M/G/1$ retrial queue with feedback. Falin (1997) discussed retrial queue with feedback and geometric loss. Kailash et al

Several results have been reported separately on retrial queueing systems with non-persistent customers, retrial queues with feedback and retrial queues with multiple vacations. However, the study of bulk retrial queueing systems taking together above mentioned features is interesting. This chapter describes a single server batch arrival feedback retrial queue with non-persistent customers and multiple vacations with N-policy. Analytical treatment of this model is obtained using the supplementary variable technique. The probability generating function of number of customers in the retrial group and various performance measures are also obtained.

In this chapter an M^X/ G/1 feedback retrial queue with non-persistent customers and multiple vacations with N-policy is considered. It is assumed that there is no waiting space and therefore if an arriving batch of customers finds the server idle, one of the arriving customers starts its service immediately and the rest joins a retrial group in order to seek the service again after a random amount of time. Otherwise, if an arriving batch of customers on arrival finds the server busy, becomes impatient and leaves the system with probability (1− α) and with probability α, they enter into the orbit (retrying pool). At a service completion epoch, if the number of customers in the orbit is zero, the server does a secondary job (vacation) repeatedly until
the retrial group size reaches N. At the secondary job (vacation) completion epoch, if the orbit size is at least N, then the server remains in the system to render service for customers from the main pool or from the retrial group. At the service completion epoch, the leaving customer may either request for another service with a probability of $f$ or leave the system forever with a probability of $d$ (where $f + d = 1$). The model under study is schematically represented in Figure 4.1.

![Figure 4.1 Schematic Representation of the Queueing Model](image)

4.2 MOTIVATION

Motivation for the above model comes from a real life situation that exists in “Simple Mail Transfer Protocol (SMTP)”. In the transfer model of e-mail system, mail system uses the Simple Mail Transfer Protocol (SMTP) to deliver the messages between mail servers. In this protocol, contacting messages arrive at the mail server following the Poisson stream. One message comprises collection of finite number of packets (i.e. batch arrival). These
contacting messages may be terminated (non-persistent customer) by a sender before arriving at the mail server. When messages arrive at the mail server, one packet is selected to serve and the remaining packets will join the buffer (i.e. retrial group). In the buffer, each packet waits a certain amount of time and request for service again. There is a daemon program implemented at mail server to manage the service requests from buffer. Each time it tries but fails, it will wait another amount of time before trying again. The messages will feedback to the server to request for the service again. In addition, to keep the mail server functioning well, some maintenance activities are needed, such as virus scan and spam filtering etc. (vacation). It can be performed when the mail server is idle and can be programmed to perform on a regular basis (till the number of packets in the buffer reaches the threshold value N). The above SMTP can be effectively modeled as an M^{X}/G/1 feedback retrial queue with non-persistent customers and multiple vacations with N-policy.

4.3 MATHEMATICAL MODEL

The customers arrive according to a Poisson process with rate \( \lambda \). The time between successive retrials in orbit is assumed to be exponentially distributed at the rate \( \gamma \). Let \( S(x) \) (\( s(x) \)) \{\( \tilde{S}(\theta) \)\} \{\( S^0(x) \)\} be the cumulative distribution function (probability density function) \{Laplace-Stieltjes transform\} [remaining service time] of service. \( V(x) \), \( v(x) \) \{\( \tilde{V}(\theta) \)\} \{\( V^0(x) \)\} be the cumulative distribution function (probability density function) \{Laplace-Stieltjes transform\} [remaining vacation time] of vacation. \( N(t) \) denotes the number of customers in the orbit at time \( t \). Let \( g_k \) be the probability that \( k \) customers arrive in a batch and \( X(z) = \sum_{k=0}^{\infty} g_k z^k \) be the probability generating function (PGF) of the batch size distribution of arrival.
The different states of the server at time ‘t’ can be defined as follows:

\[ C(t) = \begin{cases} 
0, & \text{if the server is idle} \\
1, & \text{if the server is busy} \\
2, & \text{if the server is on vacation}
\end{cases} \]

Further, it is assumed that, \( Y(t) = j \), if the server is on the \( j^{th} \) vacation.

Now the system state probabilities are defined as follows

1. \( P_{0,n}(t) = P\{C(t) = 0; N(t) = n\}; \ n \geq 1 \) is the probability that at time \( t \) the server is idle and the orbit size is \( n \).

2. \( P_{1,n}(x,t)dx = P\{C(t) = 1; N(t) = n; x < S^r(t) \leq x + dx\}; \ n \geq 0 \) is the probability that at time \( t \) the server is busy, the orbit size is \( n \) and the remaining service time of a customer under service at an arbitrary time is between \( x \) and \( x+dx \).

3. \( V_{j,n}(x,t)dx = P\{C(t) = 2; N(t) = n, Y(t) = j, x \leq V^v(t) \leq x + dx\}; \ n \geq 0; \ j \geq 1 \) is the probability that the server is on \( j^{th} \) vacation, the orbit size is \( n \) and remaining vacation time is between \( x \) and \( x+dx \).

