CHAPTER 2

ANALYSIS OF SINGLE SERVER NON-MARKOVIAN RETRIAL QUEUE WITH WORKING VACATION AND CONSTANT RETRIAL POLICY

2.1 INTRODUCTION

In the recent years, vacation models have been the subject of interest in queueing theory. This is because of their applications in real life congestion situations such as manufacturing and production, computer and communication systems and service and distribution systems. Li & Yang (1995) developed an M/G/1 retrial system with server vacations and M independent identical input sources. Later, Artalejo (1997) analyzed an M/G/1 retrial queue with exhausted server vacations that is the server avails a vacation only when there are no customers in the orbit. Doshi (1985) discussed an M/G/1 system with variable vacations. Batch arrival Markovian single server queueing systems with multiple vacations were first studied by Baba (1986). Later Senthilkumar & Arumuganathan (2008) analyzed single server batch arrival retrial queue with general vacation time under Bernoulli schedule and two phases of heterogeneous service. The variations and extensions of these vacation models can be referred to in Lee et al (1994, 1995) and in Krishna Reddy et al (1998). Arumuganathan et al (2008) analyzed a steady state non-Markovian bulk queueing system with N-policy and different types of vacations. Haridass & Arumuganathan (2012) analyzed a batch arrival, bulk service queueing system with interrupted vacation.
A queue with working vacation was first analyzed by Servi & Finn (2002) and they obtained the queue length distribution of M/M/1/Wv queue. They discussed a classical single server vacation model, in which a single server works at a different rate rather than completely stopping the service during the vacation period. Further, they applied the model for the performance evaluation of Wavelength Division Multiplexing (WDM) optical systems. However, they have assumed an exponential service time, which may not always be the case. Subsequently, Kim et al (2003) analyzed an M/G/1 queue with exponentially distributed working vacations and obtained the steady state queue length distribution through the decomposition approach. Later Wu & Takagi (2006) extended Servi and Finn’s model to an M/G/1 working vacation, in which, both regular service time and the service time in working vacation are assumed to be generally distributed. Li et al (2009) considered an M/G/1 queue with exponentially distributed working vacations, which is a special case as that in Wu & Takagi (2006). All the above contributors consider classical queueing model. Tien Van Do (2009) studied a Markovian retrial queue with working vacation. The retrial policy considered was a classical one.

In this chapter, an M/G/1 retrial queue with working vacation and constant retrial policy is analyzed. If the server is busy at the arrival time, the customers join the orbit to repeat their request later. On the other hand, if the server is idle, then the arriving customer begins its service immediately. The single server takes a working vacation at times when the customers being served depart from the system and no customers are in the orbit. In vacation models, the server does not serve any customer during vacation times. But in working vacation models, during the vacation periods the arriving customers are served with a rate smaller than the regular service rate. It is considered
that if the slow service is completed prior to the end of vacation, the server avails of the remaining vacation. On the other hand, if the service time is extended beyond the vacation time, the server switches to regular service rate and becomes idle after completion of service. The server works at different service rates instead of completely stopping service during a vacation. After completing a vacation, the server stays idle in the system until a customer arrives from main pool or from the orbit. The model under study is schematically represented in Figure 2.1.

![Figure 2.1 Schematic Representation of the Queueing Model](image-url)
2.2 MOTIVATION

In this work, a working vacation to retrial queues with generally distributed service times, which are motivated by the performance analysis of Media Access Control (MAC) function in wireless networks is discussed. Wireless MAC protocols often use collision avoidance techniques, in conjunction with a (physical or virtual) carrier sense mechanism. In carrier sense mechanism, when a node wishes to transmit a packet, it first waits until the channel is idle. Nodes hearing RTS (Request-to-Send) or CTS (Clear-to-Send) stay silent for the duration of the corresponding transmission. Once channel becomes idle, the node waits for a randomly chosen duration before attempting to transmit. This mechanism can be modelled as M/G/1 retrial queueing model with working vacation by considering the orbit as pool of packets waiting for transmission once it senses the idle channel and RTS and CTS as working vacation times. So, the above mechanism can be modelled as an M/G/1 retrial queue with a single working vacation and a constant retrial policy.

2.3 MATHEMATICAL MODEL

The customers arrive according to the Poisson process with rate $\lambda$. When the server is idle, the customers in the orbit try for service one by one with a constant retrial rate $\gamma$. The service rate is $\mu_b$ when the server is not on vacation and $\mu_v$ during a working vacation ($\mu_v < \mu_b$). Vacation durations are exponentially distributed with the parameter $\eta$. Let $S_b(x)(s_b(x)) [S_b^0(x)]$ be the cumulative distribution function (probability density function) [Laplace– Stieltjes transform] [remaining service time] of service during working vacation. Let $S_v(x)(s_v(x)) [S_v^0(x)]$ be the cumulative distribution function (probability density function) [Laplace– Stieltjes transform] [remaining
service time] of service when the server is not on working vacation. \( N(t) \) denotes the number of customers in the orbit at time \( t \). The process considered here is a semi-Markov process which become Markov by including additional random variable as the remaining service time.

