CHAPTER 5

ANALYSIS OF AN M/G/1 RETRIAL QUEUEING
SYSTEM WITH DIFFERENT CLASSES OF
ARRIVALS SUBJECT TO BREAKDOWNS

5.1 INTRODUCTION

Negative arrivals are used as a control mechanism in many telecommunication and computer Networks. When a negative customer arrives at the queue, it immediately removes the customer in service.

G-queues or queues with negative arrivals were first introduced for the purpose of modeling neural networks. A comprehensive analysis of queueing networks with negative arrivals can be found in Chao et al (1999), Galenbe & Pujolle (1998), Atentia & Moreno (2007). Recently, Jinbiao Wu & Zhaotong Lian (2013) analyzed M₁,M₂/G/1 G-Queueing systems with retrial customers. In most of queueing literature, the server is assumed to be always available. In fact, queueing systems with server breakdowns are very common in computer, manufacturing systems and communication networks. Retrial queues that take into account server failures and repairs were introduced by Aissani (1994). Zaiming Liu et al (2009) analyzed an M/G/1 retrial G-queue with server breakdowns and repairs. Queueing systems and networks with negative customers have a rich and wide range of real world applications. In practice, the negative arrivals affect the performance of system partially in most of the situations.
Partially negative arrivals are not considered in many research contributions. This work focuses on considering the effect of partially negative arrivals.

A single server retrial queueing system with three classes of customers namely positive, negative and partially negative customers with server breakdowns is discussed in this chapter. We assume that there is no waiting space and therefore, if an arriving positive customer finds the server idle, it gets the service immediately. If an arriving positive customer finds the server busy with regular service or slow service or under repair, it enters the retrial group (called orbit) to seek the service again till it finds the server ideal. If an arriving negative customer finds the server idle, it is lost. If an arriving negative customer finds the server busy with regular service or slow service, it not only removes the customer under service, but also causes the server break down. When the server fails, it is sent to repair immediately. If an arriving partially negative customer finds the server idle, it is lost. On the other hand, if an arriving partially negative customer finds the server busy with regular service, it slows down the server. If an arriving partially negative customer finds the server busy with slow service, it breaks down the server. The model under study is schematically represented in Figure 5.1.
5.2 MOTIVATION

The motivation of the model comes from a situation where negative arrivals are used as a control mechanism in many telecommunication and computer Networks. Since the introduction of the concept of negative customers by Gelenbe (1989), research on queueing systems with negative arrivals has been greatly motivated by some special practical applications such as computer, neural networks, manufacturing systems and...
communication networks. In a computer network or database, negative customers can represent viruses or commands to delete some transaction. In a neural network, negative and positive customers can represent inhibitory and excitatory signals respectively. In a manufacturing system, negative customers can represent orders of demand. In practice, the negative arrivals affect the performance of system partially in most of the situations. This partial effect of negative arrivals is not considered into account in many research contributions. In this chapter we have considered the partial effect of negative arrivals.

5.3 MATHEMATICAL MODEL

The customers namely, positive, negative and partially negative arrive according to three independent Poisson processes at the rates $\lambda_1$, $\lambda_2$ and $\lambda_3$ respectively ($\lambda = \lambda_1 + \lambda_2 + \lambda_3$). The time between two successive repeated attempts of each call in orbit is assumed to be exponentially distributed with a rate $\gamma$. Let $S(x)$ ($s(x)$) $\{\tilde{S}(\theta)\}$ $[S(\theta)]$ be the cumulative distribution function (probability density function) {Laplace – Stieltjes transform} [remaining service time] of regular service. Let $R(x)$ ($r(x)$) $\{\tilde{R}(\theta)\}$ $[R(\theta)]$ be the cumulative distribution function (probability density function) {Laplace – Stieltjes transform} [remaining service time] of repair. Let $S_s(x)$ ($s_s(x)$) $\{\tilde{S}_s(\theta)\}$ $[S_s(\theta)]$ be the cumulative distribution function (probability density function) {Laplace – Stieltjes transform} [remaining service time] of slow service. $N(t)$ denotes the number of customers in the orbit at time $t$. The server state is denoted as

$$C(t) = \begin{cases} 
0, & \text{if the server is idle} \\
1, & \text{if the server is busy with regular service} \\
2, & \text{if the server is under repair} \\
3, & \text{if the server is busy with slow service}
\end{cases}$$
Now the system state probabilities are defined as follows

1. \( P_{0,j}(t) = P\{C(t) = 0, N(t) = j\}, \ j \geq 0 \) is the probability that at time \( t \) the server is idle and the orbit size is \( j \).

2. \( P_{1,j}(x, t)dt = P\{C(t) = 1, N(t) = j, x \leq S(t) \leq x + dt\}, \ j \geq 0 \) is the probability that at time \( t \) the server is busy with regular service, the orbit size is \( j \) and the remaining service time of a customer under regular service at an arbitrary time is between \( x \) and \( x + dt \).

3. \( P_{2,j}(x, t)dt = P\{C(t) = 2, N(t) = j, x \leq R(t) \leq x + dt\}, \ j \geq 0 \) is the probability that at time \( t \) the server is under repair, the orbit size is \( j \) and the remaining repair time of a customer at an arbitrary time is between \( x \) and \( x + dt \).

4. \( P_{3,j}(x, t)dt = P\{C(t) = 3, N(t) = j, x \leq S(t) \leq x + dt\}, \ j \geq 0 \) is the probability that at time \( t \) the server is busy with slow service, the orbit size is \( j \) and the remaining service time of a customer under slow service at an arbitrary time is between \( x \) and \( x + dt \).

The above notations and probability functions are used to develop steady state queue size distribution.

