4 Generating new analytical benchmarks for non-equilibrium radiation diffusion in finite size systems

4.1 Introduction

In the earlier chapters, a fully ionized plasma was considered so that the effect of bound-bound and bound-free transitions were neglected while considering the interaction of radiation with the medium [59]. Also, the radiation intensity was assumed to be spatially uniform [58]. For partially ionized plasmas with spatial variation, the radiation transport equation needs to be solved along with the material energy equation. Non-equilibrium radiation diffusion is an important mechanism of energy transport in Inertial Confinement Fusion, astrophysical plasmas, furnaces and heat exchangers. We devote this chapter to this important topic of radiation diffusion and derive new analytical solutions to the non-equilibrium Marshak diffusion problem in a finite planar slab, spherical shell and sphere. The variation in integrated energy densities and leakage currents are also studied. In order to linearize the radiation transport and material energy equation, the heat capacity is assumed to be proportional to the cube of the material temperature [63]. The steady state energy densities show linear variation along the depth of the planar slab, whereas non-linear dependence is observed for the spherical shell. Non-equilibrium diffusion codes can be more easily validated and verified against these new benchmark results because there is no need to consider a slab or spherical medium of very large size for avoiding
boundary effects. Analytical expressions for all the quantities of interest can be obtained for finite slab/shell width and parameter values relevant to practical problems.

4.2 Analytical solution

Using two independent methods, viz., the Laplace transform method and the Eigen function expansion method, expressions for radiation and material energy densities as a function of space and time is derived for a finite planar slab, spherical shell and sphere.

4.2.1 Planar slab

We consider a planar slab of finite thickness which is purely absorbing and homogeneous occupying \(0 \leq z \leq l\). The medium is at zero temperature initially. At time \(t=0\), a constant radiative flux \((F_{inc})\) is incident on the surface at \(z=0\) as shown in figure 4.1. Neglecting hydrodynamic motion, the one group radiative transfer equation (RTE) in the diffusion approximation and the material energy balance equation (ME) are [6]

\[
\frac{\partial E_r(z,t)}{\partial t} - \frac{\partial}{\partial z}\left[3\sigma_a(T)\frac{\partial E_r(z,t)}{\partial z}\right] = c\sigma_a(T)[aT^4(z,t) - E_r(z,t)], \tag{4.1}
\]

\[
C_V(T)\frac{\partial T(z,t)}{\partial t} = c\sigma_a(T)[E_r(z,t) - aT^4(z,t)], \tag{4.2}
\]

where \(E_r(z,t)\) is the radiation energy density, \(T(z,t)\) is the material temperature, \(\sigma_a(T)\) is the opacity (absorption cross section), \(c\) is the speed of light, \(a\) is the radiation constant, and \(C_V(T)\) is the specific heat of the material.

The Marshak boundary condition on the surface at \(z = 0\) is given by

\[
E_r(0,t) - \left(\frac{2}{3\sigma_a[T(0,t)]}\right) \frac{\partial E_r(0,t)}{\partial z} = \frac{4}{c} F_{inc}. \tag{4.3}
\]
And that at \( z = l \) is

\[
E_r(l, t) + \left( \frac{2}{3\sigma_a[T(l, t)]} \right) \frac{\partial E_r(l, t)}{\partial z} = 0. \tag{4.4}
\]

The initial conditions on these two equations are

\[
E_r(z, 0) = T(z, 0) = 0. \tag{4.5}
\]

To remove the nonlinearity in the RTE (Eqn. [4.1]) and ME (Eqn. [4.2]), opacity \( \sigma_a \) is assumed to be independent of temperature and specific heat \( C_V(T) \) is assumed to be proportional to the cube of the temperature. i.e., \( C_V(T) = \alpha T^3 \). The RTE and the ME are recast into the dimensionless form by introducing the dimensionless independent variables given by

\[
x \equiv \sqrt{3\sigma_a}z, \quad \tau \equiv \left( \frac{4ac\sigma_a}{\alpha} \right)t. \tag{4.6}
\]

The new dependent variables are given by

\[
u_r(x, \tau) \equiv \left( \frac{c}{4} \right) \left[ \frac{E_r(z, t)}{F_{inc}} \right], \quad u_m(x, \tau) \equiv \left( \frac{c}{4} \right) \left[ \frac{\alpha T^4(z, t)}{F_{inc}} \right]. \tag{4.7}
\]
With these new variables, the RTE and ME take the dimensionless form

\[ \varepsilon \frac{\partial u_r(x, \tau)}{\partial \tau} = \frac{\partial^2 u_r(x, \tau)}{\partial x^2} + u_m(x, \tau) - u_r(x, \tau), \]  
(4.8)

\[ \frac{\partial u_m(x, \tau)}{\partial \tau} = u_r(x, \tau) - u_m(x, \tau), \]  
(4.9)

with the initial conditions

\[ u_r(x, 0) = 0, \]  
(4.10)

\[ u_m(x, 0) = 0. \]  
(4.11)

And the boundary conditions on the surfaces are

\[ u_r(0, \tau) - \frac{2}{\sqrt{3}} \frac{\partial u_r(0, \tau)}{\partial x} = 1, \]  
(4.12)

\[ u_r(b, \tau) + \frac{2}{\sqrt{3}} \frac{\partial u_r(b, \tau)}{\partial x} = 0, \]  
(4.13)

where \( b = \sqrt{3} \sigma_a l \) and the parameter \( \varepsilon \) is defined as

\[ \varepsilon = \frac{4a}{\alpha}. \]  
(4.14)

### 4.2.1.1 Laplace transform method

To solve Eqs. (4.8) - (4.13), we introduce the Laplace transform according to

\[ \tilde{f}(s) = \int_0^\infty d\tau e^{-s\tau} f(\tau), \]  
(4.15)

to obtain

\[ \varepsilon s \bar{u}_r(x, s) - \frac{\partial^2 \bar{u}_r(x, s)}{\partial x^2} = \bar{u}_m(x, s) - \bar{u}_r(x, s), \]  
(4.16)
\( s \bar{u}_m(x, s) = \bar{u}_r(x, s) - \bar{u}_m(x, s), \)  
\( \bar{u}_r(0, s) - \frac{2}{\sqrt{3}} \frac{\partial \bar{u}_r(0, s)}{\partial x} = \frac{1}{s}, \)
\( \bar{u}_r(b, s) + \frac{2}{\sqrt{3}} \frac{\partial \bar{u}_r(b, s)}{\partial x} = 0. \)