The server state is denoted as

\[
C(t) = \begin{cases} 
0, & \text{if the server is on working vacation and the server is not occupied} \\
1, & \text{if the server is not on working vacation and the server is not occupied} \\
2, & \text{if the server is on working vacation and the server is busy} \\
3, & \text{if the server is not on working vacation and the server is busy}
\end{cases}
\]

Now the system state probabilities are defined as follows

1. \( W_n(t) = P_r\{N(t) = n, C(t) = 0\}, \quad n \geq 0 \) is the probability that at time \( t \) the server is on working vacation when the server is not occupied and the orbit size is \( n \).

2. \( I_n(t) = P_r\{N(t) = n, C(t) = 1\}, \quad n \geq 0 \) is the probability that at time \( t \) the server is not on working vacation when the server is not occupied and the orbit size is \( n \).

3. \( Q_n(x, t)dt = P_r\{N(t) = n, C(t) = 2, x \leq S_x^o(t) \leq x + dt\}, n \geq 0 \) is the joint probability that at time \( t \) the server is busy during working vacation, the orbit size is \( n \) and the remaining service time of a customer during working vacation at an arbitrary time is between \( x \) and \( x+dt \).
(4) $P_n(x, t)dt = P_t \{ N_t = n, C_t = 3, x \leq S^0_t(t) \leq x + dt \}$, $n \geq 0$ is the joint probability that at time $t$ the server is busy when it is not on working vacation, the orbit size is $n$ and the remaining service time of a customer when the server is not on working vacation at an arbitrary time is between $x$ and $x+dt$.

### 2.4 STEADY STATE QUEUE SIZE DISTRIBUTION

To derive the steady state queue size distribution, the following equations are obtained using supplementary variable technique,

$$W_n(t+\Delta t) = W_n(t)(1-\lambda \Delta t - \eta \Delta t) + Q_n(0, t) \Delta t + P_n(0, t) \Delta t,$$

$$W_n(t+\Delta t) = W_n(t)(1-\lambda \Delta t - \gamma \Delta t - \eta \Delta t) + Q_n(0, t) \Delta t,$$

$$I_n(t+\Delta t) = I_n(t)(1-\lambda \Delta t) + W_n(t) \eta \Delta t,$$

$$I_n(t+\Delta t) = I_n(t)(1-\lambda \Delta t - \gamma \Delta t) + W_n(t) \eta \Delta t + P_n(0, t) \Delta t,$$

$$Q_n(x-\Delta t, t+\Delta t) = Q_n(x, t)(1-\lambda \Delta t - \eta \Delta t) + \lambda W_n(t)s_n(x) \Delta t + \gamma W_{n+1}(t)s_n(x) \Delta t + \lambda Q_{n-1}(x, t) \Delta t(1-\delta_{n,1}),$$

$$P_n(x-\Delta t, t+\Delta t) = P_n(x, t)(1-\lambda \Delta t) + \lambda I_n(t)s_n(x) \Delta t + \gamma P_{n+1}(t)s_n(x) \Delta t + (\int_0^\infty Q_n(y, t)dy) \eta s_n(x) \Delta t + \lambda P_{n-1}(x, t)(1-\delta_{n,1}) \Delta t,$$

where $\delta_{n,0} = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$.

In steady state, we can set $W_0 = \lim_{t \to 0^+} W_0(t)$, $I_0 = \lim_{t \to 0^+} I_0(t)$, $W_n = \lim_{t \to 0^+} W_n(t)$, $I_n = \lim_{t \to 0^+} I_n(t)$ and limiting densities $Q_n(x) = \lim_{t \to 0^+} Q_n(x, t)$ for $x > 0$ and $P_n(x) = \lim_{t \to 0^+} P_n(x, t)$ for $x > 0$. 
Now the above equations under steady state conditions can be written as follows:

\[
(\lambda + \eta) W_0 = Q_n(0) + P_n(0), \tag{2.1}
\]

\[
(\lambda + \gamma + \eta) W_n(0) = Q_n(0), \tag{2.2}
\]

\[
\lambda I_0 = \eta W_0, \tag{2.3}
\]

\[
(\lambda + \gamma) I_n(0) = W_n(0) + P_n(0), \tag{2.4}
\]

\[
-\frac{d}{dx} Q_n(x) = -(\lambda + \eta) Q_n(x) + \lambda W_n(0)s_n(x) + \gamma W_{n+1}(0)s_n(x) + \delta Q_{n+1}(x)(1 - \delta_n), \tag{2.5}
\]

\[
-\frac{d}{dx} P_n(x) = -\lambda P_n(x) + \lambda I_n(0)s_n(x) + \gamma I_{n+1}(0)s_n(x) + \delta P_{n+1}(x)(1 - \delta_n) + \left(\int_0^\infty Q_n(y)dy\right)\eta s_n(x). \tag{2.6}
\]

The Laplace-Stieltjes transforms (LST) of \(P_n(x), Q_n(x)\) are defined as

\[
\text{LST}(P_n(x)) = \tilde{P}_n(\theta) = \int_0^{\infty} e^{-\theta y} P_n(x)dx;
\]

\[
\text{LST}(Q_n(x)) = \tilde{Q}_n(\theta) = \int_0^{\infty} e^{-\theta y} Q_n(x)dx.
\]