5.4 STEADY STATE QUEUE SIZE DISTRIBUTION

To derive the steady state queue size distribution the following equations are obtained using supplementary variable technique,

\[
P_{0,j}(t + \Delta t) = P_{0,j}(t)(1 - \lambda \Delta t - j \gamma \Delta t) + P_{1,j}(0, t) \Delta t + P_{2,j}(0, t) \Delta t + P_{3,j}(0, t) \Delta t, \ j \geq 0
\]

\[
P_{1,j}(x - \Delta t, t + \Delta t) = P_{1,j}(x, t)(1 - \lambda \Delta t) + \lambda_s P_{0,j}(t)S(x) \Delta t
\]

\[\quad + (j + 1) \gamma P_{0,j+1}(t)S(x) \Delta t + \lambda_s P_{1,j-1}(x) \Delta t, \ j \geq 0
\]
From the above equations, the steady state queue size equations are obtained as follows:

\[ (\lambda_1 + j\gamma)P_{i,j} - P_{i,j}(0) + P_{i-1,j}(0) + P_{i,j+1}(0), \quad j \geq 0 \] (5.1)

\[ -\frac{d}{dx}P_{1,j}(x) = -\lambda_1 P_{1,j}(x) + \lambda_2 P_{2,j}(x) s(x) + (j + 1) \gamma P_{i,j+1}(0) s(x) + \lambda_3 P_{i,j+1}(0), \quad j \geq 0 \] (5.2)

\[ -\frac{d}{dx}P_{2,j}(x) = -\lambda_2 P_{2,j}(x) + \lambda_3 \left( \int_0^\infty p_{i,j}(x) dx \right) r(x) + \delta_{1,j} P_{i,j+1}(0), \quad j \geq 0 \] (5.3)

\[ -\frac{d}{dx}P_{3,j}(x) = -\lambda_3 P_{3,j}(x) + \lambda_4 \left( \int_0^\infty p_{i,j}(x) dx \right) s(x) + \delta_{2,j} (1 - \delta_{0,j}) P_{i,j+1}(0), \quad j \geq 0 \] (5.4)

The Laplace-Stieltjes transforms (LST) of \( P_{ij}(x), \ i = 1,2,3 \) are defined as

\[ \text{LST}\{P_{ij}(x)\} = \tilde{P}_{ij}(\theta) = \int_0^\infty e^{-\theta x} P_{ij}(x) \, dx, \quad i = 1,2,3 \]
Taking Laplace-Stieltjes Transform (LST) on steady state Equations (5.2) – (5.4) we have

\[ \theta \tilde{P}_{j,1}(\theta) - P_{j,1}(\theta) = \lambda_i \tilde{P}_{j,1}(\theta) - \lambda_i P_{j,1}(\theta) - (\gamma + 1) \gamma P_{\gamma,1} S_{j,1}(\theta) - \lambda_i P_{j,1}^{(j+1)}(\theta), \quad j \geq 0 \]  

(5.5)

\[ \theta \tilde{P}_{j,1}(\theta) - P_{j,1}(\theta) = \lambda_i \tilde{P}_{j,1}(\theta) - \lambda_i P_{j,1}(\theta) \tilde{R}(\theta) - \lambda_i (1 - \delta_{j,0}) \tilde{P}_{j,1}^{(j+1)}(\theta), \quad j \geq 0 \]  

(5.6)

\[ \theta \tilde{P}_{j,1}(\theta) - P_{j,1}(\theta) = \lambda_i \tilde{P}_{j,1}(\theta) - \lambda_i P_{j,1}(\theta) S_{j,1}(\theta) - \lambda_i (1 - \delta_{j,0}) \tilde{P}_{j,1}^{(j+1)}(\theta), \quad j \geq 0 \]  

(5.7)

**5.4.1 Probability Generating Function**

As discussed in section 2.4.1 of chapter II, to find the probability generating function (PGF) of the number of customers in the orbit at an arbitrary time epoch, the following probability generating functions are defined.

\[ \tilde{P}_{1,1}(z, \theta) = \sum_{j=0}^{\infty} \tilde{P}_{j,1}(\theta) z^j ; \quad P_{1,1}(z) = \sum_{j=0}^{\infty} P_{j,1}(\theta) z^j \]  

(5.8)

\[ \tilde{P}_{i,1}(z, \theta) = \sum_{j=0}^{\infty} \tilde{P}_{j,i}(\theta) z^j ; \quad P_{i,1}(z, 0) = \sum_{j=0}^{\infty} P_{j,i}(\theta) z^j ; \quad i = 1, 2, 3, \quad \text{where } |z| \leq 1 \]

Multiplying the Equations (5.1), (5.5) - (5.7) by \( z^n \), taking summation from \( n=0 \) to \( \infty \) and using the Equation (5.8), we get

\[ \lambda_i P_{i,1}(z) + \gamma z P'_{i,1}(z) = P_{i,1}(z, 0) + P_{i,2}(z, 0) + P_{i,3}(z, 0) \]  

(5.9)

\[ (\theta - \lambda + \lambda_i) \tilde{P}_{1,1}(z, 0) = P_{1,1}(z, 0) - (\lambda_{i} P_{i,1}(z) + \gamma P'_{i,1}(z)) \tilde{S}(\theta) \]  

(5.10)

\[ (\theta - \lambda_i + \lambda_i) \tilde{P}_{1,2}(z, 0) = P_{1,2}(z, 0) - (\lambda_i \tilde{P}_{1,1}(z, 0) + (\lambda_i + \lambda_i) \tilde{P}_{1,1}(z, 0)) \tilde{R}(\theta) \]  

(5.11)

\[ (\theta - \lambda_i + \lambda_i) \tilde{P}_{1,3}(z, 0) = P_{1,3}(z, 0) - (\lambda_i \tilde{P}_{1,1}(z, 0)) \tilde{S}_{i,1}(\theta) \]  

(5.12)
The probability generating function \( P(z) \) of number of customers in orbit at an arbitrary time instant can be expressed as follows

\[
P(z) = P_1(z) + \tilde{P}_1(z,0) + \tilde{P}_2(z,0) + \tilde{P}_3(z,0)
\]

(5.13)

In order to find \( \tilde{P}_1(z,0), \tilde{P}_2(z,0) \) and \( \tilde{P}_3(z,0) \), the following sequence of operations are done.

Substituting \( \theta = \lambda - \lambda_1z \) in the Equations (5.10) and (5.12) we have

\[
P_1(z,0) = (\lambda - \lambda_1)P_1(z) + \gamma P'_n(z)S(\lambda - \lambda_1z)
\]

(5.14)

\[
P_1(z,0) = \lambda_1 \tilde{P}_1(z,0)S(\lambda - \lambda_1z)
\]

(5.15)

Substituting \( \theta = \lambda_i - \lambda_1z \) in the Equation (5.11) we obtain

\[
P_2(z,0) = (\lambda_1 \tilde{P}_1(z,0) + (\lambda_1 + \lambda_2)\tilde{P}_2(z,0))R(\lambda_1 - \lambda_1z)
\]

(5.16)

Substituting for \( P_1(z,0), P_2(z,0), P_3(z,0) \) from the Equations (5.14), (5.16), (5.15) in the Equations (5.10), (5.11), (5.12) respectively we have,

\[
\tilde{P}_1(z,0) = \frac{(\lambda_1 P_1(z) + \gamma P'_n(z))(S(\lambda - \lambda_1z) - 1)}{(-\lambda + \lambda_1z)}
\]