The solutions of Eqs. (4.16)-(4.19) in s space are obtained as
\[
\bar{u}_r(x, s) = \frac{3 \sin[\beta(s)(b - x)] + 2\sqrt{3}\beta(s) \cos[\beta(s)(b - x)]}{s[3 \sin(\beta(s)b) + 4\sqrt{3}\beta(s) \cos(\beta(s)b) - 4\beta^2(s) \sin(\beta(s)b)]}, \]
\[
\bar{u}_m(x, s) = \frac{3 \sin[\beta(s)(b - x)] + 2\sqrt{3}\beta(s) \cos[\beta(s)(b - x)]}{s(s + 1)[3 \sin(\beta(s)b) + 4\sqrt{3}\beta(s) \cos(\beta(s)b) - 4\beta^2(s) \sin(\beta(s)b)]}. \]

where \( \beta(s) \) is given by
\[
\beta^2(s) = -\frac{s}{s + 1}[1 + \varepsilon(s + 1)]. \tag{4.22} \]

Before solving for the radiation and material energy densities by inverting \( \bar{u}_r(x, s) \) and \( \bar{u}_m(x, s) \), we first obtain the small and large \( \tau \) limits of \( u_r(x, \tau) \) and \( u_m(x, \tau) \) from the large and small s limits of eqns. [4.20] and [4.21] respectively. Using the theorems
\[
\lim_{s \to \infty} [s \tilde{f}(s)] = \lim_{\tau \to 0}[f(\tau)], \tag{4.23}
\]
\[
\lim_{s \to 0} [s \tilde{f}(s)] = \lim_{\tau \to \infty}[f(\tau)], \tag{4.24}
\]
we have
\[
u_r(x, 0) = u_m(x, 0) = 0, \]
\[
u_r(x, \tau \to \infty) \to u_m(x, \tau \to \infty) \to \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}}. \tag{4.26}
\]

Thus according to eqn. [4.25], at the initial instant, both the material and radiation energy
densities are zero inside the slab. Eqn. [4.26] asserts that at infinite time the radiation and material energy density equilibrate among themselves. However, because of the finite thickness of the slab, flux leaks out of the right edge so that the energy densities vary linearly along the length of the slab.

The solutions for \( u_r(x, \tau) \) and \( u_m(x, \tau) \) follow from \( \bar{u}_r(x, s) \) and \( \bar{u}_m(x, s) \) by inverting them using the Laplace inversion theorem

\[
f(\tau) = \frac{1}{2\pi i} \int_C ds e^{s \tau} \bar{f}(s), \tag{4.27}
\]

where the integration contour is a line parallel to the imaginary \( s \) axis to the right of all the singularities of \( \bar{f}(s) \). The contour is closed in the left half plane so that the large semi circle gives a zero contribution. Both \( \bar{u}_r(x, s) \) and \( \bar{u}_m(x, s) \) are single valued functions and hence there are no branch points. However, there are an infinite number of poles obtained from the roots of the transcendental equation

\[
3 \sin(\beta(s)b) + 4\sqrt{3} \beta(s) \cos(\beta(s)b) - 4\beta^2(s) \sin(\beta(s)b) = 0,
\]

or,

\[
\tan(\beta(s)b) = \frac{4\sqrt{3} \beta(s)}{4\beta^2(s) - 3}. \tag{4.28}
\]

For the semi infinite slab, because of the multiple valuedness of the functions obtained by Laplace transform, inverting them using the inverse Laplace transform required evaluation of contributions from all the branch cuts. This resulted in integrals which had to be computed numerically [64]. The oscillations in the integrand resulted in difficulty in their convergence. The advantage of solving the finite problem is that because of the single valuedness of the Laplace transformed functions, the inversion is very simple. The sum of the residues at the singularities (poles) give the required solution. The roots of the transcendental equation has been obtained using MATHEMATICA [97] as shown in the graph of figure 4.2.

Corresponding to each root of \( \beta(s) \), there exists two values of \( s \), i.e., two simple poles. The
Figure 4.2: Finding the roots of the transcendental equation $\tan(\beta(s)) = f(\beta) = \frac{4\sqrt{3}f(s)}{4\beta(s)-3}$

poles are obtained from solution of eqn. [4.22] as

$$s = -[\varepsilon + \beta^2(s) + 1] \pm \sqrt{[\varepsilon + \beta^2(s) + 1]^2 - 4\varepsilon\beta^2(s)} \over 2\varepsilon}.$$  

(4.29)

According to the residue theorem, $\int_C ds e^{s\tau} f(s) = 2\pi i \times \text{(sum of the residues at the singularities)}$. The residue at $s=0$ gives the asymptotic (steady state) solution for the radiation and material energy densities as $u_r(x, \infty) = u_m(x, \infty) = \frac{3b+2\sqrt{3}-3x}{3b+4\sqrt{3}}$ which is also obtained by equating $\frac{\partial u_r(x, \tau)}{\partial \tau}$ and $\frac{\partial u_m(x, \tau)}{\partial \tau}$ in Eqs. (4.8) and (4.9) to zero, solving $\frac{\partial^2 u_r(x, \tau)}{\partial x^2} = 0$ and obtaining the values of the constants from the BC given by Eqs. (4.12) and (4.13).