Taking Laplace transform on steady state Equations (2.5) and (2.6), we have

\[
\theta \tilde{Q}_n(\theta) - Q_n(0) - (\lambda + \eta) \tilde{Q}_n(\theta) - \lambda W_n(0)\tilde{S}_n(\theta) - \gamma W_{n+1}(0)\tilde{S}_n(\theta) - \delta Q_{n+1}(1 - \delta_n), \tag{2.7}
\]

\[
\theta \tilde{P}_n(\theta) - P_n(0) = \lambda \tilde{P}_n(\theta) - \lambda I_n(0)\tilde{S}_n(\theta) - \gamma I_{n+1}(0)\tilde{S}_n(\theta) - \lambda \tilde{P}_{n+1}(1 - \delta_n) - \tilde{Q}_n(0)\eta \tilde{S}_n(\theta). \tag{2.8}
\]
2.4.1 Probability Generating Function

Lee (1991) developed a technique to find the steady state probability generating function (PGF) of the number of customers in the queue at an arbitrary time epoch. To apply the technique, the following probability generating functions are defined.

\[
W(z,0) = \sum_{n=0}^{\infty} W_n(0) z^n; \quad I(z,0) = \sum_{n=0}^{\infty} I_n(0) z^n; \\
Q(z,0) = \sum_{n=0}^{\infty} Q_n(0) z^n; \quad P(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n; \\
\tilde{W}(z,0) = \sum_{n=0}^{\infty} \tilde{W}_n(0) z^n; \quad \tilde{I}(z,0) = \sum_{n=0}^{\infty} \tilde{I}_n(0) z^n; \quad \tilde{Q}(z,0) = \sum_{n=0}^{\infty} \tilde{Q}_n(0) z^n; \quad \tilde{P}(z,0) = \sum_{n=0}^{\infty} \tilde{P}_n(0) z^n
\]

Multiplying Equations (2.1) and (2.3) by \( z^0 \), Equations (2.2),(2.4), (2.7) and (2.8) by \( z^n \), taking summation from \( n=0 \) to \( \infty \) and using (2.9), we get

\[
(\lambda + \eta) W(z,0) + \gamma (W(z,0) - W_0) = Q(z,0) + P_0 \quad \text{(2.10)}
\]

\[
\lambda I(z,0) + \gamma (I(z,0) - I_0) = \eta W(z,0) + (P(z,0) - P_0) \quad \text{(2.11)}
\]

\[
(\theta - (\lambda + \eta) + \lambda z) \tilde{Q}(z,0) = Q(z,0) - \lambda W(z,0) \tilde{S}_n(0) - (\gamma / z)(W(z,0) - W_0) \tilde{S}_n(0) \quad \text{(2.12)}
\]

\[
(\theta - \lambda + \lambda z) \tilde{P}(z,0) = P(z,0) - (\lambda + (\gamma / z)) \tilde{S}_n(0) I(z,0) + (\gamma / z) \tilde{S}_n(0) I_0 - \tilde{Q}(z,0) \eta \tilde{S}_n(0) \quad \text{(2.13)}
\]

The probability generating function \( P(z) \) of number of customers in orbit at an arbitrary time instant can be expressed as follows

\[
P(z) = W(z,0) + I(z,0) + \tilde{P}(z,0) + \tilde{Q}(z,0). \quad \text{(2.14)}
\]
In order to find \(W(z,0), I(z,0), \tilde{P}(z,0)\) and \(\tilde{Q}(z,0)\), the following sequence of operations are done.

Substituting \(\theta = (\lambda + \eta - \lambda z)\) in the Equation (2.12) we have

\[
Q(z,0) = \lambda \cdot W(z,0) \tilde{S}_c \cdot (\lambda + \eta - \lambda z) + (\gamma / z) (W(z,0) - W_0) \tilde{S}_c \cdot (\lambda + \eta - \lambda z).
\]

(2.15)

Substituting for \(Q(z,0)\) from Equation (2.15) into Equation (2.12) we have

\[
\tilde{Q}(z,0) = \frac{\left(\lambda W(z,0) + (\gamma / Z) (W(z,0) - W_0) \right) \tilde{S}_c \cdot (\lambda + \eta - \lambda z) - 1}{(\lambda z - (\lambda + \eta))}.
\]

(2.16)

Substituting for \(Q(z,0)\) from Equation (2.15) into Equation (2.10) we get

\[
W(z,0) = \frac{\omega \omega - (\gamma / Z) \tilde{S}_c \cdot (\lambda + \eta - \lambda z)) + P_0}{(\lambda + \eta + \gamma) - (\lambda + (\gamma / Z)) \tilde{S}_c \cdot (\lambda + \eta - \lambda z)}
\]

(2.17)

Substituting \(\theta = \lambda - \lambda z\) in the Equation (2.13) we obtain

\[
P(z,0) = (\lambda + (\gamma / z)) \tilde{S}_c \cdot (\lambda - \lambda z) I(z,0) - (\gamma / z) \tilde{S}_n \cdot (\lambda - \lambda z) I_n + \tilde{Q}(z,0) \eta \tilde{S}_c \cdot (\lambda - \lambda z).
\]

(2.18)

Substituting for \(\tilde{Q}(z,0)\) from Equation (2.16) into Equation (2.18) we get

\[
P(z,0) = \frac{(\lambda + (\gamma / z)) \tilde{S}_c \cdot (\lambda - \lambda z) I(z,0) - (\gamma / z) \tilde{S}_n \cdot (\lambda - \lambda z) I_n +
(\lambda W(z,0) + (\gamma / Z) (W(z,0) - W_0) \right) \tilde{S}_c \cdot (\lambda + \eta - \lambda z) - 1) \eta \tilde{S}_c \cdot (\lambda - \lambda z))}{(\lambda z - (\lambda + \eta))}.
\]