(5.17)

\[
\tilde{P}_2(z,0) = \frac{(\lambda_1 \tilde{P}_1(z,0) + (\lambda_1 + \lambda_2)\tilde{P}_2(z,0))(R(\lambda_1 - \lambda_1z) - 1)}{(-\lambda_1 + \lambda_1z)}
\]

(5.18)

\[
\tilde{P}_3(z,0) = \frac{\lambda_3 \tilde{P}_3(z,0)S(\lambda - \lambda_1z) - 1}{(-\lambda + \lambda_1z)}
\]

(5.19)

Substituting for \( \tilde{P}_1(z,0) \) from the Equation (5.17) in the Equation (5.19) we have
\[ \tilde{P}_1(z,0) = \frac{\lambda_1 (\tilde{S}_1(\lambda - \lambda_1 z) - 1)(\lambda_1 P_0(z) + \gamma P'_c(z))(\tilde{S}(\lambda - \lambda_1 z) - 1)}{(-\lambda + \lambda_1 z)^2} \quad (5.20) \]

Substituting for \( \tilde{P}_1(z,0) \) and \( \tilde{P}_3(z,0) \) from the Equations (5.17) and (5.20) respectively in the Equation (5.16) we have

\[ (\lambda_2 \tilde{R}(\lambda_1 - \lambda_1 z) (\lambda_1 P_0(z) + \gamma P'_c(z))(\tilde{S}(\lambda - \lambda_1 z) - 1) \\
(-\lambda - \lambda_1 z + \lambda_1 \lambda_2) \tilde{R}(\lambda_1 - \lambda_1 z) \lambda_1 \tilde{S}(\lambda - \lambda_1 z) - 1)) \\
\tilde{P}_2(z,0) = \frac{(\lambda_1 P_0(z) + \gamma P'_c(z)))(\tilde{S}(\lambda - \lambda_1 z) - 1)}{(-\lambda - \lambda_1 z)^2} \quad (5.21) \]

Substituting for \( \tilde{P}_1(z,0) \) from the Equation (5.17) into the Equation (5.15) we have

\[ \tilde{P}_3(z,0) = \frac{\lambda_3 (\tilde{S}_1(\lambda - \lambda_1 z))(\lambda_1 P_0(z) + \gamma P'_c(z))(\tilde{S}(\lambda - \lambda_1 z) - 1)}{(-\lambda - \lambda_1 z)} \quad (5.22) \]

Substituting for \( \tilde{P}_1(z,0) \) and \( \tilde{P}_3(z,0) \) from the Equations (5.17) and (5.20) respectively in the Equation (5.18) we have

\[ (\lambda_2 (\lambda_1 P_0(z) + \gamma P'_c(z))(\tilde{S}(\lambda - \lambda_1 z) - 1)(\tilde{R}(\lambda_1 - \lambda_1 z) - 1) \\
+ (\lambda_2 + \lambda_3) \lambda_1 (\tilde{S}_1(\lambda - \lambda_1 z) - 1)) \lambda_1 P_0(z) + \gamma P'_c(z) \\
\tilde{P}_4(z,0) = \frac{(\tilde{S}(\lambda - \lambda_1 z) - 1))}{(\tilde{R}(\lambda_1 - \lambda_1 z) - 1))} \quad (5.23) \]

The following theorem is proved by substituting for \( \tilde{P}_1(z,0), \tilde{P}_2(z,0) \) and \( \tilde{P}_3(z,0), \) from the Equations (5.17), (5.20) and (5.23) in the Equation (5.13).
Theorem: 5.1

The probability generating function $P(z)$ of number of customers in the orbit is given by

\[ P(z) = \frac{\lambda_1 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}{(-\lambda + \lambda_1 z)(-\lambda + \lambda_2 z)(-\lambda + \lambda_3 z)} \]

\[ \text{(5.24)} \]

Substituting the Equations (5.14) – (5.16) in (5.9) we get

\[ P'(z) = \lambda_1 \lambda_2 \lambda_3 \frac{(z) + \gamma P'(z)}{(\lambda_1 - \lambda_2, z) - 1}(\lambda + \lambda_2, z)\]

\[ \text{(5.25)} \]

Substituting for $P'(z)$ from the Equation (5.25) in the Equation (5.24) we get

\[ \text{(5.26)} \]

where

\[ M_i = (\lambda_1 \lambda_2 \lambda_3)(\lambda_1 + \lambda_2, z)(\lambda_2 + \lambda_3, z)(\lambda_3 + \lambda_1, z) \]

\[ \text{(5.27)} \]
\[ M_1 = (z(-\lambda + \lambda_i z)^2 - \tilde{S}(\lambda - \lambda_i z)(-\lambda + \lambda_i z)^2 - \lambda_i \tilde{R}(\lambda - \lambda_i z)(\tilde{S}(\lambda - \lambda_i z) - 1) \\
+ (-\lambda + \lambda_i z)(\tilde{R}(\lambda - \lambda_i z) - 1)(\tilde{S}(\lambda - \lambda_i z) - 1)(-\lambda_i + \lambda_i z)) \]  
(5.28)

\[ M_3 = (\tilde{S}(\lambda - \lambda_i z) - 1)(-\lambda + \lambda_i z)(-\lambda_i + \lambda_i z) + \lambda_i (\tilde{R}(\lambda - \lambda_i z) - 1)(\tilde{S}(\lambda - \lambda_i z) - 1) \\
+ (-\lambda + \lambda_i z) + \lambda_i (\lambda_i + \lambda_i z)(\tilde{R}(\lambda - \lambda_i z) - 1)(\tilde{S}(\lambda - \lambda_i z) - 1)(\tilde{S}(\lambda - \lambda_i z) - 1)(-\lambda_i + \lambda_i z)) \]  
(5.29)

\[ M_4 = (-\lambda_i + \lambda_i z)^2 + \lambda_i (\tilde{S}(\lambda - \lambda_i z))(\tilde{S}(\lambda - \lambda_i z)); (-\lambda + \lambda_i z) \\
+ \lambda_i \tilde{R}(\lambda - \lambda_i z)\tilde{S}(\lambda - \lambda_i z) - 1)(-\lambda + \lambda_i z) \\
+ (\lambda_i + \lambda_i z)(\lambda - \lambda_i z)(\tilde{S}(\lambda - \lambda_i z) - 1)(\tilde{S}(\lambda - \lambda_i z) - 1)(\tilde{S}(\lambda - \lambda_i z) - 1)(\tilde{S}(\lambda - \lambda_i z) - 1)(-\lambda_i + \lambda_i z)) \]  
(5.30)