The contribution to the time dependent part comes from the higher order poles. Adding residues from all the poles give us the complete space and time dependence of the radiation energy density as

$$u_r(x, \tau) = \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}} + \sum_n \frac{e^{s_n\tau}[3\sin(\beta(s_n)(b-x)) + 2\sqrt{3}\beta(s_n)\cos(\beta(s_n)(b-x))] + \frac{s_n[Q(s_n)\cos(\beta(s_n)b) - R(s_n)\sin(\beta(s_n)b)]}{ds}}{s_n[Q(s_n)\cos(\beta(s_n)b) - R(s_n)\sin(\beta(s_n)b)]}$$  

(4.30)
with

\[ Q(s_n) = 3b + 4\sqrt{3} - 4\beta^2(s_n)b, \]  
\[ R(s_n) = 4\sqrt{3}\beta(s_n)b + 8\beta(s_n). \]

Similarly, the solution for the material energy density is

\[ u_m(x, \tau) = \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}} + \sum_n \frac{e^{s_n\tau}[3\sin(\beta(s_n)(b - x)) + 2\sqrt{3}\beta(s_n)\cos(\beta(s_n)(b - x))]}{s_n(s_n + 1)[Q(s_n)\cos(\beta(s_n)b) - R(s_n)\sin(\beta(s_n)b)]} d\beta(s_n), \]

We also consider the \( \varepsilon=0 \) case which arises when the speed of light is taken to be infinite so that radiation is not retarded initially. At infinite time, the radiation and material energy densities assume the same spatial dependence as for \( \varepsilon \neq 0 \) case.

\[ u_r(x, \tau \rightarrow \infty) \rightarrow u_m(x, \tau \rightarrow \infty) \rightarrow \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}}. \]

However, for \( \tau = 0 \), as \( s \rightarrow \infty \) for \( \varepsilon = 0 \), we obtain \( \beta = i \) where \( i = \sqrt{-1} \). Thus,

\[ u_r(x, 0) = \frac{3\sinh(b - x) + 2\sqrt{3}\cosh(b - x)}{7\sinh(b) + 4\sqrt{3}\cosh(b)}, \]

\[ u_m(x, 0) = 0. \]

Thus the material energy density is zero at \( \tau = 0 \) as predicted by the initial condition. However, because of the absence of retardation effects, the radiation energy density attains a finite value consistent with the incoming flux of radiation. This behavior is in agreement with that obtained in the case of a semi infinite planar slab for the no retardation case.

The solution \( u_r(x, \tau) \) and \( u_m(x, \tau) \) for \( \varepsilon = 0 \) is obtained by inverting Eqs. (4.20) and (4.21).
using inverse Laplace transform as in the general case with \( \varepsilon = 0 \). The difference from the \( \varepsilon \neq 0 \) case is that only one pole is obtained corresponding to a value of beta i.e., \( s = -\frac{\beta^2(s)}{\beta^2(s) + 1} \).

### 4.2.1.2 Eigenfunction expansion method

To solve eqns. [4.8] - [4.13] using eigenfunction expansion method, we write the solution as the sum of an asymptotic (i.e., infinite time) and a transient part. Let us denote the asymptotic solutions for radiation and material energy densities by \( u^0_r(x) \) and \( u^0_m(x) \) respectively. Similarly the transient parts are denoted by \( u^1_r(x, \tau) \) and \( u^1_m(x, \tau) \). Then,

\[
\begin{align*}
  u_r(x, \tau) &= u^0_r(x) + u^1_r(x, \tau), \\
  u_m(x, \tau) &= u^0_m(x) + u^1_m(x, \tau).
\end{align*}
\]

#### Obtaining the asymptotic solution

After infinite time, both \( u_r(x, \tau) \) and \( u_m(x, \tau) \) attain the asymptotic value so that \( \frac{\partial u_r(x, \tau)}{\partial \tau} = 0 \) and \( \frac{\partial u_m(x, \tau)}{\partial \tau} = 0 \). Therefore, \( u_r(x, \tau) = u_m(x, \tau) \) and hence from eqn. [4.8],

\[
\frac{\partial^2 u_r(x, \tau)}{\partial x^2} = 0.
\]

The solution is \( u^0_r(x, \tau) = c + dx \). The values of the constants \( c \) and \( d \) can be obtained from the BC given by eqns. [4.12] and [4.13]. Omitting the algebra, the obtained solution is

\[
u^0_r(x, \tau) = \frac{2 + \sqrt{3}b - \sqrt{3}x}{4 + \sqrt{3}b}.
\]

#### Obtaining the transient solution
The equations for the transient parts \( u^1_r(x, \tau) \) and \( u^1_m(x, \tau) \) are

\[
\varepsilon \frac{\partial u^1_r(x, \tau)}{\partial \tau} = \frac{\partial^2 u^1_r(x, \tau)}{\partial x^2} + u^1_m(x, \tau) - u^1_r(x, \tau), \tag{4.41}
\]

\[
\frac{\partial u^1_m(x, \tau)}{\partial \tau} = u^1_r(x, \tau) - u^1_m(x, \tau). \tag{4.42}
\]

with the initial conditions

\[
u^1_r(x, 0) = -u^0_r(x) \tag{4.43} \]
\[
u^1_m(x, 0) = -u^0_m(x) \tag{4.44}
\]

and the homogeneous BC on the surfaces are

\[
u^1_r(0, \tau) - \frac{2}{\sqrt{3}} \frac{\partial \nu^1_r(0, \tau)}{\partial x} = 0, \tag{4.46} \]
\[
u^1_r(b, \tau) + \frac{2}{\sqrt{3}} \frac{\partial \nu^1_r(b, \tau)}{\partial x} = 0. \tag{4.47}
\]

The eigen value equation (EVE) is given by

\[
\frac{\partial^2 \phi}{\partial x^2} + \beta^2 \phi = 0 \tag{4.48}
\]

where \( \phi \) is the eigenvector and \( \beta \) is the eigenvalue. BCs on \( \phi \) are

\[
\phi(0, \tau) - \frac{2}{\sqrt{3}} \frac{\partial \phi(0, \tau)}{\partial x} = 0, \tag{4.49} \]
\[
\phi(b, \tau) + \frac{2}{\sqrt{3}} \frac{\partial \phi(b, \tau)}{\partial x} = 0. \tag{4.50}
\]

The EVE can be solved and we can determine an infinite set of normalized and orthogonal eigen functions and corresponding eigen values. Thus corresponding to a particular eigen value we
have

\[ \frac{\partial^2 \phi_n}{\partial x^2} + \beta_n^2 \phi_n = 0, \]  
(4.51)

\[ \int_{0}^{b} \phi_m(x) \phi_n(x) dx = \delta_{mn}, m, n = 1, 2, 3, \ldots \]  
(4.52)