The above notations and probability functions are used to develop steady state queue size distribution.

**4.4 STEADY STATE QUEUE SIZE DISTRIBUTION**
To derive the steady state queue size distribution, the following equations are obtained using supplementary variable technique,

\[ P_{n,j}(t + \Delta t) = P_{n,j}(t)(1 - \lambda \Delta t - j \gamma \Delta t) + dP_{n,j}(0, t) \Delta t, \quad 1 \leq j \leq N - 1 \]

\[ P_{n,j}(t + \Delta t) = P_{n,j}(t)(1 - \lambda \Delta t - j \gamma \Delta t) + dP_{n,j}(0, t) \Delta t + \sum_{k=1}^{\infty} V_{n,k}(0, t) \Delta t, \quad j \geq N \]

\[ P_{l,j}(x - \Delta t, t + \Delta t) = P_{l,j}(x, t)(1 - \lambda \alpha \Delta t) + \lambda \sum_{k=1}^{\infty} g_k P_{l,j-k}(s(x) \Delta t) + P_{l,j}(0, t) s(x) \Delta t + (j + 1) \gamma P_{l,j+1}(s(x) \Delta t) \]

\[ V_{l,0}(x - \Delta t, t + \Delta t) = V_{l,0}(x, t)(1 - \lambda \alpha \Delta t) + dP_{l,0}(0, t) v(x) \Delta t \]

\[ V_{l,j}(x - \Delta t, t + \Delta t) = V_{l,j}(x, t)(1 - \lambda \alpha \Delta t) + \sum_{k=1}^{\infty} V_{l,j-k}(x, t) \lambda \alpha g_k \Delta t, \quad j \geq 1 \]

\[ V_{l,0}(x - \Delta t, t + \Delta t) = V_{l,0}(x, t)(1 - \lambda \alpha \Delta t) + \sum_{k=1}^{\infty} V_{l,0-k}(x, t) \lambda \alpha g_k \Delta t, \quad l \geq 2 \]

\[ V_{l,j}(x - \Delta t, t + \Delta t) = V_{l,j}(x, t)(1 - \lambda \alpha \Delta t) + \sum_{k=1}^{\infty} V_{l,j-k}(x, t) \lambda \alpha g_k \Delta t \]

From the above equations, the steady state queue size equations are obtained as follows:

\[ (\lambda + j\gamma)P_{n,j} = dP_{n,j}(0), \quad 1 \leq j < N \quad (4.1) \]

\[ (\lambda + j\gamma)P_{n,j} = dP_{n,j}(0) + \sum_{k=1}^{\infty} V_{n,k}(0) \quad j \geq N \quad (4.2) \]
\[-P_{1,j}'(x) = -\lambda \alpha P_{1,j}(x) + \lambda \sum_{k=1}^{l} g_{k} P_{n-k,1}(x) + P_{1,j}(0)s(x)f \]
\[+ (j+1)\gamma P_{n-j+1} s(x) + \sum_{k=1}^{l} P_{n-k+1}(x) \lambda \alpha a_{k}, \quad j \geq 0 \quad (4.3)\]

\[-V_{l,j}'(x) = -\lambda \alpha V_{l,j}(x) + dP_{l,j}(0)v(x) \quad (4.4)\]

\[-V_{l,j}'(x) = -\lambda \alpha V_{l,j}(x) + \sum_{k=1}^{l} V_{l,j-k}(x) \lambda \alpha a_{k}, \quad j \geq 1 \quad (4.5)\]

\[-V_{l,j}'(x) = -\lambda \alpha V_{l,j}(x) + V_{l,j-1}(0)v(x), \quad l \geq 2 \quad (4.6)\]

\[-V_{l,j}'(x) = -\lambda \alpha V_{l,j}(x) + V_{l,j-1}(0)v(x) + \sum_{k=1}^{l} V_{l,j-k}(x) \lambda \alpha a_{k}, \quad 0 \leq j < N \quad (4.7)\]

\[-V_{l,j}'(x) = -\lambda \alpha V_{l,j}(x) + \sum_{k=1}^{l} V_{l,j-k}(x) \lambda \alpha a_{k}, \quad j \geq N \quad (4.8)\]

The Laplace-Stieltjes transforms (LST) of \( P_{1,j}(x) \), \( V_{l,j}(x) \) are defined as

\[\text{LST}(P_{1,j}(x)) = \tilde{P}_{1,j}(\theta) = \int_{0}^{\theta} e^{-\theta x} P_{1,j}(x) \, dx;\]
\[\text{LST}(V_{l,j}(x)) = \tilde{V}_{l,j}(\theta) = \int_{0}^{\theta} e^{-\theta x} V_{l,j}(x) \, dx.\]