(2.19)

Substituting for \(P(z,0)\) from Equation (2.19) into Equation (2.13) we obtain
\[
(\lambda, W(z,0) + (\gamma / z)(W(z,0) - W_c))((\tilde{S}_v(\lambda + \eta - \lambda z) - 1)\eta(\tilde{S}_v(\lambda - \lambda z) - 1) + \\
(\lambda + (\gamma / z))I(z,0))((\tilde{S}_v(\lambda - \lambda z) - 1)(\lambda z - (\lambda + \eta)) - (\gamma / z)(\tilde{S}_v(\lambda - \lambda z) - 1))
\]

\[
\tilde{P}(z,0) = \frac{(\lambda z - (\lambda + \eta))}{(\lambda z - (\lambda + \eta))(-\lambda + \lambda z)}
\]

(2.20)

Substituting for \( P(z,0) \) from Equation (2.19) in the Equation (2.11) we get

\[
(W(z,0)[\eta(\lambda z - \lambda - \eta) + (\lambda + (\gamma / z))\tilde{S}_v(\lambda + \eta - \lambda z) \\
\eta\tilde{S}_v(\lambda - \lambda z)] + (\gamma \lambda - p_o - (\gamma / z)\tilde{S}_v(\lambda - \lambda z)I,)
\]

\[
(\lambda z - \lambda - \eta) - (\gamma / z)W_c(\tilde{S}_v(\lambda + \eta - \lambda z) - 1)
\]

\[
I(z,0) = \frac{\tilde{S}_v(\lambda - \lambda z)\eta)}{(\lambda z - \lambda - \eta)((\lambda + \gamma) - (\lambda + (\gamma / z))\tilde{S}_v(\lambda - \lambda z))}
\]

(2.21)

The following theorem is proved by substituting the Equations (2.16), (2.17), (2.20) and (2.21) in (2.14).

**Theorem : 2.1**

The probability generating function \( P(z) \) of number of customers in the orbit is given as

\[
(P(z,0))((\lambda z - (\lambda + \eta))(-\lambda + \lambda z) + (\lambda + (\gamma / z))((\tilde{S}_v(\lambda + \eta - \lambda z)) \\
\eta(\tilde{S}_v(\lambda - \lambda z) - 1) + (\lambda + (\gamma / z))(\tilde{S}_v(\lambda + \eta - \lambda z))(-\lambda + \lambda z)) \\
+I(z,0)((\lambda z - (\lambda + \eta))(-\lambda + \lambda z) + (\lambda + (\gamma / z))(\tilde{S}_v(\lambda - \lambda z) - 1) \\
(\lambda z - (\lambda + \eta))) + W_c((\gamma / z)\eta(\tilde{S}_v(\lambda + \eta - \lambda z) - 1)) \\
(1 - \tilde{S}_v(\lambda - \lambda z)) + (\gamma / z)(\tilde{S}_v(\lambda + \eta - \lambda z) - 1)(\lambda z - \lambda))
\]

\[
P(z) = \frac{+W_c((\gamma / z)\eta(\tilde{S}_v(\lambda + \eta - \lambda z) - 1))((\lambda z - (\lambda + \eta)))}{(\lambda z - (\lambda + \eta))(-\lambda + \lambda z)},
\]

(2.22)
where

$$W(z,0) = \frac{W_0 \left( \frac{\gamma}{z} \right) S_0 (\lambda + \eta - \lambda z)}{\left( (\lambda + \eta - \lambda z) - (\lambda + (\gamma / z)) S_0 (\lambda + \eta - \lambda z) \right)},$$

and

$$W(z,0) (\lambda - \lambda z - \eta) + (\lambda + (\gamma / z)) S_0 (\lambda + \eta - \lambda z)$$

$$\eta S_b (\lambda - \lambda z) + (\gamma I_0 - p_o - (\gamma / z) S_b (\lambda - \lambda z) I_0)$$

$$I(z,0) = \frac{(\lambda z - \lambda - \eta - (\gamma / z) W_0 (S_0 (\lambda + \eta - \lambda z) - 1) (S_{b} (\lambda - \lambda z) \eta)}{(\lambda z - \lambda - \eta - (\lambda + \gamma) \eta) S_0 (\lambda + \eta - \lambda z)}.$$

### 2.5 STABILITY CONDITION

The probability generating function P(z) has to satisfy the condition

$$\lim_{z \to 1} P(z) = 1.$$ 

In order to satisfy this condition L’Hôpital’s rule is applied twice to Equation (2.22). Since W_0 and I_0 are probabilities the numerator of P(z) is positive when z→1. So

$$\lim_{z \to 1} P(z) = 1$$

is satisfied only if

$$2\eta^2 \lambda (\lambda + \eta + \gamma - \lambda + \gamma) S_0 (\eta)^2 (\gamma - (\lambda + \gamma) S_b > 0),$$

that is only when

$$\gamma > \frac{\lambda^2}{\mu_b - \lambda}.$$ 

So we derive the steady state condition as

$$\gamma > \frac{\lambda^2}{\mu_b - \lambda}$$

which coincides with the stability condition given by Tien Van Do (2009).