As these form a complete set, we expand the solutions in terms of these eigen functions:

\[ u_1^r(x, \tau) = \sum_n a_n(\tau) \phi_n(x), \]  
(4.53)

\[ u_1^l(x, \tau) = \sum_n b_n(\tau) \phi_n(x), \]  
(4.54)

where the expansion coefficients \( a_n(\tau) \) and \( b_n(\tau) \) have to be determined. From the orthogonal and normalization conditions of \( \phi_n(x) \) we have

\[ a_n(\tau) = \int_{0}^{b} \phi_n(x) u_1^r(x, \tau) dx, \]  
(4.55)

\[ b_n(\tau) = \int_{0}^{b} \phi_n(x) u_1^l(x, \tau) dx. \]  
(4.56)

Multiplying both sides of eqns. [4.41] and [4.42] with \( \phi_n(x) \), integrating over x from 0 to b, and using the boundary conditions at the surfaces viz. eqns. [4.46], [4.47], [4.49] and [4.50] along with eqns. [4.55] and [4.56], we obtain ODEs involving the expansion coefficients \( a_n(\tau) \) and \( b_n(\tau) \).

\[ \varepsilon \frac{da_n(\tau)}{d\tau} + (1 + \beta_n^2)a_n(\tau) - b_n(\tau) = 0, \]  
(4.57)

\[ \frac{db_n(\tau)}{d\tau} + b_n(\tau) - a_n(\tau) = 0, \]  
(4.58)
with the initial condition on the expansion coefficients as

\[ a_n(0) = - \int_0^b \phi_n(x) u_r^0(x) dx, \]  
\[ b_n(0) = - \int_0^b \phi_n(x) u_m^0(x) dx. \]  

(4.59)  

(4.60)

The solution of the EVE, i.e., eqn. [4.98] is given by

\[ \phi(x) = A \sin(\beta x + B) = C^* \left[ \sin(\beta x) + \frac{2\beta}{\sqrt{3}} \cos(\beta x) \right]. \]  

(4.61)

From the normalization condition of the eigenfunction \( \phi \) i.e., \( \int_0^b \phi^2(x) dx = 1 \) and the BC on \( \phi \) at \( x=0 \) (Eqn. [4.49]), the value of the normalization constant is obtained as

\[ C^* = \sqrt{\frac{12\beta}{4\sqrt{3}\beta + (6\beta + 8\beta^2)b + (4\beta^2 - 3) \sin(2\beta b) - 4\sqrt{3}\beta \cos(2\beta b)}}. \]  

(4.62)

The eigenvalues are obtained by applying the BC on \( \phi \) i.e., eqns. [4.49] and [4.50]. The conditions are

\[ \sin B - \frac{2}{\sqrt{3}} \beta \cos B = 0, \]  
\[ (\cos(\beta b) - \frac{2}{\sqrt{3}} \beta \sin(\beta b)) \sin B + (\sin(\beta b) + \frac{2}{\sqrt{3}} \beta \cos(\beta b)) \cos B = 0. \]  

(4.63)  

(4.64)

We will have nontrivial solutions if the system in \( \sin B \) and \( \cos B \) is singular so that the determinant of coefficients vanishes. This condition gives us the same transcendental equation viz. eqn. [4.28] for the eigenvalue \( \beta \). As in the Laplace transform method, the eigenvalues are obtained as roots of this equation. For a particular eigenvalue \( \beta_n \), the ODEs involving the expansion coefficients \( a_n(\tau) \) and \( b_n(\tau) \) are solved using MATHEMATICA and \( \phi_n(x) \) is obtained from eqn. [4.61]. Summing the contribution from all the eigenvalues give the transient solutions for scaled radiation and material energy densities using eqns. [4.53] and [4.54]. Adding the steady state
4.2.2 Spherical shell

Analogous to the planar slab problem, in spherical geometry we consider a spherical shell of inner and outer radii $R_1$ and $R_2$ respectively (figure 4.3). Under the same assumptions, with a time independent radiative flux ($F_{\text{inc}}$) incident on the inner surface of the shell, the one group radiative transfer equation (RTE) in the diffusion approximation and the material energy balance equation (ME) in spherical geometry are

$$\frac{\partial E_r(r,t)}{\partial t} - \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 c \frac{\partial E_r(r,t)}{\partial r} \right] = c \sigma_a(T) \left[ a T^4(r,t) - E_r(r,t) \right], \quad (4.65)$$

$$C_V(T) \frac{\partial T(r,t)}{\partial t} = c \sigma_a(T) \left[ E_r(r,t) - a T^4(r,t) \right], \quad (4.66)$$

with the same notations as used in Subsec. 4.2.1

The Marshak boundary condition on the inner surface at $r = R_1$ is given by

$$E_r(R_1, t) - \left( \frac{2}{3 \sigma_a[T(R_1, t)]} \right) \frac{\partial E_r(R_1, t)}{\partial r} = \frac{4}{c} F_{\text{inc}}. \quad (4.67)$$
And that at \( r = R_2 \) is

\[
E_r(R_2, t) + \left( \frac{2}{3\sigma_a[T(R_2, t)]} \right) \frac{\partial E_r(R_2, t)}{\partial r} = 0. \tag{4.68}
\]

With new dimensionless variables introduced in Subsec. 4.2.1, the RTE and ME take the dimensionless form

\[
\varepsilon \frac{\partial u_r(x, \tau)}{\partial \tau} = \frac{1}{x^2} \frac{\partial}{\partial x} \left( x^2 \frac{\partial u_r(x, \tau)}{\partial x} \right) + u_m(x, \tau) - u_r(x, \tau), \tag{4.69}
\]

\[
\frac{\partial u_m(x, \tau)}{\partial \tau} = u_r(x, \tau) - u_m(x, \tau), \tag{4.70}
\]

with the initial conditions

\[
u_r(x, 0) = 0, \tag{4.71}
\]

\[
u_m(x, 0) = 0. \tag{4.72}
\]