**Theorem 5.2:**

The probability generating function \( R(z) \) of number of customers in the system is given by

\[ R(z) = \frac{P_r(z)(R M_3 + R_1 M_4)}{(-\lambda + \lambda_i z)^2 (-\lambda_i + \lambda_i z) M_1} \]  
(5.31)

where

\[ R_1 = ((-\lambda + \lambda_i z)^2 (-\lambda_i + \lambda_i z) + z\lambda_i (\tilde{S}(\lambda - \lambda_i z) - 1)(-\lambda + \lambda_i z)(-\lambda_i + \lambda_i z) \\
+ \lambda_i (\tilde{R}(\lambda - \lambda_i z) - 1)(\tilde{S}(\lambda - \lambda_i z) - 1)(-\lambda_i + \lambda_i z)) \]  
(5.32)

\[ R_3 = ((\tilde{S}(\lambda - \lambda_i z) - 1)\tilde{S}(\lambda - \lambda_i z)(\tilde{R}(\lambda - \lambda_i z) - 1)(\tilde{S}(\lambda - \lambda_i z) - 1) \\
+ \lambda_i z(\tilde{S}(\lambda - \lambda_i z) - 1)(\tilde{R}(\lambda - \lambda_i z) - 1)(\tilde{S}(\lambda - \lambda_i z) - 1)(\tilde{R}(\lambda - \lambda_i z) - 1)) \]  
(5.33)

and \( M_2, M_3 \) are given by the Equations (5.28) and (5.30).
Proof

The probability generating function $R(z)$ of number of customers in
the system at an arbitrary time instant can be expressed as follows

$$R(z) = P_1(z) + z\tilde{P}_1(z,0) + \tilde{P}_2(z,0) + z\tilde{P}_3(z,0). \quad (5.34)$$

Substituting for $\tilde{P}_1(z,0), \tilde{P}_2(z,0)$ and $\tilde{P}_3(z,0)$ from the Equations
(5.17), (5.20) and (5.23) in the Equation (5.34) we get the Equation (5.31).

5.4.2 Computational Aspects of Unknown Function

The unknown function $P_0(z)$ can be obtained from equation (5.25).

On integrating equation (5.25), $P_0(z)$ is obtained as

$$P_0(z) = P_1(1)\exp\left(-\frac{1}{\gamma}\int \frac{k(u)}{g(u)}du\right) \quad (5.35)$$

where

$$k(u) = (-\lambda_1(-\lambda_1 + \lambda_2)u^2 + \lambda_1(S(-\lambda_1) - \lambda_2)u(-\lambda_2 + \lambda_2 u)^2 + \lambda_2\lambda_3 R(\lambda_1 - \lambda_2)u(S(-\lambda_1) - \lambda_2) - 1)(-\lambda + \lambda_2 u) + (\lambda_2 + \lambda_3)\lambda_1 R(\lambda_1 - \lambda_2)u(S(-\lambda_1) - \lambda_2)(-\lambda + \lambda_3 u) - 1)$$

$$g(u) = (u(-\lambda + \lambda_2)u^2 - S(-\lambda_1 + \lambda_2)u(-\lambda_2 + \lambda_2 u)^2 - \lambda_2 R(\lambda_2 - \lambda_2 u)(S(-\lambda_1) - \lambda_2) - 1)(-\lambda + \lambda_2 u) - (\lambda_2 + \lambda_3)\lambda_1 R(\lambda_1 - \lambda_2)u(S(-\lambda_1) - \lambda_2)(\lambda_2 - \lambda_2 u) - 1)$$

$$-\lambda_3(S(-\lambda_1) - \lambda_2)(-\lambda + \lambda_2 u) - 1)(-\lambda + \lambda_2 u)$$
5.5 PERFORMANCE CHARACTERISTICS

In this section, some useful performance measures of the proposed model such as probability that the server is idle, probability that the server is busy with regular service, probability that the server is under repair, the probability that the server is busy with slow service, expected busy period, expected busy cycle, mean number of customers in the orbit and mean waiting time in the retrial queue have been derived.

5.5.1 Probability that the Server is Idle

Let I be the idle period random variable and let $P(I)$ be the probability that the server is idle at time $t$. Using the Equation (5.35) and applying limit $z \to 1$ we get the probability that the server is idle as

$$P(I) = \frac{((\lambda + \lambda_r)^2 T_s)}{(T_i T_s + T_i T_s)}$$

(5.36)

where

$$T_i = (-(\lambda + \lambda_r)^2 + \lambda_r (\tilde{S}(\lambda - \lambda_r) - 1)(\lambda + \lambda_r)(1 + \lambda_r E(R))$$

$$+ \lambda_r \lambda_r (\tilde{S}(\lambda - \lambda_r) - 1)(\tilde{S}(\lambda - \lambda_r) - 1)((\lambda_r + \lambda_r) E(R) + 1))$$

(5.37)

$$T_s = ((-\lambda + \lambda_r)^2 + 2\lambda_r (\lambda + \lambda_r) - 2\lambda r (\lambda + \lambda_r) \tilde{S}(\lambda - \lambda_r) - \lambda_r \lambda_r (\tilde{S}(\lambda - \lambda_r) - 1)$$

$$- \lambda_r \lambda_r (\lambda + \lambda_r) E(R) (\tilde{S}(\lambda - \lambda_r) - 1) + \lambda_r \lambda_r (\lambda + \lambda_r) E(R) (\tilde{S}(\lambda - \lambda_r) - 1)$$

$$(\tilde{S}(\lambda - \lambda_r) - 1) - \lambda_r \lambda_r (\lambda - \lambda_r) (\tilde{S}(\lambda - \lambda_r) - 1))$$

(5.38)

$$T_s = (\lambda_r (\lambda - \lambda_r) (\tilde{S}(\lambda - \lambda_r) - 1) + \lambda_r \lambda_r (\lambda + \lambda_r) E(R) (\tilde{S}(\lambda - \lambda_r) - 1) - \lambda_r \lambda_r (\lambda + \lambda_r)$$

$$E(R) (\tilde{S}(\lambda - \lambda_r) - 1) (\tilde{S}(\lambda - \lambda_r) - 1) + \lambda_r \lambda_r (\tilde{S}(\lambda - \lambda_r) - 1)(\tilde{S}(\lambda - \lambda_r) - 1))$$