### 2.6 PERFORMANCE CHARACTERISTICS

This section discusses some useful performance measures of the proposed model such as probability that the server is not occupied during working vacation and the probability that the server is not occupied when the server is not on working vacation, the probability that the server is busy during working vacation and the probability that the server is busy when the
server is not on working vacation are derived. Further, the mean orbit size $L_v$ during a working vacation and the mean orbit size $L_b$ when the server is not on a working vacation are also studied.

### 2.6.1 Probability that the Server is Not Occupied during Working Vacation

Using Equation (2.17) and applying limit $z \to 1$ we get $P_{iv}$, probability that the server is not occupied during working vacation as

$$P_{iv} = \frac{w \gamma (1 - S_v(\eta)) + p_v}{(\lambda + \gamma)(1 - S_v(\eta)) + \eta}. \quad (2.23)$$

### 2.6.2 Probability that the Server is Not Occupied when it is not on Working Vacation

Using Equation (2.21) and applying limit $z \to 1$ we get $P_{inv}$, probability that the server is not occupied when it is not on working vacation as

$$P_{inv} = \frac{(W_0 \gamma \eta ((\lambda S_{v1} + \eta S_{b1})(1 - S_v(\eta)) - \eta) + I_v \gamma \eta ((\lambda + \gamma)(1 - S_v(\eta)) + \eta)(S_{b1} - 1))}{\eta \gamma ((\lambda + \gamma)(1 - S_v(\eta)) + \eta)(S_{b1} - 1)}. \quad (2.24)$$

### 2.6.3 Probability that the Server is Busy During Working Vacation

Using Equation (2.16) and applying limit $z \to 1$ we get $P_{bv}$, the probability that the server is busy during working vacation as
\[
P_{bv} = \frac{(1 - \tilde{S}_v(\eta))(\lambda + \gamma)p_0 - \eta \gamma W_v)}{\eta((\lambda + \gamma)(1 - \tilde{S}_v(\eta)) + \eta)}.
\] (2.25)

### 2.6.4 Probability that the Server is Busy when it is Not on Working Vacation

Using Equation (2.20) and applying limit \( z \to 1 \) we get \( P_{bv} \), the probability that the server is busy when not on working vacation as

\[
P_{bv} = \frac{\lambda + \gamma(1 - \tilde{S}_v(\eta)) + \eta_0)(1 - \tilde{S}_v(\eta)) + (\lambda + \gamma)P_{nv} - \gamma(S_{bv} / \lambda),}{(2.26)}
\]

where

\[
S_{bl} = \lambda E(S_b)
\]

\[
S_{v1} = \lambda \int_0^\eta e^{-\eta \tilde{S}_v(t)} dt.
\]

### 2.6.5 The Mean Orbit Size \( L_Q \)

Let \( L_v \) and \( L_b \) denote the mean orbit size during working vacation and regular busy periods respectively. Since the expressions are too large, the numerical values of \( L_v \) and \( L_b \) are calculated using the software Mathematica.

\[
L_v = \lim_{z \to 1} \frac{d}{dz} W(z,0) + \tilde{Q}(z,0),
\]

\[
L_b = \lim_{z \to 1} \frac{d}{dz} I(z,0) + \tilde{P}(z,0).
\]

Hence the mean orbit size is given by \( L_Q = L_v + L_b \).
2.6.6 **Mean Waiting Time in the Retrial Queue**

Using Little’s formula, the mean waiting time in the retrial queue is 

\[ W = \frac{L}{\lambda}. \]

2.7 **SPECIAL CASES**

The model so developed is general in nature as the service time is arbitrary. But for practical purposes, service time with a particular distribution is required. In this section, some special cases of the proposed model by specifying service time random variables as Exponential, Erlang and Hyper Exponential distribution are discussed.

**Case (i) :** \( M / M / 1 \) retrial queue with working vacation (Exponential service time both for the service during working vacation and for the service when the server is not on working vacation) and constant retrial policy.

If the service times are assumed to be Exponential with probability density functions \( s_i(x) = u_i e^{-u_i x} \) where \( u_i (i = 1, 2) \) is the parameter and \( u_i > 0, x \geq 0 \) then

\[
\tilde{S}_h(\lambda - \lambda x) = \left( \frac{u_1}{u_1 + (\lambda - \lambda x)} \right), \quad \tilde{S}_h(\lambda + \eta - \lambda x) = \left( \frac{u_2}{u_2 + (\lambda + \eta - \lambda x)} \right).
\]