And the boundary conditions on the surfaces are

\[
u_r(X_1, \tau) - \frac{2}{\sqrt{3}} \frac{\partial u_r(X_1, \tau)}{\partial x} = 1, \tag{4.73}
\]

\[
u_r(X_2, \tau) + \frac{2}{\sqrt{3}} \frac{\partial u_r(X_2, \tau)}{\partial x} = 0, \tag{4.74}
\]

where \( x = \sqrt{3}\sigma_a r \). Changing variable \( u_r(x, \tau) \) to \( w(x, \tau) = u_r(x, \tau)x \) and \( u_m(x, \tau) \) to \( g(x, \tau) = u_m(x, \tau)x \), the equations simplify to

\[
\varepsilon \frac{\partial w(x, \tau)}{\partial \tau} = \frac{\partial^2 w(x, \tau)}{\partial x^2} + g(x, \tau) - w(x, \tau), \tag{4.75}
\]

\[
\frac{\partial g(x, \tau)}{\partial \tau} = w(x, \tau) - g(x, \tau). \tag{4.76}
\]
4.2.2.1 Laplace transform method

Applying Laplace transform, the solution in s space are obtained as

\[
\bar{u}_r(x, s) = \frac{A}{\beta(s)x} \sin(\beta(s)x + B),
\]

\[\bar{u}_m(x, s) = \bar{u}_r(x, s) \frac{s}{s+1},\]

with the constants A and B obtained from the BCs

\[
\bar{u}_r(X_1, s) - \frac{2}{\sqrt{3}} \frac{\partial \bar{u}_r(X_1, s)}{\partial x} = \frac{1}{s},
\]

\[
\bar{u}_r(X_2, s) + \frac{2}{\sqrt{3}} \frac{\partial \bar{u}_r(X_2, s)}{\partial x} = 0.
\]

Then the Laplace transformed radiation energy density is given by

\[
\bar{\bar{u}} = \frac{\sqrt{3}X_1^2[2 - \sqrt{3}X_2 \sin(\beta(s)(X_2 - x)) - 2\beta(s)X_2 \cos(\beta(s)(X_2 - x))]}{sx[S(s) \sin(\beta(s)(X_2 - X_1)) - T(s) \cos(\beta(s)(X_2 - X_1))]},
\]

with

\[
S(s) = (4\beta^2(s) - 3)X_1X_2 - 2\sqrt{3}(X_2 - X_1) + 4,
\]

and

\[
T(s) = 4\beta(s)(X_2 - X_1) + 4\sqrt{3}\beta(s)X_1X_2.
\]

As in the case of the finite planar slab, the solutions for \(u_r(x, \tau)\) and \(u_m(x, \tau)\) follow from \(\bar{u}_r(x, s)\) and \(\bar{u}_m(x, s)\) by inverting them using the Laplace inversion theorem. An infinite num-
ber of poles are obtained from the roots of the transcendental equation

\[
\tan(\beta(s)(X_2 - X_1)) = \frac{4\sqrt{3}\beta(s)X_2X_1 + 4\beta(s)(X_2 - X_1)}{(4\beta(s)^2 - 3)X_1X_2 - 2\sqrt{3}(X_2 - X_1) + 4}. \tag{4.84}
\]

Summing over the residues at all the poles, the radiation energy density is obtained as

\[
u_r(x, \tau) = \frac{\sqrt{3}X_2^2X_1^2 + X_2^2x(2 - \sqrt{3}X_2)}{x[2X_2^2 - \sqrt{3}X_2^2X_2 + \sqrt{3}X_1X_2^2 + 2X_2^2]}
+ \sum_n \left( \frac{[(2 - \sqrt{3}X_2) \sin(\beta(s_n)(X_2 - x)) - 2\beta(s_n)X_2 \cos(\beta(s_n)(X_2 - x))]}{[Y(s_n) \sin(\beta(s_n)(X_2 - X_1)) + Z(s_n) \cos(\beta(s_n)(X_2 - X_1))]} \times \frac{e^{s_n\tau}\sqrt{3}X_1^2}{s_n x \frac{d\beta(s_n)}{ds}} \right) \tag{4.85}
\]

with

\[
Y(s_n) = 4\beta^2(s_n)(X_2^2 + X_1^2) + 4\sqrt{3}\beta(s_n)X_1X_2(X_2 - X_1), \tag{4.86}
\]

and

\[
Z(s_n) = 4\beta^2(s_n)X_1X_2(X_2 - X_1) - 3X_1X_2(X_2 - X_1) - 2\sqrt{3}(X_1^2 + X_2^2). \tag{4.87}
\]

Similarly, the solution for the material energy density follows the same form as that for the radiation energy density with an extra \((s_n + 1)\) in the denominator of the second term.

### 4.2.2.2 Eigenfunction expansion method

In a manner similar to the finite planar slab, the solution is assumed to be the sum of an asymptotic (i.e., infinite time) and a transient part given by eqns. [4.37] and [4.38].

**Obtaining the asymptotic solution**
The asymptotic solution is

\[ u^0_\tau(x) = u^0_m(x) = \frac{\sqrt{3} X^2_1 X^2_3 + X^2_1 x(2 - \sqrt{3} X_2)}{x(2X^2_1 - \sqrt{3}X^2_1 X_2 + \sqrt{3}X^2_1 X^2_3 + 2X^2_2)} \]  

(4.88)

\[ \frac{\partial u^0_\tau(x)}{\partial x} = \frac{\partial u^0_m(x)}{\partial x} = \frac{3X^2_1 X^2_3(X_1 - X_2) - 2\sqrt{3}X^2_1 X^2_2(X^2_1 + X^2_2)}{x^2(2X^2_1 - \sqrt{3}X^2_1 X_2 + \sqrt{3}X^2_1 X^2_3 + 2X^2_2)^2} \]  

(4.89)

The outgoing flux from the surface of the sphere \( j+ \) in the asymptotic limit is

\[ u^0_\tau(X^2_2) - \frac{2}{\sqrt{3}} \frac{\partial u^0_\tau(X^2_2, \tau)}{\partial x} = \frac{4X^2_1}{2X^2_1 - \sqrt{3}X^2_1 X_2 + \sqrt{3}X^2_1 X^2_3 + 2X^2_2}. \]  