(5.39)
\[ T_1 = (\lambda_1 (\tilde{S}_1 (\lambda - \lambda_s) - 1) (2(\lambda + \lambda_s) + \lambda_s + \lambda_s \tilde{S}_1 (\lambda_s)) + ((-\lambda + \lambda_s) E(R) (\tilde{S}(\lambda - \lambda_s) - 1) (\lambda_s + \lambda_s (\tilde{S}_1 (\lambda_s - \lambda_s) - 1))) \]  

(5.40)

### 5.5.2 Probability that the Server is Busy with Regular Service

Let \( B_3 \) be the busy period random variable when the server is busy with regular service and \( P(B_3) \) be the probability that the server is busy with regular service at time \( t \). Using the Equation (5.17) and applying limit \( z \to 1 \) we get the probability that the server is busy with regular service as

\[ P(B_3) = \hat{P}_1(1, 0) = \frac{((-\lambda + \lambda_s) (\lambda_s T_1 + T_3) (\tilde{S}(\lambda - \lambda_s) - 1))}{(T_1 T_2 + T_3 T_4)} \]  

(5.41)

where \( T_1, T_2, T_3, \) and \( T_4 \) are given by the Equations (5.37) - (5.40) respectively.

### 5.5.3 Probability that the Server is Under Repair

Let \( R \) be the repair time random variable and \( P(R) \) be the probability that the server is under repair at time \( t \). Using the Equation (5.23) and applying limit \( z \to 1 \) we get the probability that the server is under repair as

\[ P(R) = \hat{P}_2(1, 0) = \frac{((\lambda_s T_1 + T_4) (\tilde{S}_1 (\lambda - \lambda_s) - 1) (\lambda - \lambda_s) E(R) + \lambda_s \lambda_s (\tilde{S}_1 (\lambda - \lambda_s) - 1) (\tilde{S}_1 (\lambda - \lambda_s) - 1) E(R))}{\lambda_s (T_1 T_2 + T_3 T_4)} \]  

(5.42)

where \( T_1, T_2, T_3, \) and \( T_4 \) are given by the Equations (5.37) - (5.40) respectively.
5.5.4 Probability that the Server is Busy with Slow Service

Let $B_4$ be the busy period random variable when the server is busy with slow service and $P(B_4)$ be the probability that the server is busy with slow service at time $t$. Using the Equation (5.20) and applying limit $z \to 1$ we get the probability that the server is busy with slow service as

$$P(B_4) = \frac{(\lambda_4(T_2 + T_4)(S_4(\lambda - \lambda_4) - 1)(S_4(\lambda - \lambda_4) - 1))}{(T_1T_2 + T_3T_4)}$$  \hspace{1cm} (5.43)

5.5.5 The Mean Number of Customers in the Orbit

The expected number of customers in the orbit is obtained by using probability generating function given in equation (5.24) and $L_Q = \lim_{z \to 1} \frac{d}{dz} P(z)$. Since the expression for $P(z)$ is too large the numerical values of $L_Q$ are calculated using the software mathematica.

5.5.6 The Mean Waiting Time in Retrial Queue

Using Little’s formula, the mean waiting time in the retrial queue ($W_Q$) is obtained as

$$W_Q = E(W) = \frac{L_Q}{\lambda_1}$$

Theorem 5.3:

If $T_b$ and $T_C$ are the length of busy period and busy cycle, then under steady state conditions, we have

$$E(T_b) = \frac{(T_1T_2 + T_3T_4)}{\lambda_1((\lambda + \lambda_4)^2)} \mathbb{E} \left( \frac{1}{\gamma \cdot g \cdot u} \right)$$

and

$$E(T_c) = \frac{((\lambda + \lambda_4)^2)}{\lambda_1(T_1T_2 + T_3T_4)} \mathbb{E} \left( -\frac{1}{\gamma \cdot g \cdot u} \right)$$
Proof

By applying the argument of alternating renewal process, it is known that

\[
E(T_b) = \frac{1}{\lambda} \left( \begin{array}{c} \lambda \\ p_0 \end{array} \right) \quad \text{and} \quad E(T_c) = \frac{1}{\lambda} \left( \begin{array}{c} 1 \\ p_0 \end{array} \right)
\]  \hspace{1cm} (5.44)

On substituting \( z=0 \) in Equation (5.33) we get

\[
p_0 = \frac{(-\lambda + \lambda_r)^2 T_r}{T_1 T_2 + T_3 T_4} \left( \frac{1}{\gamma} \frac{1}{g(u)} \right. \left. \int_0^1 k(u) \, du \right)
\]  \hspace{1cm} (5.45)

Substituting for \( p_0 \) from the Equation (5.45) in the Equation (5.44) we get

\[
E(T_b) = \frac{(T_1 T_2 + T_3 T_4)}{\lambda (\lambda + \lambda_r)^2 T_r} \left( \frac{1}{\gamma} \frac{1}{g(u)} \int_0^1 k(u) \, du \right)
\]  \hspace{1cm} (5.46)

\[
E(T_c) = \frac{(-\lambda + \lambda_r)^2 T_r}{\lambda (T_1 T_2 + T_3 T_4)} \left( \frac{-1}{\gamma} \frac{1}{g(u)} \int_0^1 k(u) \, du \right)
\]  \hspace{1cm} (5.47)

5.6 STOCHASTIC DECOMPOSITION

In this section, the stochastic decomposition property of the system size distribution for the proposed model is derived. Stochastic decomposition has been developed for M/G/1 type queues with server vacations by Falin (1997) and Doshi (1985). Later, Yang & Templeton (1987) extended this property to some retrial queues. Atencia et al (2006) discussed the applications derived from it.

In order to emphasize the dependence of the characteristics of the single server retrial queueing system with three classes of customers subject to server breakdowns and repairs on the retrial rate \( \gamma \), the variables \( C=C(t) \) and \( N=N(t) \) are denoted as \( C_\gamma \) and \( N_\gamma \), respectively. Here \( C_\gamma \) and \( N_\gamma \) represent the
corresponding variables for the standard single server queueing system with three classes of customers subject to server breakdowns and repairs. (i.e., when $\gamma \to \infty$ or high rate of retrials, the proposed model behaves as standard single server queueing system with three classes of customers subject to server breakdowns and repairs). The state of standard single server queueing system with three classes of customers subject to server breakdowns and repairs is described by the single variable $K_{\gamma}$, which represents the total number of customers in the system rather than by the vector $(C_{\gamma}, N_{\gamma})$.