Substituting in (2.22), the PGF of the retrial queue size distribution for single server retrial queue with working vacation and constant retrial policy is given as
\[
(W(z,0) ((\lambda z - (\lambda + \eta))(-\lambda + \lambda z) + (\lambda + (\gamma / z)))
+ (u_2 / (u_2 + (\lambda + \eta - \lambda z))))\eta((u_1 / (u_1 + (\lambda - \lambda z))) - 1)
+ (u_2 / (u_2 + (\lambda + \eta - \lambda z))) (-\lambda + \lambda z))
+ I(z,0)((\lambda z - (\lambda + \eta))(-\lambda + \lambda z) + (\lambda + (\gamma / z)))
+ W_0((\eta / z)u_2 / (u_2 + (\lambda + \eta - \lambda z)) - 1)
(1 - (u_1 / (u_1 + (\lambda - \lambda z)))) + (\gamma / z)((u_2 / (u_2 + (\lambda + \eta - \lambda z))) - 1)
\]

\[
P(z) = \frac{(\lambda z - (\lambda + \eta))(-\lambda + \lambda z)}{((\lambda + \eta - (\lambda + (\gamma / z))) + p \eta((\lambda + (\gamma / z))) + (u_2 / (u_2 + (\lambda + \eta - \lambda z))))}
\]

where

\[
W(z,0) = \frac{W_0((\eta / z)(u_2 / (u_2 + (\lambda + \eta - \lambda z)))) + p \eta((\lambda + (\gamma / z))) + (u_2 / (u_2 + (\lambda + \eta - \lambda z))))}{((\lambda + \eta - (\lambda + (\gamma / z))) + (u_2 / (u_2 + (\lambda + \eta - \lambda z))))}
\]

Case (ii) : M / E_k / 1 retrial queue with working vacation (Erlang service time both for the service during working vacation and for the service when the server is not on working vacation) and constant retrial policy.

If the server times are assumed to be Erlang with probability density function \( s_i(x) = \frac{x^{k_i - 1} e^{-\lambda x}}{(k_i - 1)!}, i = 1, 2; u_i > 0, \) \( k_i \) is a positive integer and \( u_i \) is the parameter then

\[
\tilde{S}_b(\lambda - \lambda z) = \left(\frac{u_{i_1}}{u_{i_1} + (\lambda - \lambda z)}\right)^{k_{i_1}}
\]

\[
\tilde{S}_s(\lambda + \eta - \lambda z) = \left(\frac{u_{i_2}}{u_{i_2} + (\lambda + \eta - \lambda z)}\right)^{k_{i_2}}
\]
Substituting in (2.22), the PGF of the retrial queue size distribution for single server retrial queue with working vacation and constant retrial policy is given by

\[
(W(z,0)((\lambda z-(\lambda+\eta))(-\lambda+\lambda z)+(\lambda+(\gamma/z)))
\]
\[
((u_k, k_z / (u_k, k_z + (\lambda + \eta - \lambda z)))^{k_z} - 1)\eta((u_k, k_1 / (u_k, k_1 + (\lambda - \lambda z)))^{k_1} - 1)
\]
\[
+((\lambda + (\gamma/z))((u_k, k_z / (u_k, k_z + (\lambda + \eta - \lambda z)))^{k_z} - 1)(-\lambda+\lambda z))
\]
\[
+I(z,0)((\lambda z-(\lambda+\eta))(-\lambda+\lambda z)+(\lambda+(\gamma/z))((u_k, k_1 / (u_k, k_1 + (\lambda - \lambda z)))^{k_1} - 1)
\]
\[
+((\lambda z-(\lambda+\eta)))W_{(z,0)}((\gamma/z)\eta((u_k, k_z / (u_k, k_z + (\lambda + \eta - \lambda z)))^{k_z} - 1)
\]
\[
(1-(u_k, k_1 / (u_k, k_1 + (\lambda - \lambda z)))^{k_1} + (\gamma/z)((u_k, k_z / (u_k, k_z + (\lambda + \eta - \lambda z)))^{k_z} - 1)
\]
\[
P(z) = \frac{(\lambda z-(\lambda+\eta))I_{(\gamma/z)}(1-(u_k, k_1 / (u_k, k_1 + (\lambda - \lambda z)))^{k_1}(\lambda z-(\lambda+\eta))))}{(\lambda z-(\lambda+\eta))(-\lambda+\lambda z)}.
\]

(2.28)

Case (iii) : M / Hyper exponential / 1 retrial queue with working vacation (Hyper Exponential service time both for the service during working vacation and for the service when the server is not on working vacation) and constant retrial policy.

If the service times are assumed to be Hyper Exponential with probability density function \( s(x) = c u e^{-ux} + (1-c)w e^{-wx} \) where \( x > 0, u > 0, w > 0, 0 \leq c \leq 1 \), then

\[
\hat{S}_b(\lambda - \lambda z) = \left( \frac{u_1 c}{u_1 + (\lambda - \lambda z)^2} \right) + \left( \frac{w_1 (1-c)}{w_1 + (\lambda - \lambda z)^2} \right)
\]
\[
\hat{S}_s(\lambda + \eta - \lambda z) = \left( \frac{u_1 c}{u_1 + (\lambda + \eta - \lambda z)^2} \right) + \left( \frac{w_1 (1-c)}{w_2 + (\lambda + \eta - \lambda z)^2} \right)
\]

Substituting in (2.22), the PGF of the retrial queue size distribution for single server retrial queue with working vacation and constant retrial policy is obtained.
2.8 NUMERICAL RESULTS

This section presents certain numerical results to justify the theoretical results obtained. To study the effect of arrival rate \( \lambda \) and retrial rate \( \gamma \) on the mean orbit size and the mean waiting time \( W \), the following notations are used and some assumptions are made:

- Average arrival rate \( \lambda \)
- Service rate during working vacation \( \mu_v \)
- Regular service rate (when the server is not on working vacation) \( \mu_b \)
- Vacation duration is exponential with parameter \( \eta \)
- Retrial rate \( \gamma \)

Table 2.1 shows the way the mean orbit size \( L_Q \) and the mean waiting time \( W \) change for increasing values of retrial rate \( \gamma \). Figures 2.2 and 2.3 show the influence of retrial rate \( \gamma \) on mean orbit size \( L_Q \) and the mean waiting time \( W \). The service times are considered as exponential, Erlang-2 and Hyper exponential with parameters \( \lambda=0.3, \mu_v=0.2, \mu_b=1, \) and \( \eta=2 \).