(4.90)

and the flux \( j- \) coming out of the inner surface is

\[ u^0_\tau(X^2_1) + \frac{2}{\sqrt{3}} \frac{\partial u^0_\tau(X^2_1, \tau)}{\partial x} = \frac{4X^6 + 3X^4 X^2_3 - 4\sqrt{3}X^4_1 X_2 + 4\sqrt{3}X^3 X^2_2 - 6X^3 X^3_3 + 3X^2 X^4_2 - 4X^4_2}{(2X^2_1 - \sqrt{3}X^2_1 X_2 + \sqrt{3}X^2_1 X^2_3 + 2X^2_2)^2}. \]  

(4.91)

Obtaining the transient solution

The equations for the transient parts \( u^1_\tau(x, \tau) \) and \( u^1_m(x, \tau) \) are

\[ \varepsilon \frac{\partial u^1_\tau(x, \tau)}{\partial \tau} = \frac{1}{x^2} \frac{\partial}{\partial x} \left( x^2 \frac{\partial u^1_\tau(x, \tau)}{\partial x} \right) + u^1_m(x, \tau) - u^1_\tau(x, \tau), \]  

(4.92)

\[ \frac{\partial u^1_m(x, \tau)}{\partial \tau} = u^1_\tau(x, \tau) - u^1_m(x, \tau), \]  

(4.93)

with the initial conditions

\[ u^1_\tau(x, 0) = -u^0_\tau(x), \]  

(4.94)

\[ u^1_m(x, 0) = -u^0_m(x), \]  

(4.95)
with the homogeneous BC on the inner and outer surface

\[ u_1^r(X_1, \tau) - \frac{2}{\sqrt{3}} \frac{\partial u_1^r(X_1, \tau)}{\partial x} = 0, \quad (4.96) \]
\[ u_1^r(X_2, \tau) + \frac{2}{\sqrt{3}} \frac{\partial u_1^r(X_2, \tau)}{\partial x} = 0. \quad (4.97) \]

The eigen value equation (EVE) is given by

\[ \frac{1}{x^2} \frac{\partial}{\partial x} \left( x^2 \frac{\partial \phi}{\partial x} \right) + \beta^2 \phi = 0. \quad (4.98) \]

BCs on \( \phi \) are

\[ \phi(X_1) - \frac{2}{\sqrt{3}} \frac{\partial \phi(X_1)}{\partial x} = 0, \quad (4.99) \]
\[ \phi(X_2) + \frac{2}{\sqrt{3}} \frac{\partial \phi(X_2)}{\partial x} = 0. \quad (4.100) \]

The EVE can be solved and we can determine an infinite set of normalized and orthogonal eigen functions and corresponding eigen values. Thus corresponding to a particular eigen value we have

\[ \frac{1}{x^2} \frac{\partial}{\partial x} \left( x^2 \frac{\partial \phi_n}{\partial x} \right) + \beta_n^2 \phi_n = 0, \quad (4.101) \]
\[ \int_{X_1}^{X_2} \phi_m(x) \phi_n(x) 4\pi x^2 dx = \delta_{mn}, \quad m, n = 1, 2, 3, \ldots \quad (4.102) \]

Following the same steps as for the finite planar slab, by integrating over the volume \( \frac{4\pi x^3}{3} \) with \( x \) going from \( X_1 \) to \( X_2 \), we obtain the solution of the eigenvalue equation as

\[ \phi(x) = \frac{C^*}{x} \left[ \sin(\beta x) + \cos(\beta x) \left( \frac{X_2 - \frac{2}{\sqrt{3}}}{\frac{2X_2 \beta}{\sqrt{3}}} \sin(\beta X_2) + \frac{2X_2 \beta}{\sqrt{3}} \cos(\beta X_2) \right) \right], \quad (4.103) \]
Figure 4.4: Radiation flux incident on the outer surface of a sphere.

with $C^* = A \cos B$. From the normalization condition of the eigenfuncion $\phi$ i.e., $\int_{X_1}^{X_2} \phi^2(x) 4\pi x^2 dx = 1$, the value of the normalization constant is obtained as

$$C^* = \sqrt{\frac{1}{4\pi \int_{X_1}^{X_2} \left[ \left( X_2 - \frac{x}{\sqrt{3}} \right) \sin \beta (X_2 - x) + \frac{2X_2 \beta}{\sqrt{3}} \cos \beta (X_2 - x) \right] \left[ \frac{2X_2 \beta}{\sqrt{3}} \sin (\beta X_2) - (X_2 - \frac{x}{\sqrt{3}}) \cos (\beta X_2) \right]^2 dx}}. \quad (4.104)$$

From the b.c. on $\phi$, the same transcendental equation (eqn. [4.84]) as in the Laplace transform method is obtained for the eigenvalue $\beta$. Finally, the scaled radiation and material energy densities are obtained.

### 4.2.3 Sphere

We consider a sphere of radius $R$ with a radiative flux incident on the outer surface as shown in figure 4.4. The radiation transport and material equation are the same as eqns. [4.69] and [4.70]. The boundary conditions on the surface and centre are given by

$$u_r(X, \tau) + \frac{2}{\sqrt{3}} \frac{\partial u_r(X, \tau)}{\partial x} = 0, \quad (4.105)$$

$$u_r(0, \tau) = \text{finite}. \quad (4.106)$$
4.2.3.1 Laplace transform method

In a manner similar to the spherical shell, the Laplace transformed radiation energy density is given by

\[ \bar{u}_r(x, s) = \frac{\sqrt{3}X^2 \sin(\beta(s_n)x)}{sx[(\sqrt{3}X - 2) \sin(\beta(s_n)X) + 2\beta(s_n)X \cos(\beta X)]}. \]  
(4.107)

The transcendental equation in this case is

\[ \tan(\beta(s_n)X) = \frac{2\beta(s_n)X}{2 - \sqrt{3}X}. \]  
(4.108)

The radiation energy density is obtained as

\[ u_r(x, \tau) = 1 + \sum_n e^{sn\tau} \frac{\sqrt{3} \sin(\beta(s_n)X)}{s_n x[\sqrt{3} \cos(\beta(s_n)X) - 2\beta(s_n) \sin(\beta(s_n)X)]}. \]  
(4.109)