This section investigates the stochastic decomposition law. First, it is important to observe the following relationship between the generating functions

\[
\lim_{z \to \infty} R(z) = E[z^{K_{\gamma}}] = \frac{((-\lambda_1 + \lambda_2)^2T_2(R_1M_2 + R_3M_4))}{((-\lambda_1 + \lambda_2)^2(-\lambda_1 + \lambda_2)M_1(T_1T_2 + T_3T_4))}
\]

where $M_2, M_4, R_1, R_3, T_1, T_2, T_3$ and $T_4$ are given by the Equations (5.28), (5.30), (5.32), (5.33) and (5.37)-(5.40) respectively. $E[z^{K_{\gamma}}]$ is the generating function of the system size in a standard queueing system with three classes of customers subject to server breakdowns and repairs.

At this junction a random variable, $R_{\gamma}$ representing the number of customers in the orbit given that the server is idle is introduced with the generating function

\[
E[z^{R_{\gamma}}] = \exp \left( \frac{1}{\gamma} \int \frac{k(u)}{g(u)} du \right)
\]

\[
= \frac{P_{\gamma}(z)(T_1T_2 + T_3T_4)}{(-\lambda + \lambda_2)^2T_2}
\]

\[
= \frac{E(z^{N_{\gamma}(t)}, C_{\gamma}(t) = 0)}{\Pr(C_{\gamma}(t) = 0)} = E(z^{N_{\gamma}(t)}, C_{\gamma}(t) = 0)
\]
Thus the distribution of the random variable $R_\gamma$ is the conditional distribution of the number of the repeated requests given that the server is free. It is observed that the vector $(C_\gamma, N_\gamma)$ can be represented as a sum of two independent random vectors as follows:

$$(C_\gamma, N_\gamma) = (C_\gamma, N_\gamma) + (0, R_\gamma).$$

In particular, the number of customers in the proposed single server retrial queueing system with three classes of customers subject to server breakdowns and repairs $K_\gamma$, can be represented as the sum of two independent random variables as follows, $K_\gamma = K_\gamma + R_\gamma$. Therefore the probability generating function of the number of customers in the system $K_\gamma$ can be expressed as follows:

$$R(z) = \frac{((-\lambda + \lambda_\kappa)^2 T_r (R_1 T_1 + R_2 T_2))}{((-\lambda + \lambda_\kappa) z ((-\lambda + \lambda_\kappa) z M_\gamma (T_1 T_2 + T_1 T_2)) } \frac{P_\gamma(z)}{P_\gamma(1)}$$

$$= E z^{K_\gamma} E z^{R_\gamma}$$

$$= E z^{K_\gamma} E (z^{N_\gamma(1)}) | C(t) = 1).$$

Thus, the generating function of the system size distribution can be written as $R(z) = E z^{K_\gamma} \cdot \frac{P_\gamma(z)}{P_\gamma(1)}$ where the fraction corresponds to the probability generating function to the system size given that the server is idle. In fact, the above equality provides the stochastic decomposition property for our queueing system in an immediate way. i.e., the number of customers in our system is the sum of two independent random variables: one is the number of customers in the corresponding standard single server queueing system with three classes of customers subject to server breakdowns and repairs and the other is the number of repeated customers given that the server is idle.

5.7 RELIABILITY ANALYSIS
This section discusses the reliability analysis of the queueing system under study. The server is available when it is either idle or working on a customer. The following results concern the availability of the server.

**Theorem 5.4:**

If $A$ is the random variable that the server is available in the system, then the probability that the server is available is given by

\[
P(A) = \frac{\left(1 - \frac{1}{a} \right)^2 + \lambda_s (1 - \frac{1}{a} - 1) (-\lambda + \lambda_s) + \lambda_s}{(1 - \frac{1}{a})^2 + \lambda_s (1 - \frac{1}{a} - 1) (-\lambda + \lambda_s)}
\]

**Proof**

Server is available when it is idle or when it is busy with regular service or when it is busy with slow service. From the Equations (5.17) and (5.20), the generating function of the orbit size when the server is available is given as follows:

\[
P_n(z) + P_1(z,0) + P_3(z,0) = P_n(z) + \frac{\lambda_s P_n(z) + \gamma P'_n(z) (\tilde{S}(\lambda - \lambda_s z) - 1)}{(-\lambda + \lambda_s z)}
\]

\[
+ \frac{\lambda_s (\tilde{S}_s P_n(z) + \gamma P'_n(z)) (\tilde{S}(\lambda - \lambda_s z) - 1)(\tilde{S}_s (\lambda - \lambda_s z) - 1)}{(-\lambda + \lambda_s z)^2}
\]

(5.48)
Substituting for $P_n(z)$ from the Equation (5.25) we have

\[
\begin{align*}
P_n(z) &= M_z((-\lambda + \lambda_z z)^2 + \lambda_z \{S(\lambda - \lambda_z z) - 1\}(-\lambda + \lambda_z z) + \lambda_z \lambda_i \\
&+ (S(\lambda - \lambda_z z) - 1)(S(\lambda - \lambda_z z) - 1)) + M_i ((S(\lambda - \lambda_z z) - 1) \\
&\times M_z((-\lambda + \lambda_z z)^2) \\
\end{align*}
\]  

(5.49)

Substituting $z=1$ in the Equation (5.47), we get the probability that the server is available as

\[
\begin{align*}
P(1) &= (1-\lambda_i)(S(\lambda - \lambda_i) - 1)(-\lambda + \lambda_i) + \lambda_i \lambda_i \\
&+ (S(\lambda - \lambda_i) - 1)(S(\lambda - \lambda_i) - 1)) + T_i ((S(\lambda - \lambda_i) - 1) \\
&\times (T_i, T + T_i, T_i) \\
P(A) &= \frac{(-\lambda + \lambda_i) + \lambda_i ((S(\lambda - \lambda_i) - 1)(S(\lambda - \lambda_i) - 1)))}{(T_i, T + T_i, T_i)} \\
5.8 \quad \textbf{PARTICULAR CASE}
\end{align*}
\]

In this section a particular case of the proposed model is discussed. In the proposed model if there are no negative and partially negative arrivals

$\lambda_z = \lambda_i = 0$, then from the Equations (5.24) and (5.25) we have

\[
P(z) = \frac{(1-z)P_n(z)}{(S(\lambda_i - \lambda_i z) - z)}
\]

This equation exactly coincides with the result of orbit size distribution of $M^o/G/1$ retrial queueing system (when the arrival is single) by Falin and Templeton (1997).
5.8.1 Special Cases

The model so developed is general in nature as the service time is arbitrary. But for practical purposes, service time with particular distribution is required. In this section, some special cases of the proposed model by specifying service time random variables as Erlang and hyper exponential distributions are discussed.

Case (i) : M/E_k/1 retrial queue with three classes of customers (Erlang service time both for regular service and slow service) subject to server breakdowns and repairs.