It is observed that

- Mean orbit size decreases when retrial rate increases
- Mean waiting time decreases when retrial rate increases.

From Table 2.2, we can see the effect of arrival rate \( \lambda \) on the mean orbit size \( L_Q \) and the mean waiting time \( W \). In Figures 2.4 and 2.5, the mean orbit size \( L_Q \) and the mean waiting time \( W \) are compared for different arrival rates. The service times are considered as Exponential, Erlang-2 and Hyper exponential with parameters \( \gamma=0.6, \mu_v=0.2, \mu_b=1, \) and \( \eta=2 \).
It is observed that

- Mean orbit size increases when arrival rate increases.
- Mean waiting time increases when arrival rate increases.

Table 2.3, Figures 2.6 and 2.7 represent the effect of service rate $\mu_e$ on the mean orbit size and the mean waiting time $W$. The service times are considered as Exponential, Erlang-2 and Hyper exponential with parameters $\lambda=.03, \gamma=0.6, \mu_v=0.2$, and $\eta=2$.

It is observed that

- Mean orbit size decreases when service rate $\mu_e$ increases.
- Mean waiting time decreases when service rate $\mu_e$ increases.

Table 2.4 gives the effect of retrial rate $\gamma$ on the probability that the server is busy during working vacation with $\lambda=0.1, \mu_v=0.2, \mu_b=0.9$, and $\eta=2$ when the service time distribution follows Exponential, Erlang-2 and Hyper exponential respectively.

It is observed that

- Probability that the server is busy during working vacation increases when retrial rate increases.

Table 2.5 represents the effect of retrial rate $\gamma$ on the probability that the server is busy and not on working vacation with $\lambda=0.1, \mu_v=0.2, \mu_b=0.9$, and $\eta=2$ when the service time distribution follows Exponential, Erlang-2 and Hyper exponential respectively.
It is observed that

- Probability that the server is busy and not on working vacation increases when retrial rate increases.

Table 2.6 represents the effect of retrial rate \( \gamma \) on the probability that the server is not occupied during working vacation with \( \lambda = 0.1, \mu_v = 0.2, \mu_b = 0.9 \), and \( \eta = 2 \) when the service time distribution follows Exponential, Erlang-2 and Hyper exponential respectively.

It is observed that

- Probability that the server is not occupied during working vacation decreases when retrial rate increases

Table 2.7 represents the effect of retrial rate \( \gamma \) on the probability that the server is not occupied when not on working vacation with \( \lambda = 0.1, \mu_v = 0.2, \mu_b = 0.9 \), and \( \eta = 2 \) when the service time distribution follows Exponential, Erlang-2 and Hyper exponential respectively.

It is observed that

- Probability that the server is not occupied when not on working vacation is decreasing when retrial rate increases.

2.9 CONCLUSION

In this chapter, a single server retrial queue with general retrial time, single working vacation and constant retrial policy is analyzed under the condition of stability. The probability generating function for the queue size at an arbitrary time epoch is derived. Some system performance measures are computed in steady state. The theoretical development of the model has been justified with numerical results.
Table 2.1 Retrial Rate $\gamma$ (Vs) Mean Orbit Size $L_Q$ and Mean Waiting Time $W$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Exponential</th>
<th>Erlang-2</th>
<th>Hyper Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L_Q$</td>
<td>$W$</td>
<td>$L_Q$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5972</td>
<td>1.9907</td>
<td>0.5321</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4828</td>
<td>1.6093</td>
<td>0.4249</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4058</td>
<td>1.3525</td>
<td>0.3525</td>
</tr>
<tr>
<td>0.8</td>
<td>0.3495</td>
<td>1.1650</td>
<td>0.2994</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3061</td>
<td>1.0201</td>
<td>0.2583</td>
</tr>
<tr>
<td>1.0</td>
<td>0.2710</td>
<td>0.9032</td>
<td>0.2249</td>
</tr>
<tr>
<td>1.1</td>
<td>0.2418</td>
<td>0.8059</td>
<td>0.1971</td>
</tr>
<tr>
<td>1.2</td>
<td>0.2168</td>
<td>0.7227</td>
<td>0.1733</td>
</tr>
<tr>
<td>1.3</td>
<td>0.195</td>
<td>0.6501</td>
<td>0.1524</td>
</tr>
<tr>
<td>1.4</td>
<td>0.1757</td>
<td>0.5857</td>
<td>0.1338</td>
</tr>
<tr>
<td>1.5</td>
<td>0.1583</td>
<td>0.5278</td>
<td>0.1171</td>
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</tbody>
</table>

Table 2.2 Arrival Rate $\lambda$ (Vs) Mean Orbit Size $L_Q$ and Mean Waiting Time $W$