4.2.3.2 Eigenfunction expansion method

The asymptotic solution is obtained as

\[ u_r^0(x) = u_m^0(x) = 1, \]  
(4.110)

and as in the finite spherical shell, we obtain the solution of the eigenvalue equation as

\[ \phi(x) = \frac{\beta}{\pi[2\beta X - \sin(2\beta X)]} \frac{\sin(\beta x)}{x}, \]  
(4.111)

with the same transcendental equation 4.108 for the eigenvalue \( \beta \). The scaled radiation and material energy densities are also obtained in a manner similar to the planar slab and spherical shell.
4.3 Results and discussions

4.3.1 Planar slab

For the finite planar slab, at early stages ($\tau=0.01$) the radiation energy density falls rapidly from the left surface where radiation is incident as shown in figure 4.5. As time proceeds, the values of energy densities increase and the variation with distance keeps on attaining linearity. At infinite time, the steady state values are linear with position as given by eqn. [4.26]. Similarly, the material energy density initially exhibits slight non-linear variation and finally attains the linearity (figure 4.6). The non-linear variation at early stages occurs due to net absorption of energy by the initially cold material (as $u_r(x,0) = u_m(x,0) = 0$). Initially, the material energy density is found to lag behind the radiation energy densities and finally equilibrate as time proceeds (beyond $\tau=10$). In this work, all the results have been obtained by considering contribution from the first 30 roots of the transcendental equation. The value of opacity $\sigma_a$ is chosen to be 100 and $\varepsilon$ equals 0.1. For a heat wave traveling into a thin plate and composite planar slab, a similar linear variation in temperature with distance was observed though difference existed in the space and time dependent behaviour due to heat conduction approximation [98],[99].

The first derivatives w.r.t. position of the analytical radiation and material energy density are plotted in figures 4.7 and 4.8. As the radiation and material energy densities decrease with $x$, the derivative has negative values. The derivative has a greater negative value at the left compared to the right zone. As both radiation and material energy densities keep on increasing with time due to radiation diffusion, magnitude of the gradient decreases for the left and increases for the right sides. The gradient of both radiation and material energy densities obtain a constant value of $-\frac{3}{3+4\sqrt{2}} = -0.30217$ after infinite time showing that there is a constant leakage of flux from the right surface due to the finite thickness. This result is different from the semi-infinite slab result where at infinite time, the entire halfspace is at a constant temperature with a uniform radiation field and hence there is no gradient and no flux [63].
Figure 4.5: Scaled radiation energy density \( u_r(x, \tau) \) Vs. position (x) in the slab of scaled thickness \( b = 1 \) at different times for \( \varepsilon = 0.1 \).

Figure 4.6: Scaled material energy density \( u_m(x, \tau) \) Vs. position (x) in the slab at different times for \( \varepsilon = 0.1 \).
Figure 4.7: Space derivative of scaled radiation energy density $\partial u_r(x, \tau)/\partial x$ Vs. position (x) in the slab at different times.

Figure 4.8: Space derivative of scaled material energy density $\partial u_m(x, \tau)/\partial x$ Vs. position (x) in the slab at different times.
Figure 4.9: Leakage currents $J_-(\tau)$ and $J_+(\tau)$ from the left and right surfaces of the slab respectively.

The current of radiation leaking out from the left and right surfaces of the slab are

$$J_-(\tau) = u_r(0, \tau) + \frac{2}{\sqrt{3}} \frac{\partial u_r(0, \tau)}{\partial x}$$
$$J_+(\tau) = u_r(b, \tau) - \frac{2}{\sqrt{3}} \frac{\partial u_r(b, \tau)}{\partial x}.$$ 

The leakage currents are plotted as a function of time in figure 4.9. It is found that though $J_-(\tau)$ is negative initially, it attains a constant positive value of 0.30217 after saturation. $J_+(\tau)$ is zero initially as the incident flux has not reached the right face. However it rises rapidly and reaches a saturation value of 0.6978. The energy densities and leakage currents at the left and right surfaces are also related as $u_r(0, \tau) + u_r(b, \tau) = 1$ and $J_-(\tau) + J_+(\tau) = 1$.

The averaged or integrated radiation and material energy densities are given by

$$\psi_r(\tau) = \int_0^b u_r(x, \tau) dx$$
$$\psi_m(\tau) = \int_0^b u_m(x, \tau) dx$$

respectively. The steady state integrated value is 0.5 as seen from figure 4.10. The integrated material energy density is also found to lag the radiation energy density at early times but finally the two equilibrate.

To check the consistency of the final results, we add Eqs. (4.8) and (4.9) and integrate over $x$ from 0 to $b$, yielding

$$\int_0^b \left( \varepsilon \frac{\partial u_r(x, \tau)}{\partial \tau} + \frac{\partial u_m(x, \tau)}{\partial \tau} \right) dx = \int_0^b \frac{\partial^2 u_r(x, \tau)}{\partial x^2} dx = \frac{\partial u_r(b, \tau)}{\partial x} - \frac{\partial u_r(0, \tau)}{\partial x}.$$
Figure 4.10: Integrated radiation ($\psi_r(\tau)$) and material energy densities ($\psi_m(\tau)$) in the slab as a function of scaled time $\tau$.

Using the expressions for the energy densities, their first derivatives in space and the integrated quantities, we find that both the left and right hand sides reduce to 

$$
\varepsilon \frac{\partial \psi_r(\tau)}{\partial \tau} + \frac{\partial \psi_m(\tau)}{\partial \tau} = \frac{\partial u_r(b, \tau)}{\partial x} - \frac{\partial u_r(0, \tau)}{\partial x}.
$$

(4.112)

As there are infinite number of residues, the exact solution is obtained only on adding all of them. However, the contribution from the poles decrease very sharply. To study convergence, we plot percentage error as a function of number of roots of the transcendental equation considered. As seen from figure 4.11, 2.1% error in the value of $u_r(0, 2.5)$ is observed on considering only the first two roots i.e., the steady state result and residue for the two non zero poles. The errors arising due to non inclusion of higher order terms is more initially as the higher order poles
Figure 4.11: Percentage error in the radiation energy density $u_r(x, \tau)$ in the slab as a function of number of roots considered (N).

contribute only at very small times because of the exponential term. The error falls sharply to a negligible value (0.005%) on considering the contribution from the first 6 roots i.e., first 11 poles. More accurate results can be obtained by adding residues from higher order poles.