Taking Laplace-Stieltjes transform on steady state Equations (4.3) - (4.8), we have

\[\theta \tilde{P}_{1,j}(\theta) - P_{1,j}(0) - \lambda \alpha \tilde{P}_{1,j}(\theta) - \lambda \sum_{k=1}^{l} g_{k} P_{n-j-k+1}(\theta) \tilde{S}(\theta) - P_{1,j}(0) \tilde{S}(\theta)f \]
\[= (j+1)\gamma \tilde{P}_{n-j+1}(\theta) \tilde{S}(\theta) - \sum_{k=1}^{l} \tilde{P}_{n-k}(\theta) \lambda \alpha g_{k}, \quad j \geq 1 \quad (4.9)\]

\[\theta \tilde{V}_{l,0}(\theta) - V_{l,0}(0) = \lambda \alpha \tilde{V}_{l,0}(\theta) - d\tilde{P}_{l,0}(0) \tilde{V}(\theta) \quad (4.10)\]

\[\theta \tilde{V}_{l,j}(\theta) - V_{l,j}(0) = \lambda \alpha \tilde{V}_{l,j}(\theta) - \sum_{k=1}^{l} \tilde{V}_{l,j-k}(\theta) \lambda \alpha g_{k}, \quad j \geq 1 \quad (4.11)\]
4.4.1 Probability Generating Function

As discussed in section 2.4.1 of chapter II, to find the probability generating function (PGF) of the number of customers in the orbit at an arbitrary time epoch, the following probability generating functions are defined.

\[ P_0(z) = \sum_{j=0}^{\infty} P_{j,0} z^j, \]
\[ P_1(z,0) = \sum_{j=0}^{\infty} P_{j,1}(0) z^j, \]
\[ P_1(z,\theta) = \sum_{j=0}^{\infty} P_{j,1}(\theta) z^j, \]
\[ V_0(z) = \sum_{j=0}^{\infty} V_{j,0} z^j, \]
\[ V_1(z,0) = \sum_{j=0}^{\infty} V_{j,1}(0) z^j, \]
\[ V_1(z,\theta) = \sum_{j=0}^{\infty} V_{j,1}(\theta) z^j. \]

where \(|\theta| \leq 1 \)

The probability generating function \( P(z) \) of number of customers in orbit at an arbitrary time instant can be expressed as follows

\[ P(z) = P_0(z) + P_1(z,0) + \sum_{j=1}^{\infty} V_1(z,0) \]

In order to find \( \tilde{P}_1(z,0) \), and \( \sum_{j=1}^{\infty} \tilde{V}_1(z,0) \) the following sequence of operations are done.

\[ \theta \tilde{V}_{l,0}(\theta) - V_{l,0}(\theta) = \lambda \alpha \tilde{V}_{l,0}(\theta) - V_{l-1,0}(\theta) \tilde{V} \theta, \quad l \geq 2 \quad (4.12) \]

\[ \theta \tilde{V}_{l,0}(\theta) - V_{l,0}(\theta) = \lambda \alpha \tilde{V}_{l,0}(\theta) - V_{l-1,0}(\theta) \tilde{V} \theta - \sum_{j=0}^{\infty} \tilde{V}_{l-1,j}(\theta) \lambda \alpha g_j, \quad 0 \leq j < N \quad (4.13) \]

\[ \theta \tilde{V}_{l,0}(\theta) - V_{l,0}(\theta) = \lambda \alpha \tilde{V}_{l,0}(\theta) - \sum_{j=0}^{\infty} \tilde{V}_{l-1,j}(\theta) \lambda \alpha g_j, \quad j \geq N \quad (4.14) \]
Multiplying the Equations (4.1) and (4.2) by \(z^0\), Equations (4.9) - (4.14) by \(z^n\), taking summation from \(n=0\) to \(\infty\) and using the Equation (4.15), we get

\[
\lambda P_0(z) + \gamma z P_0'(z) = dP_1(z,0) - dP_{1,0}(0) + \sum_{i=1}^{\infty} (V_i(z,0) - \sum_{j=1}^{N_j} V_{i,j}(0) z^j) \tag{4.17}
\]

\[
(0 - \lambda \alpha + \lambda \alpha X(z)) \tilde{P}_1(z,0) = \tilde{P}_1(z,0) - \tilde{S}(\theta) \tilde{P}_1(z,0) - \gamma \tilde{S}(\theta) \tilde{P}_0'(z) - (\lambda X(z) \tilde{P}_1(z) \tilde{S}(\theta) / z) \tag{4.18}
\]

\[
(0 - \lambda \alpha + \lambda \alpha X(z)) \tilde{V}_1(z,0) = \tilde{V}_1(z,0) - d\tilde{P}_{1,0}(0) \tilde{V}(0) \tag{4.19}
\]

\[
(0 - \lambda \alpha + \lambda \alpha X(z)) \tilde{V}_1(z,0) = \tilde{V}_1(z,0) - \tilde{V}(\theta) \sum_{j=1}^{N-1} V_{i,j}(0) z^j \tag{4.20}
\]