<table>
<thead>
<tr>
<th>$\lambda$</th>
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<th>Hyper Exponential</th>
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<tbody>
<tr>
<td></td>
<td>$L_Q$</td>
<td>$W$</td>
<td>$L_Q$</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0165</td>
<td>0.165</td>
<td>0.0112</td>
</tr>
<tr>
<td>0.15</td>
<td>0.0685</td>
<td>0.4567</td>
<td>0.0566</td>
</tr>
<tr>
<td>0.20</td>
<td>0.1664</td>
<td>0.8321</td>
<td>0.1429</td>
</tr>
<tr>
<td>0.25</td>
<td>0.3362</td>
<td>1.3448</td>
<td>0.2929</td>
</tr>
<tr>
<td>0.30</td>
<td>0.6309</td>
<td>2.1033</td>
<td>0.5522</td>
</tr>
<tr>
<td>0.35</td>
<td>1.1784</td>
<td>3.3669</td>
<td>1.0312</td>
</tr>
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<td>2.3793</td>
<td>5.9482</td>
<td>2.0759</td>
</tr>
<tr>
<td>0.45</td>
<td>6.4691</td>
<td>14.3757</td>
<td>5.6154</td>
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</table>
Table 2.3  Service Rate $\mu_b$ (Vs) Mean Orbit Size $L_Q$ and Mean Waiting Time $W$

<table>
<thead>
<tr>
<th>$\mu_b$</th>
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<th>Hyper Exponential</th>
</tr>
</thead>
<tbody>
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<td>$L_Q$</td>
<td>$W$</td>
<td>$L_Q$</td>
</tr>
<tr>
<td>1.1</td>
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</tr>
<tr>
<td>1.2</td>
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</tr>
<tr>
<td>1.3</td>
<td>0.2629</td>
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</tr>
<tr>
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<td>0.1924</td>
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<td>0.1719</td>
</tr>
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<td>0.5579</td>
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</tr>
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<tr>
<td>2.0</td>
<td>0.1029</td>
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</table>

Table 2.4  Retrial Rate $\gamma$ (Vs) Probability that the Server is Busy during Working Vacation $P_{iv}$

<table>
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<th>Hyper Exponential</th>
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<td>0.2</td>
<td>0.0393</td>
<td>0.0392</td>
<td>0.0393</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0400</td>
<td>0.0399</td>
<td>0.0399</td>
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<td>0.4</td>
<td>0.0403</td>
<td>0.0403</td>
<td>0.0403</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0404</td>
<td>0.0404</td>
<td>0.0404</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0404</td>
<td>0.0404</td>
<td>0.0404</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0404</td>
<td>0.0404</td>
<td>0.0404</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0404</td>
<td>0.0404</td>
<td>0.0404</td>
</tr>
<tr>
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<td>0.0404</td>
<td>0.0404</td>
<td>0.0404</td>
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</table>
Table 2.5  Retrial Rate $\gamma$ (Vs) Probability that the Server is Busy but not on Working Vacation $P_{inv}$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$P_{inv}$</th>
<th></th>
<th></th>
<th></th>
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<tbody>
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<td>Hyper Exponential</td>
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</tr>
<tr>
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<td>0.0098</td>
<td>0.0097</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
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<td>0.0149</td>
<td>0.0148</td>
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</tr>
<tr>
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<td>0.0187</td>
<td>0.0193</td>
<td>0.0192</td>
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</tr>
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</tr>
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</tr>
<tr>
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<td></td>
</tr>
<tr>
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<td>0.0306</td>
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Table 2.6  Retrial Rate $\gamma$ (Vs) Probability that the Server is Not Occupied During Working Vacation $P_{bv}$.

<table>
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<th>$P_{bv}$</th>
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</thead>
<tbody>
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<td>Hyper Exponential</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
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<td>0.0034</td>
<td>0.0033</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.0031</td>
<td>0.0033</td>
<td>0.0033</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.0030</td>
<td>0.0032</td>
<td>0.0032</td>
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<tr>
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<td>0.0030</td>
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<td>0.0032</td>
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<td>0.0030</td>
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Table 2.7 Retrial Rate $\gamma$ (Vs) the Probability that the Server is Not Occupied and Not on Working Vacation $P_{bnv}$

<table>
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<th>$\gamma$</th>
<th>$P_{bnv}$</th>
<th>$P_{bnv}$</th>
<th>$P_{bnv}$</th>
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</thead>
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<td>Exponential</td>
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<td>Hyper Exponential</td>
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<tr>
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<td>0.1030</td>
<td>0.0944</td>
</tr>
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<td>0.1030</td>
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</tr>
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</tr>
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<td>0.0857</td>
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<tr>
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<tr>
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<td>0.0056</td>
<td>0.0968</td>
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</table>

Figure 2.2 Retrial Rate $\gamma$ (Vs) Mean Orbit Size $L_Q$
Figure 2.3: Retrial Rate $\gamma$ (Vs) Mean Waiting Time $W$

Figure 2.4: Arrival Rate $\lambda$ (Vs) Mean Orbit Size $L_Q$
Figure 2.5 Arrival Rate $\lambda$ (Vs) Mean Waiting Time $W$

Figure 2.6 Service Rate $\mu_b$ (Vs) Mean Orbit Size $L_Q$
Figure 2.7 Service Rate $\mu_b$(Vs) Mean Waiting Time $W$