Figure 4.12 shows the plot of radiation energy density $u_r(x, \tau)$ as a function of space and time for $\varepsilon = 0$. Contrary to the results for finite $\varepsilon$, the radiation energy density attains a finite value even at very early times due to the absence of retardation effects. However, the material energy density shows the same trend as for finite $\varepsilon$.

4.3.2 Spherical shell

For the spherical shell, initially ($\tau=0.01$) the radiation energy density falls rapidly from the inner surface (scaled radius $X_1 = 1$) where radiation is incident towards the outer surface (scaled radius $X_2 = 2$) as shown in figure 4.13. Though the trend is similar to the planar slab, the values of the scaled energy densities are less. Also, contrary to the planar case, the variation in energy densities remain sharper in the inner meshes compared to the outer ones and the variations in
energy densities are not linear with position even after attaining steady state. This is evident because the mass of the material to be heated in the radially outward direction increases. Similar to the planar slab, the material energy density lags behind the radiation energy densities at early stages and finally reaches equilibrium (beyond $\tau=10$) [figure 4.14]. Magnitude of derivative of analytical radiation and material energy densities remain higher in the inner meshes as compared to outer ones at all times [figures 4.15 and 4.16]. The leakage currents from the inner and outer surfaces of the spherical shell are $J_-(\tau) = u_r(X_1, \tau) + \frac{2}{\sqrt{3}} \frac{\partial u_r(X_1, \tau)}{\partial x}$, $J_+(\tau) = u_r(X_2, \tau) - \frac{2}{\sqrt{3}} \frac{\partial u_r(X_2, \tau)}{\partial x}$.

The variation in $J_+(\tau)$ is similar to planar slab though the values are less. However, $J_-(\tau)$ remains negative throughout as radiation always diffuses outwards in order to maintain the flux boundary conditions (figure 4.17). As the derivative $\partial u_r(x, \tau)/\partial x$ is more negative for inner radii, $u_r(X_1, \tau) + u_r(X_2, \tau) < 1$ which leads to $J_+(\tau) + J_-(\tau) < 1$. For the case considered, $X_1 = 1$ and $X_2 = 2$, it is found that $2u_r(X_1, \tau) < 1$. As $J_-(\tau) = 2u_r(X, \tau) - 1$, hence $J_-(\tau)$ is negative. The averaged or integrated radiation and material energy densities are given by
ψ_r(τ) = \int_{X_1}^{X_2} u_r(x, τ)4\pi x^2 dx \quad \text{and} \quad ψ_m(τ) = \int_{X_1}^{X_2} u_m(x, τ)4\pi x^2 dx. \quad \text{and plotted in figure 4.18.}

The integrated material energy density is also found to lag the radiation energy density at early times but finally the two equilibrate to a value of 0.25.

To check the consistency of the final results, we add Eqs. (4.69) and (4.70) and integrate over x from \(X_1\) to \(X_2\), yielding

\[
\int_{X_1}^{X_2} \left( ε \frac{∂u_r(x, τ)}{∂τ} + \frac{∂u_m(x, τ)}{∂τ} \right) 4\pi x^2 dx = \frac{4\pi}{X_2^2} \frac{∂u_r(X_2, τ)}{∂x} - X_1^2 \frac{∂u_r(X_1, τ)}{∂x}. \quad (4.113)
\]

Using the expressions for the energy densities, we find that both the left and right hand sides reduce to the same expression proving the consistency of the obtained solutions.

As for the planar slab, convergence of relative error in radiation energy density for spherical shell on increasing contribution from higher order poles is found to follow the same trend. However, the values of relative errors are slightly higher (3.4% for \(u_r(0, 2.5)\) for contribution from first 2 roots) than the planar slab as shown in figure 4.19. Thus for these finite systems,
Figure 4.14: Scaled material energy density $u_m(x, \tau)$ Vs. position in a spherical shell of scaled inner radius $X_1 = 1$ and outer radius $X_2 = 2$ at different times for $\varepsilon = 0.1$.

Figure 4.15: Space derivative of scaled radiation energy density $\partial u_r(x, \tau)/\partial x$ Vs. position (x) in the spherical shell at different times.
Figure 4.16: Space derivative of scaled material energy density $\frac{\partial u_m(x, \tau)}{\partial x}$ Vs. position ($x$) in the spherical shell at different times.

Figure 4.17: Leakage currents $J_-(\tau)$ and $J_+ (\tau)$ from the inner and outer surfaces of the spherical shell respectively.
Figure 4.18: Integrated radiation ($\psi_r(\tau)$) and material energy densities ($\psi_m(\tau)$) in the spherical shell as a function of scaled time $\tau$.

Figure 4.19: Percentage error in the radiation energy density $u_r(x, \tau)$ in the spherical shell as a function of number of roots considered (N).
4.3.3 Sphere

The scaled radiation and material energy densities for a sphere of scaled radius $x=0.5$ are shown in figures 4.20 and 4.21. Both the scaled radiation and material energy densities attain a steady state value of 1 implying that the sphere finally attains the temperature of the incident radiation as expected. The material energy density lags behind the radiation energy density as usual and the results are found to be consistent.

4.4 Summary

In this chapter, the time dependent non equilibrium radiation diffusion problem has been solved analytically for finite planar slab, spherical shell and sphere with a constant radiation flux inci-
Figure 4.21: Scaled material energy density $u_r(x, \tau)$ Vs. position (x) in a sphere of scaled radius $X=0.5$.

dent on the surface. The observed trend in temporal and spatial variation of energy densities, leakage currents, integral quantities, etc. has been explained physically. The results obtained in this work can serve as new and useful benchmarks for non equilibrium radiation diffusion codes in both planar and spherical geometries. The same methodology can be applied to any other finite size systems like layered media with various boundary conditions.