Substituting \(\theta = \lambda \alpha - \lambda \alpha X(z)\) in the Equations (4.18) - (4.20), the PGF’s are rewritten as,

\[
P_1(z,0) = \frac{\gamma \gamma V_1(z) \tilde{S}(\lambda \alpha - \lambda \alpha X(z)) + \lambda X(z) \tilde{P}_1(z) \tilde{S}(\lambda \alpha - \lambda \alpha X(z))}{z(1 - \tilde{S}(\lambda \alpha - \lambda \alpha X(z)))} \tag{4.21}
\]

\[
\tilde{V}_1(z,0) = \tilde{V}(\lambda \alpha - \lambda \alpha X(z)) \tag{4.22}
\]

\[
\tilde{V}_1(z,0) = \tilde{V}(\lambda \alpha - \lambda \alpha X(z)) \sum_{j=0}^{N-1} V_{i,j}(0) z^j \tag{4.23}
\]

Substituting (4.21) in the Equation (4.17), Equation (4.17) becomes,

\[
P_1'(z) = \frac{\gamma(z) - \gamma(z)(1 - \tilde{S}(\lambda \alpha - \lambda \alpha X(z))) - dX(z) \tilde{S}(\lambda \alpha - \lambda \alpha X(z))}{\gamma(z^2(1 - \tilde{S}(\lambda \alpha - \lambda \alpha X(z))) - dX(z) \tilde{S}(\lambda \alpha - \lambda \alpha X(z)))} \tag{4.24}
\]
where

\[
g(z) = \left( \sum_{l=1}^{\infty} (V_i(z,0) - \sum_{p=0}^{N-1} V_{i,p}(0) z) - d P_{i,0}(0) \right) (1 - f \tilde{S}(\lambda \alpha - \lambda \alpha X(z)))
\]

\[
= (V(\lambda \alpha - \lambda \alpha X(z)) - 1) (d P_{i,0}(0) + \sum_{l=1}^{\infty} \sum_{p=0}^{N-1} V_{i,p}(0) z) (1 - f \tilde{S}(\lambda \alpha - \lambda \alpha X(z)))
\]

On integrating the Equation (4.24), \( P_0(z) \) is obtained as,

\[
P_0(z) = P_0(1) K(z)
\]

\[
+ K(z) \int_1^{g(t)} (1 - f \tilde{S}(\lambda \alpha - \lambda \alpha X(t))) (1 - f \tilde{S}(\lambda \alpha - \lambda \alpha X(t))) - d \tilde{S}(\lambda \alpha - \lambda \alpha X(t)) dt
\]

(4.25)

where

\[
K(z) = \exp \left\{ - \frac{\lambda}{\gamma} \int_{1}^{g(t)} \tilde{S}(\lambda \alpha - \lambda \alpha X(u)) \frac{d(X(u)/u) \tilde{S}(\lambda \alpha - \lambda \alpha X(u))}{(u - \gamma \tilde{S}(\lambda \alpha - \lambda \alpha X(u))) - d \tilde{S}(\lambda \alpha - \lambda \alpha X(u))} du \right\}
\]

Substituting (4.21) – (4.23) respectively in the Equations (4.18) – (4.20), the following probability generating functions of service and vacation times can be obtained.

\[
(\theta - \lambda \alpha + \lambda \alpha X(z)) \tilde{P}_i(z,0) = \left[ \frac{S(\lambda \alpha - \lambda \alpha X(z))(1 - S(\theta))}{1 - f \tilde{S}(\lambda \alpha - \lambda \alpha X(z))} - S(\theta) \right] \gamma P_i(z) + \lambda X(z) P_i(z) \tilde{S}(z)/z
\]

(4.26)

\[
(\theta - \lambda \alpha + \lambda \alpha X(z)) \tilde{V}_i(z,0) = (V(\lambda \alpha - \lambda \alpha X(z)) - V(\theta)) d P_{i,0}(0)
\]

(4.27)

\[
(\theta - \lambda \alpha + \lambda \alpha X(z)) \tilde{V}_i(z,0) = (V(\lambda \alpha - \lambda \alpha X(z)) - V(\theta)) \sum_{j=0}^{N-1} V_{i,j}(0) \tilde{S}(z)/z
\]

(4.28)
From the Equations (4.26) - (4.28), after some algebra we have

\[ \tilde{P}_i(z,0) = \frac{(\tilde{S}(\lambda \alpha - \lambda \alpha X(z)))d - (1 - \tilde{S}(\lambda \alpha - \lambda \alpha X(z)))) \gamma P_i'(z) + \lambda X(z) P_i(z)/z}{(-\lambda + \lambda \alpha X(z))(1 - \tilde{S}(\lambda \alpha - \lambda \alpha X(z)))} \]  

(4.29)

\[ \sum_{i=1}^{\infty} \tilde{V}_i(z,0) = \frac{\tilde{V}(\lambda \alpha - \lambda \alpha X(z)) - 1)(d P_{i,0}(0) + \sum_{j=1}^{N-1} \sum_{k=0}^{\infty} V_{i,j}(0)z^j)}{(-\lambda + \lambda \alpha X(z))} \]  

(4.30)

The following theorem is proved by substituting for \( \tilde{P}_i(z,0) \) and \( \sum_{i=1}^{\infty} \tilde{V}_i(z,0) \) from the Equations (4.29) and (4.30) in the Equation (4.16).

