Chapter 5

Infrared Instabilities in Graviton-Higgs Theory

In this chapter quantum vacuum instability of a theory containing spin-2 gravitons coupled to Higgs field is considered. Gravitons minimally coupled to a massless scalar field in a background Minkowski spacetime is shown to develop an instability in their propagators in presence of a spacetime-independent Higgs field background. The instability is indicated in the graviton propagator due to appearance of a tachyonic pole. The one loop effective potential for this theory is shown to develop an infrared instability in the form of acquiring an imaginary part, which can be traced to the tachyonic pole in the graviton propagator. This instability is analogous to the finite temperature infrared instability of a gas of gravitons coupled to fermions found by Gross et. al. [75], even though it already exists at zero temperature; it is thus reminiscent of the Jeans instability thought to be at the heart of structure formation in the early Universe. A finite temperature analysis of the effective potential at one loop shows that in the high temperature limit, the zero-temperature instability is in fact reinforced by finite temperature effects. In the low temperature limit, the finite temperature contribution to the imaginary part of the effective potential exhibits a damped oscillatory behaviour; all thermal effects are damped out as the temperature
vanishes, consistent with the zero-temperature result. This chapter is based on ref. [96].

5.1 Jeans instability and tachyons at finite temperature

Stability of flat spacetime under quantum gravitational fluctuations has been studied extensively since the incipient work on the Euclidean path integral formulation of gravity [97, 98]. Employing a saddle-point approximation in the Euclidean partition function, Gross et. al. [75] show that flat space is stable at zero temperature both classically and quantum mechanically under perturbative quantum fluctuations of Euclidean 4-space. However, when the system is kept in contact with a heat bath, the self-gravitating system becomes unstable, both by itself (vacuum) and in the presence of massless spinor fields. This is unlike in the case of an electrical plasma where charge carriers produce a screening effect over the fluid (Debye screening). This distinct feature of gravity is the source of several instabilities. In classical Newtonian gravity one such instability occurs when we treat the Universe as being filled with a static, homogeneous nonrelativistic fluid. For long-wavelength gravitational perturbations the system develops an instability. This instability is very often be related to classical Jeans instability [69]. Jeans’ Universe is filled with a non viscous fluid with mass density $\rho$, pressure $p$, and velocity $\vec{v}$ satisfying the usual continuity and Navier-Stokes equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad Eqn. of \text{continuity}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \vec{g} \quad Euler \text{ equation}$$
The gravitational field $g$ satisfies the following equations

\begin{align}
\nabla \times g &= 0 \quad (5.3) \\
\nabla \cdot g &= -4\pi G \rho \quad \text{gravitational field eqn.} \quad (5.4)
\end{align}

To analyze the dynamics of the system one considers small perturbations $\rho_1, p_1, v_1, g_1$ around an equilibrium configuration which is taken to be a homogeneous, static fluid. The effect of gravitation are also ignored in the unperturbed solution. The density perturbation satisfies the following equation of motion

$$
\frac{\partial^2 \rho_1}{\partial t^2} - V_s^2 \nabla^2 \rho_1 = 4\pi G \rho \rho_1 
$$

(5.5)

where $V_s$ is the speed of sound in the fluid. The solution of the equation for the density perturbation gives the usual plane waveform except the fact the right hand side of eq. (5.5) has a mass like term with wrong sign.

$$
\rho_1 \propto \exp\{i k \cdot x - i \omega t\} \quad (5.6)
$$

leading to a dispersion relation,

$$
\omega^2 = v_s^2 k^2 - 4\pi G \rho 
$$

(5.7)

where $\omega$ is imaginary for wave numbers below the critical value

$$
k_J = \left(\frac{4\pi G \rho}{v_s^2}\right)^{1/2} \quad (5.8)
$$

So the perturbation $\rho_1$ has a runaway mode below this value and can result in an exponential growth or decay of the disturbances [99].

In a classic work, Gross et. al. [75] consider a gas of gravitons in thermal contact at
finite spatial volume and interacting with thermally excited fermions. Integrating over the fermionic degrees of freedom, the graviton is shown to acquire an imaginary mass leading to a tachyonic instability. The presence of a heat bath as a source for inducing thermal fluctuations is crucial in this work, as is evident from the fact that the induced masses have power law dependence on the temperature.

In ref. [75] it is shown that flat space is stable both quantum mechanically and classically under small perturbations due to gravitons and spinors at zero temperature. They also showed that when a gas of gravitons is kept at finite temperature, an instability, stemming from thermally generated graviton modes, appears. This induces a Jeans-like instability since the thermally excited modes interact with gravitons. In a theory with gravitons coupled to thermally excited fermions, the one-loop graviton propagator contains a tachyonic term [75] which can be interpreted as a mass term for the longitudinal mode of the graviton $h_{00}$; this mass is of magnitude

$$m_g^2 = -\frac{14}{15\pi^3 GT^4}$$  \hspace{1cm} (5.9)$$

The generation of an imaginary mass term when gravitons couple with thermally excited matter field is a generic feature. This also holds for the case of scalar fields at finite temperature. In fact the mass induced for the case of scalar field can again be traced from the self energy component $\Pi_{00,00}$. The longitudinal part of graviton $h_{00}$ here again develops a mass term due to thermal fluctuations [100]. The value in this case is

$$m_g^2 = -\frac{4}{5 \pi^3 GT^4}$$  \hspace{1cm} (5.10)$$

All these effects are purely thermal, implying that hot flat space is unstable and leads to Jeans instability.
5.2 Tachyonic mode at zero temperature propagator