If the service times are assumed to be Erlang with probability density functions

\[
s_1(x) = \frac{\lambda_1 u_1^{k_1} x^{k_1 - 1} e^{-x u_1}}{(k_1 - 1)!}, \quad \text{where } u_1 > 0, \quad k_1 \text{ is a positive integer and } u_1 \text{ is a parameter,}
\]

and

\[
s_2(x) = \frac{\lambda_2 u_2^{k_2} x^{k_2 - 1} e^{-x u_2}}{(k_2 - 1)!}, \quad \text{where } u_2 > 0, \quad k_2 \text{ is a positive integer and } u_2 \text{ is a parameter then}
\]

\[
\begin{align*}
\tilde{S}(\lambda - \lambda_v z) &= \left( \frac{n_1 k_1}{n_1 + (\lambda - \lambda_v z)} \right)^{k_1}, \\
\tilde{S}_3(\lambda - \lambda_v z) &= \left( \frac{n_2 k_2}{n_2 + \lambda_v + (\lambda - \lambda_v z)} \right)^{k_2}
\end{align*}
\]

(5.50)

Substituting for \( \tilde{S}(\lambda - \lambda_v z) \) and \( \tilde{S}_3(\lambda - \lambda_v z) \) from the Equation (5.48) in (5.26), the PGF of the retrial queue size distribution for single server retrial queue with three classes of customers subject to server breakdowns and repairs is given as follows:
\[ P(z) = \frac{P_n(z) (M_1 + M_2, M_3)}{(-\lambda + \lambda_2 z)^2 (-\lambda_1 + \lambda_2, z) M_4} \]

where

\[
M_1 = \begin{vmatrix}
(-\lambda + \lambda_2 z)^2 (-\lambda_1 + \lambda_2) + \lambda_1 \left( \frac{u_1 k_1}{u_1 k_1 + (\lambda - \lambda_2 z)} \right)^{k_2} \\
\lambda_2 \lambda_1 \left( \frac{u_1 k_1}{u_1 k_1 + (\lambda - \lambda_2 z)} \right)^{k_1} - 1 \left( -\lambda + \lambda_2 z (R_1 (\lambda_2 - \lambda_2 z) - 1) \right) \\
\lambda_2 \lambda_1 \left( \frac{u_1 k_1}{u_1 k_1 + (\lambda - \lambda_2 z)} \right)^{k_1} - 1 \left( -\lambda + \lambda_2 z \right) \\
\end{vmatrix}
\]

\[
M_2 = \begin{vmatrix}
z (-\lambda + \lambda_2 z)^2 \\
-\lambda_2 R (\lambda_2 - \lambda_2 z) \frac{u_1 k_1}{u_1 k_1 + (\lambda - \lambda_2 z)} ^{k_2} \\
-\lambda_2 R (\lambda_2 - \lambda_2 z) \left( \frac{u_1 k_1}{u_1 k_1 + (\lambda - \lambda_2 z)} \right)^{k_2} \\
\end{vmatrix}
\]

\[
M_3 = \begin{vmatrix}
-\lambda_1 \left( \frac{u_1 k_1}{u_1 k_1 + (\lambda - \lambda_2 z)} \right)^{k_1} \\
\lambda_2 \lambda_1 \left( \frac{u_1 k_1}{u_1 k_1 + (\lambda - \lambda_2 z)} \right)^{k_1} - 1 \left( -\lambda + \lambda_2 z \right) \\
\lambda_2 \lambda_1 \left( \frac{u_1 k_1}{u_1 k_1 + (\lambda - \lambda_2 z)} \right)^{k_1} - 1 \left( -\lambda + \lambda_2 z \right) \\
\end{vmatrix}
\]
Case (ii) : M/Hyper exponential/1 retrial queue with three classes of customers (Hyper Exponential service time both for regular service and slow service) subject to server breakdowns and repairs.

If the service times are assumed to be hyper exponential with probability density functions

\[ s(x) = c u_1 e^{-u_1 x} + (1 - c) w_1 e^{-w_1 x} \quad \text{where} \ x > 0, \ u_1 > 0, \ w_1 > 0, \ 0 \leq c \leq 1 \]

\[ s_3(x) = c u_2 e^{-u_2 x} + (1 - c) w_2 e^{-w_2 x} \quad \text{where} \ x > 0, \ u_2 > 0, \ w_2 > 0, \ 0 \leq c \leq 1 \]
then

\[
\tilde{S}(\lambda - \lambda_c, z) = \frac{\eta_1}{n_1 + (\lambda - \lambda_c)z} + \frac{w_1 \eta_1 (1 - c)}{w_1 (\lambda - \lambda_c)z} \tag{5.51}
\]

\[
\tilde{S}_3(\lambda - \lambda_c, z) = \frac{\eta_1}{n_1 + (\lambda - \lambda_c)z} + \frac{w_2 \eta_1 (1 - c)}{w_2 (\lambda - \lambda_c)z} \tag{5.52}
\]

Substituting for \( \tilde{S}(\lambda - \lambda_c, z) \) and \( \tilde{S}_3(\lambda - \lambda_c, z) \) from the Equations (5.49) and (5.50) in (5.26) the PGF of the retrial queue size distribution for single server retrial queue with three classes of customers subject to server breakdowns and repairs can be obtained.

5.9 NUMERICAL RESULTS

In this section, some numerical results are presented to justify the theoretical results obtained. To study the effects of positive arrival rate \( \lambda_1 \), negative arrival rate \( \lambda_2 \), partially negative arrival rate \( \lambda_3 \) and retrial rate \( \gamma \) on the mean orbit size \( L_Q \) and mean waiting time \( W_Q \), the following notations are used:

- Average arrival rate
- Positive arrival rate
- Negative arrival rate
- Partially negative arrival rate
- Regular service rate
- Slow service rate
- Retrial rate
- Repair time distribution is exponential with rate

\( \lambda = \lambda_1 + \lambda_2 + \lambda_3 \)

\( \lambda_1 \)

\( \lambda_2 \)

\( \lambda_3 \)

\( \mu_r \)

\( \mu_s \)

\( \gamma \)

\( \eta_1 \)
Table 5.1, Figure 5.2 and Figure 5.3 display the effect of positive arrival rate $\lambda_1$ on the mean orbit size $L_Q$ and mean waiting time $W_Q$. The service times are considered as Erlang-2 and hyper exponential with parameters $\lambda_2 = 0.2, \lambda_3 = 0.3, \mu_r = 0.4, \mu_h = 0.3, \gamma = 2, \eta_l = 0.3$. It can be observed that the mean orbit size $L_Q$ and the mean waiting time $W_Q$ increase when the positive arrival rate increases.