**Theorem : 4.1**

The probability generating function \( P(z) \) of number of customers in the orbit is given by

\[ P(z) = P_i(z) + \frac{\left( \frac{\tilde{S}(\lambda \alpha - \lambda \alpha X(z))d}{1 - \tilde{S}(\lambda \alpha - \lambda \alpha X(z))} - 1 \right) - \left( \gamma P_i'(z) + \frac{\lambda X(z)}{z} P_i(z) \right)}{(-\lambda + \lambda \alpha X(z))} \]  

(4.31)

\[ + \frac{\left( \tilde{V}(\lambda \alpha - \lambda \alpha X(z)) - 1 \right) \left( \gamma P_{i,0}(0) + \sum_{j=1}^{N-1} \sum_{k=0}^{\infty} V_{i,j}(0)z^j \right)}{(-\lambda + \lambda \alpha X(z))} \]

Assuming \( \sum_{i=1}^{\infty} V_{i,0}(0) = q_j \) and \( P_{i,0}(0) = p_c \), Equation (4.31), after some algebra, can be rewritten as,

\[ P(z) = H(Z)/M(Z) \]  

(4.32)
where,

\[ H(Z) = P_r(z)(\lambda (X(z))-1) + \alpha (z-zf S(\lambda - \lambda X(z))) - S(\lambda - \lambda X(z))d) + (S(\lambda X(z))-1) \]

\[ + (\lambda S(\lambda X(z)) - 1) + \tau S(\lambda - \lambda X(z)) - dS(\lambda - \lambda X(z)) \]

\[ (V(\lambda X(z))-1)(dp + \sum_{j=0}^{N} q_i z^j) \]

\[ M(z) = (-\lambda \alpha + \lambda \alpha X(z))(z - zf S(\lambda - \lambda X(z)) - dS(\lambda - \lambda X(z))) \]

where \( q_i \) is the probability of \( j \) customers being in the orbit at vacation completion epoch.

4.4.2 Computational Aspects of Unknown Probabilities

In this section, the unknown probabilities are expressed in terms of known constant \( p_0 \). The probability generating function \( P(z) \) in the Equation (4.32) has \( (N+1) \) constants \( (q_j : j=0 \text{ to } N-1) \). Since the model under consideration is an \( M^X/G/1 \) retrial queue it should have only one unknown constant in \( P(z) \). So to express the excess unknowns in terms of known constant, \( p_0 \) the following theorem is proved.

**Theorem 4.2**

If \( \alpha_n \) in the probability of \( n \) customers arriving during a vacation, then

\[ q_n = \frac{dp_i \alpha_n + \sum_{i=0}^{n-1} q_i \alpha_{n-i}}{1 - \alpha \alpha} , \ n = 1, 2, 3, \ldots, N-1 \]

\[ q_0 = \frac{dp \alpha}{1 - \alpha_0} \]
Proof

Using \( \sum_{j=1}^{\infty} V_{i,j}(0) = q_j, \ p_{1,0}(0) = p_0, \) and Equations (4.22) – (4.23),

\[
\sum_{j=1}^{\infty} V_j(z,0)
\]

simplifies to

\[
\sum_{j=1}^{\infty} V_j(z,0) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} V_{i,n}(0) z^n
\]

\[
= V(\lambda \alpha - \lambda \alpha X(z)) \left( d p_0 + \sum_{j=0}^{N-1} q_j z^j \right)
\]

\[
= \sum_{n=0}^{\infty} \alpha_n z^n \left( d p_0 + \sum_{i=0}^{N-1} (q_i z^i) \right)
\]

\[
= d p_0 \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{n=0}^{N-1} \left( \sum_{j=0}^{n} (q_j \alpha_{n-j}) z^n \right) + \sum_{n=0}^{\infty} \left( \sum_{i=0}^{N-1} q_i \alpha_{n-i} \right) z^n
\]

Equating the coefficients of \( z^n \), \( n = 0, 1, 2, \ldots, N-1 \) on both sides of the above equation we have,

\[
q_n = \frac{d p_0 \alpha_n + \sum_{i=0}^{n-1} q_i \alpha_{n-i}}{1 - \alpha_0}, \quad n = 1, 2, 3, \ldots, N-1
\]

\[
q_0 = \frac{d p_0 \alpha_0}{1 - \alpha_0}
\]

Hence the theorem.
4.5 STABILITY CONDITION

The probability generating function P(z) has to satisfy the condition \( \lim_{z \to 1} P(z) = 1 \). In order to satisfy this condition, L'Hôpital's rule is applied twice to Equation (4.32). Since \( p_0 \) and \( q_j \) are probabilities, the numerator of \( P(z) \) is positive when \( z \to 1 \). So \( \lim_{z \to 1} P(z) = 1 \) is satisfied only if

\[
2\gamma \alpha \lambda \left( \right) > 0,
\]

that is only when \((d - \lambda \alpha F(X) E(S)) > 0\). Define \( \rho \) as \( \frac{\lambda \alpha E(X) E(S)}{E(X)} \). Thus \( \rho < 1 \) is the condition to be satisfied for the existence of steady state for the model under consideration.

4.6 PERFORMANCE CHARACTERISTICS

In this section, some useful performance measures of the proposed model such as expected number of customers in the orbit, probability that the server is idle, probability that the server is busy and the probability that the server is on vacation are derived.

4.6.1 The Mean Number of Customers in the Orbit

The expected number of customers in the orbit is derived using PGF given in Equation (4.32) and

\[
L_Q = \lim_{z \to 1} \frac{d}{dz} P(z)
\]

\[
= \frac{d(1-\rho)M_1 - (E(X) - M \alpha d(1-\rho) + d) (P_j(1)) (\alpha d(1-\rho) + d \rho) + M_2 (d \rho + \sum q_j)}{2 \lambda \alpha E(X) (d(1-\rho))^2}
\]

(4.33)
where,

\[ M_1 = (\lambda p_{_1}(1) E(X) (\alpha(-2\gamma \lambda \alpha E(X) E(S) - S) + S) \\
+ (\alpha d (1 - \rho) + d \rho) (2p'_{_1}(1) E(x) + p_{_1}(1) E(X^2))) \]

\[ M_2 = -2d^2 f \lambda \alpha p (1 - \rho) E(2X^3) E(V) + d^2 (1 - \rho) V - \alpha \lambda E(V) E(V^2) d(l - \rho) \]
\[ + \alpha \lambda E(X) E(V) (2f \rho d + S) \]

\[ M_3 = 2d'(1 - \rho) \lambda \alpha E(X) E(V) \]

\[ S_2 = \lambda \alpha E(S) E(X^2) + \lambda^2 \alpha^2 \Gamma \Gamma E(V) E(S^2) \]

\[ V_2 = \lambda \alpha E(V) E(X^2) + \lambda^2 \alpha^2 \Gamma \Gamma E(V^2) \]

and

\[ P_0(I) = \frac{(d(l - \rho) - E(V) d(dp_{_1} + \sum_{j=0}^{j=N} q_j))}{d(l - \rho) + \lambda E(X) E(S)} \quad (4.34) \]

\[ P'_0(I) = \frac{\lambda E(V)(1 - E(X) + \alpha E(X))(dp_{_1} + \sum_{j=0}^{j=N} q_j) - \lambda (1 - \rho - E(X))}{\gamma ((l - \rho) + (\lambda / d) F(X) E(S))} \quad (4.35) \]

4.6.2 Probability that the Server is Idle

Let I be the idle period random variable and let \( P(I) \) be the probability that the server is idle at time \( t \). Using the Equation (4.25) and applying limit \( z \to 1 \) we get the probability that the server is idle as

\[ P_0(I) = \frac{d(l - \rho) - E(V) d(dp_{_1} + \sum_{j=0}^{j=N} q_j)}{d(l - \rho) + \lambda E(X) E(S)} \quad (4.36) \]
4.6.3 Probability that the Server is Busy

Let $B$ be the busy period random variable and $P(B)$ be the probability that the server is busy at time $t$. Using the Equation (4.29) and applying limit $z \to 1$ we get the probability that the server is busy as

$$P(B) = \frac{E(S)((dp_0 + \sum_{j=0}^{N-1} q_j)(d\lambda E(X) E(V)(\alpha - 1) + \lambda \rho (1 - \rho)) - d\lambda (1 - \rho - E(X)))}{d(d(1 - \rho) + \lambda E(X) E(S))}$$

(4.37)

4.6.4 Probability that the Server is on Vacation

Let $V$ be the vacation time random variable and $P(V)$ be the probability that the server is on vacation at time $t$. Using the Equation (4.30) and applying limit $z \to 1$ we get the probability that the server is on vacation as

$$P(V) = \sum_{i=1}^{\infty} \tilde{V}_i(1,0) = E(V)(dp_0 + \sum_{j=0}^{N-1} q_j)$$

(4.38)

4.7 PARTICULAR CASE

In this section, a particular case of the proposed model has been discussed.