Table 5.2 represents the effect of negative arrival rate $\lambda_2$ on the mean orbit size $L_Q$ and the mean waiting time $W_Q$. Figure 5.4 and Figure 5.5 compare the behavior of mean orbit size $L_Q$, mean waiting time $W_Q$ against the effect of negative arrival rate $\lambda_2$. The service times are considered as Erlang-2 and hyper exponential with parameters $\lambda_1 = 0.02, \lambda_3 = 0.3, \mu_r = 0.4, \mu_h = 0.3, \gamma = 2, \eta_l = 0.3$. It can be observed that the mean orbit size $L_Q$ mean waiting time $W_Q$ decrease when the negative arrival rate increases.

Table 5.3 gives the effect of partially negative arrival rate $\lambda_3$ on the mean orbit size $L_Q$ and the mean waiting time $W_Q$. In Figure 5.6 and Figure 5.7 mean orbit size $L_Q$ and mean waiting time $W_Q$ are compared for different arrival rates of partially negative customers. The service times are considered as Erlang-2 and hyper exponential with parameters $\lambda_1 = 0.02, \lambda_3 = 0.2, \mu_r = 0.4, \mu_h = 0.3, \gamma = 2, \eta_l = 0.3$. In this case, the mean orbit size $L_Q$ and mean waiting time $W_Q$ first increase and then decrease when the partially negative arrival rate increases.

Table 5.4 shows the way the mean orbit size $L_Q$ and mean waiting time $W_Q$ change for increasing values of retrial rate $\gamma$. Figure 5.8 and Figure 5.9 show the influence of retrial rate $\gamma$ on mean orbit size $L_Q$ and mean waiting time $W_Q$. The service times are considered as exponential, Erlang-2 and hyper exponential with parameters $\lambda_1 = 0.02, \lambda_3 = 0.2, \lambda_5 = 0.3, \mu_r = 0.4,$
It is noted that the mean orbit size $L_Q$ and mean waiting time $W_Q$ decrease when the retrial rate $\gamma$ increases.

5.10 CONCLUSION

This chapter analyses a single server retrial queue with three classes of customers subject to server breakdowns and repairs. The probability generating function for the queue size at an arbitrary time epoch is derived. Some system performance measures are computed in steady state. A particular case and some special cases are discussed. Stochastic decomposition property and Reliability measures of this model are also discussed. The theoretical development of the model has been justified with numerical results.

Table 5.1 Positive Arrival Rate $\lambda_1$ (Vs) Mean Orbit Size $L_Q$ and Mean Waiting Time $W_Q$

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>Erlang- 2</th>
<th>Hyper Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L_Q$</td>
<td>$W_Q$</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0048</td>
<td>0.4804</td>
</tr>
<tr>
<td>0.02</td>
<td>0.0209</td>
<td>1.0432</td>
</tr>
<tr>
<td>0.03</td>
<td>0.0514</td>
<td>1.7119</td>
</tr>
<tr>
<td>0.04</td>
<td>0.1008</td>
<td>2.5193</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1757</td>
<td>3.5136</td>
</tr>
<tr>
<td>0.06</td>
<td>0.2861</td>
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</tr>
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<td>0.07</td>
<td>0.4480</td>
<td>6.4005</td>
</tr>
<tr>
<td>0.08</td>
<td>0.6889</td>
<td>8.6116</td>
</tr>
<tr>
<td>0.09</td>
<td>1.0598</td>
<td>11.7756</td>
</tr>
<tr>
<td>0.10</td>
<td>1.6678</td>
<td>16.6776</td>
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</table>
Table 5.2  Negative Arrival Rate $\lambda_2$ (Vs) Mean Orbit Size $L_Q$ and Mean Waiting Time $W_Q$

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>Erlang- 2</th>
<th>Hyper Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L_Q$</td>
<td>$W_Q$</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0212</td>
<td>1.0609</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0209</td>
<td>1.0432</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0205</td>
<td>1.0265</td>
</tr>
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<tr>
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<td>0.0192</td>
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</tr>
<tr>
<td>1.0</td>
<td>0.0191</td>
<td>0.9548</td>
</tr>
</tbody>
</table>

Table 5.3  Partially Negative Arrival Rate $\lambda_3$ (Vs) Mean Orbit Size $L_Q$ and Mean Waiting Time $W_Q$

<table>
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<th>$\lambda_3$</th>
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<th>Hyper Exponential</th>
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<tbody>
<tr>
<td></td>
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<td>$W_Q$</td>
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<tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>0.0211</td>
<td>1.0550</td>
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<td>1.0512</td>
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<tr>
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<td>0.0209</td>
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<td>0.0206</td>
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</table>
Table 5.4  Retrial Rate $\gamma$ (Vs) Mean Orbit Size $L_Q$ and Mean Waiting Time $W_Q$

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<th>Hyper Exponential</th>
</tr>
</thead>
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<td>$W_Q$</td>
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</tr>
<tr>
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</tr>
<tr>
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<tr>
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<td>0.0200</td>
<td>1.0003</td>
</tr>
<tr>
<td>5</td>
<td>0.0198</td>
<td>0.9917</td>
</tr>
<tr>
<td>6</td>
<td>0.0197</td>
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<td>0.9788</td>
</tr>
<tr>
<td>9</td>
<td>0.0195</td>
<td>0.9764</td>
</tr>
<tr>
<td>10</td>
<td>0.0195</td>
<td>0.9764</td>
</tr>
</tbody>
</table>

Figure 5.2 Positive Arrival Rate $\lambda_4$ (Vs) Mean Orbit Size $L_Q$
Figure 5.3: Positive Arrival Rate \( \lambda_1 \) (Vs) Mean Waiting Time \( W_Q \)

Figure 5.4: Negative Arrival Rate \( \lambda_2 \) (Vs) Mean Orbit Size \( L_Q \)
Figure 5.5 Negative Arrival Rate $\lambda_2$ (Vs) Mean Waiting Time $W_Q$

Figure 5.6: Partially Negative Arrival Rate $\lambda_3$ (Vs) Mean Orbit Size $L_Q$
Figure 5.7 Partially Negative Arrival Rate $\lambda_3$ (Vs) Mean Waiting Time $W_Q$

Figure 5.8 Retrial Rate $\gamma$ (Vs) Mean Orbit Size $L_Q$
Figure 5.9 Retrial Rate $\gamma$ (Vs) Mean Waiting Time $W_Q$