In the proposed model, if the server does not avail multiple vacation, all customers are persistent and when they do not request for re-service (i.e. there is no multiple vacation, there are no impatient customers and no feedback), ($\alpha = 1$, $\tilde{V}(\lambda - \lambda X(z)) = 1$, $d = 1$, $f = 0$), then Equation (4.31) reduces to the form
This equation exactly coincides with the result of orbit size distribution of $M^X/G/1$ retrial queuing system by Falin & Templeton (1997).

### 4.7.1 Special Cases

The model so developed is general in nature as the service time is arbitrary. But for practical purposes, service time with particular distribution is required. In the following paragraphs some special cases of the proposed model by specifying service time random variables as exponential, K-Erlang and hyper exponential distribution are discussed.

**Case (i):** $M^X/E_k/1$ retrial queue with K-Erlang service time, non persistent customers, feedback and N-policy multiple vacations.

If the service time is assumed to be K-Erlang with probability density function, 
\[ s(x) = \frac{(\lambda_k x)^k \lambda^k}{(k-1)!}, \quad k > 0 \]
where $\lambda$ is the parameter, then
\[ \tilde{S}(\lambda, -\lambda, X(z)) = \left( \frac{uk}{uk + \lambda(1 - X(z))} \right)^k \]
(4.40)

Substituting (4.40) in (4.32), the PGF of the retrial queue size distribution for the single server batch arrival retrial queue with non-persistent, feedback and N-policy multiple vacations can be given as follows.

\[ P(z) = \frac{\mathcal{H}(\lambda)}{M_z(z)} \]
(4.41)
where

\[
\begin{align*}
H_t(z) &= P_t(z) \lambda (X(z) - 1)\alpha (z - z_f (uk / (uk + \lambda (1 - X(z))))^k + \alpha (uk / (uk + \lambda (1 - X(z))))^k - 1)) \\
&+ (\tilde{V}(\lambda \alpha - \lambda X(z)) - 1)((uk / (uk + \lambda (1 - X(z))))^k - 1) \\
&+ (z - (zf + d)(uk / (uk + \lambda (1 - X(z))))^k \gamma d_p + \sum_{j=0}^{N-1} q_j z^j)
\end{align*}
\]

\[
M_t(z) = (\lambda \alpha - \lambda X(z)) (z - (zf + d)(uk / (uk + \lambda (1 - X(z))))^k)
\]

\[
g(z) = (\tilde{V}(\lambda \alpha - \lambda X(z)) - 1)(d_p + \sum_{j=0}^{N-1} q_j z^j) (1 - f(u k / (u k + \lambda (1 - X(z))))^k)
\]

\[
P_0(z) = P_0(1) k(z) \int \frac{-g(t)}{(k(t))^\gamma (1 - f(u k / (u k + \lambda (1 - X(t))))^k)} dt \\
- d(u k / (u k + \lambda (1 - X(z))))^k)
\]

\[
K(z) = \exp \left\{ \frac{(1 - f(u k / (u k + \lambda (1 - X(y))))^k}{\gamma \left( (1 - f(u k / (u k + \lambda (1 - X(y))))^k) - d(u k / (u k + \lambda (1 - X(y))))^k \right)} \right\}
\]

**Case (ii): M^X/Hyper exponential/1** retrial queue with hyper exponential service time, non-persistent customers, feedback and N-policy multiple vacations.

If the service time is assumed to be hyper exponential with probability density function, \( s(x) = cu \exp(-ux) + (1-c) w \exp(-wx) \) where \( u \) and \( w \) are parameters, then

\[
\tilde{S}(\lambda \alpha - \lambda X(z)) = (uc / (u + \lambda (1 - X(z)))) + ((w(1-c)) / (w + \lambda (1 - X(z))))
\]

(4.42)
Substituting (4.42) in (4.32), the PGF of the retrial queue size distribution for the single server batch arrival retrial queue with non-persistent, feedback and N-policy multiple vacations can be given as follows.

\[
P_0(z)\lambda(1-X(z))(\alpha(z-z\Gamma(u/(u+\lambda\alpha(1-X(z))))+
+((w(1-c))/(w+\lambda\alpha(1-X(z))))+((uc/(u+\lambda\alpha(1-X(z))))-
+((w(1-c))/(w+\lambda\alpha(1-X(z)))-1)))
\]

\[
+((w(1-c))/(w+\lambda\alpha(1-X(z)))-1)((uc/(u+\lambda\alpha(1-X(z))))-
+((w(1-c))/(w+\lambda\alpha(1-X(z)))-1)+(z-(z\Gamma+d)(uc/(u+\lambda\alpha(1-X(z)))))
\]

\[
P(z) = \frac{(-\lambda\alpha+\lambda\alpha X(z))(z-(z\Gamma+d)(uc/(u+\lambda\alpha(1-X(z))))-
+((w(1-c))/(w+\lambda\alpha(1-X(z))))(d\sum_{i=0}^{\infty} q_i z^i)\}
\]

Case (iii): M^X/M/1 retrial queue with exponential service time, non-persistent customers, feedback and N-policy multiple vacation.

If the service time is assumed to be exponential with probability density function, \( s(x) = u e^{-ux} \) where u is the parameter, then

\[
\tilde{S}_\lambda\alpha - \lambda\alpha X(z) = (u/(u+\lambda\alpha(1-X(z)))) \quad (4.43)
\]

Substituting (4.43) in (4.32), the PGF of the retrial queue size distribution for the single server batch arrival retrial queue with non-persistent, feedback and N-policy multiple vacations can be given as follows.

\[
P_0(z)\lambda(1-X(z))(\alpha(z-z\Gamma(u/(u+\lambda\alpha(1-X(z))))+
+((w(1-c))/(w+\lambda\alpha(1-X(z))))+((uc/(u+\lambda\alpha(1-X(z))))-
+((w(1-c))/(w+\lambda\alpha(1-X(z)))-1)))
\]

\[
+((w(1-c))/(w+\lambda\alpha(1-X(z)))-1)((uc/(u+\lambda\alpha(1-X(z))))-
+((w(1-c))/(w+\lambda\alpha(1-X(z)))-1)+(z-(z\Gamma+d)(uc/(u+\lambda\alpha(1-X(z)))))
\]

\[
P(z) = \frac{(-\lambda\alpha+\lambda\alpha X(z))(z-(z\Gamma+d)(uc/(u+\lambda\alpha(1-X(z))))-
+((w(1-c))/(w+\lambda\alpha(1-X(z))))(d\sum_{i=0}^{\infty} q_i z^i)\}
\]
4.8 NUMERICAL RESULTS

This section presents certain numerical results to justify the theoretical results obtained. To study the effect of various parameters on the system performance measures, the following notations are used and some assumptions are made:

- Average arrival rate $\lambda$
- Service time distribution is exponential or Erlang-2 with parameter $\mu$
- Vacation duration is exponential with parameter $\eta$
- Batch arrival size distribution is geometric with mean $2$
- Retrial rate $\gamma$
- Threshold value $N$
- Bernoulli schedule probability $\alpha$
- Feedback probability $f$

Table 4.1 and Figure 4.2 represent the effect of retrial rate $\gamma$ on the mean orbit size $L_Q$ for different threshold values of $N$. It is assumed that $\lambda = 0.8$, $\mu = 7$, $\eta = 2$, $\alpha = 0.4$ and $f = 0.3$. Considering the service time as exponential, Erlang-2 and hyper exponential, it is observed that,

- Mean number of customers in the orbit decreases with an increase in retrial rate.
- Mean number of customers in the orbit increases with an increase in the threshold value $N$. 
Figure 4.3 depicts the effect of retrial rate $\gamma$ on the mean orbit size for a fixed $N = 3$, considering the service time as exponential, Erlangian -2 and hyper-exponential.

Table 4.2 and Figure 4.4 represent the effect of $\alpha$ on the mean number of customers in the orbit for different threshold values of $N$. It is assumed that $\lambda = 0.8$, $\mu = 7$, $\eta = 2$, $\gamma = 5$ and $f = 0.3$. Considering the service time as exponential, Erlang-2 and hyper-exponential, it is observed that,

- Mean number of customers in the orbit increases when $\alpha$ increases.
- Mean number of customers in the orbit increases when the threshold value $N$ increases.

Thus, the theoretical development of the model is justified with the numerical results, which are consistent with the fact that the mean orbit size increases when $N$ increases and the mean orbit size decreases when retrial rate $\gamma$ increases, and also mean orbit size increases when $\alpha$ increases.

4.9 CONCLUSION

This chapter described a single server batch arrival feedback retrial queue with non-persistent customers where the server avails multiple vacations with N-policy. The probability generating function of number of customers in the retrial group was obtained. Various system performance measures are computed. A particular case of the proposed model has also been discussed. The theoretical development of the model has also been justified with the numerical results.
Table 4.1  Retrial rate $\gamma$ (Vs) Mean Orbit Size $L_Q$ for Different Values of N (Threshold Value - N)

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<th>$\gamma$</th>
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Table 4.2  Probability $\alpha$ (Vs) Mean Orbit Size $L_0$ for Different Values of $N$ (Threshold Value - N)

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Figure 4.2 Retrial rate $\gamma$ (Vs) Mean Orbit Size $L_Q$ (for Different Values of $N$, Exponential Service Time Distribution)

Figure 4.3 Retrial rate $\gamma$ (Vs) Mean Orbit Size $L_Q$ (for Different Service Time Distributions, $N=3$)
Figure 4.4  Probability $\alpha$ (Vs) Mean Orbit Size $L_Q$(for Different Values of N, Exponential Service Time Distribution)