In this section it will be shown that even at zero temperature, when gravitons couple to massless scalar field backgrounds which are spacetime independent, a similar instability appears, with the effective one-loop graviton propagator acquiring a tachyonic pole. This, in turn, leads to the appearance of an imaginary contribution in the one-loop effective action for a wide class of theories involving graviton and scalars, when evaluated using the Euclidean path integral saturated at a saddle point characterized by a flat Euclidean metric and a constant scalar background. This implies that graviton fluctuations coupled to constant scalar field background at \( T = 0 \) in flat spacetime plays a role similar to gravitons in a finite temperature heat-bath inducing an instability in flat spacetime. It is perhaps not inappropriate to state that this phenomenon has been an issue not particularly well-understood \([45, 46, 48]\) as to how the instability resolves itself. It is not unlikely that the instability will involve decay to a de Sitter spacetime, but the actual proof of this has not been addressed here. Let us now go back to the case without an external heat bath, and consider a minimally coupled graviton-Higgs theory at \( T = 0 \). Assuming that the theory has a vacuum characterised by a constant value of the Higgs field which plays the role of an external background for the gravitons, an instability similar to the finite temperature case is discerned. In other words, the scalar field background plays the role of a heat bath which induces a tachyonic instability in the graviton modes even at zero temperature.

To evaluate the effective propagator for Higgs-graviton theory let us start from the Einstein-Hilbert action and expand it around the flat space i.e. employ weak field approximation.

\[
\sqrt{-g} \mathcal{L}_g = \sqrt{-g} \left[ \frac{1}{\kappa^2} R \right] \tag{5.11}
\]

where \( \kappa^2 = 16\pi G \); \( g = \text{det} g_{\mu\nu} \) and \( R = g^{\mu\nu} R_{\mu\nu} \).
The Lagrangian for Gravity coupled to a massless scalar field,

\[ \sqrt{-g} \mathcal{L} = \frac{1}{\kappa^2} R + \sqrt{-g} \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \sqrt{-g} V(\phi) \]  

(5.12)

Expanding the metric around a flat background we get,

\[ g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \]  

(5.13)

where the fluctuations \( h_{\mu\nu} \) are small, \(|h_{\mu\nu}| < 1\) and \( \eta_{\mu\nu} = \text{diag}(-1,1,1,1) \). For the decomposition (6.45), the inverse of the metric is

\[ g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^\lambda_\mu h^\lambda_{\nu} + \ldots \]  

(5.14)

Furthermore, the determinant of the metric, which will be needed in the following, will be given by:

\[ (-g)^\frac{1}{2} = 1 + \kappa\frac{1}{2} h^\alpha_\alpha - \kappa^2 \frac{1}{4} h^\alpha_\beta h^\beta_\alpha + \kappa^2 \frac{1}{8} (h^\alpha_\alpha)^2 + \ldots \]  

(5.15)

The quadratic part of the Lagrangian from pure gravity sector is given by

\[ \mathcal{L}_g = -\frac{1}{4} \partial_\lambda h_{\mu\nu} \partial^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{2} \partial_\mu h^\alpha_\nu \partial_\alpha h^{\nu\alpha} + \frac{1}{4} \partial_\mu h \partial^\mu h. \]  

(5.16)

This Lagrangian is invariant under linear gauge transformations

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \]  

(5.17)

A gauge fixing term has to be added here to break this gauge invariance in order to get the propagator. The following gauge fixing Lagrangian is used here,

\[ \mathcal{L}_{gf} = -\frac{1}{2} \left[ \partial_\mu h^{\mu\nu} - \frac{1}{2} \partial^\nu h \right]^2 \]  

(5.18)
In this gauge the graviton propagator is finally determined from the surviving quadratic part of the pure gravity Lagrangian, which is

$$\mathcal{L}_g = -\frac{1}{4} \partial_\mu h_{\alpha\beta} \partial^\mu h^{\alpha\beta} + \frac{1}{8} (\partial_\mu h^\alpha_\alpha)^2$$  \hspace{1cm} (5.19)

The latter can be conveniently re-written in terms of a matrix $O$

$$\mathcal{L}_g = -\frac{1}{2} \partial_\lambda h_{\alpha\beta} O^{\alpha\beta\mu\nu} \partial^\lambda h_{\mu\nu}$$  \hspace{1cm} (5.20)

with

$$O_{\alpha\beta\mu\nu} = \frac{1}{2} \eta_{\alpha\mu} \eta_{\beta\nu} - \frac{1}{4} \eta_{\alpha\beta} \eta_{\mu\nu}$$  \hspace{1cm} (5.21)

The matter sector is also expanded around a space-time constant background $\phi = \phi_0 + \Phi$

$$\sqrt{-g} \mathcal{L}_m = \frac{1}{2} \Phi(\Box - V''(\phi_0))\Phi - \frac{\kappa}{2} h V'(\phi_0) \Phi - \Phi V'(\phi_0) - \frac{\kappa}{2} h V(\phi_0) + \frac{1}{2} \kappa^2 h_{\alpha\beta} O^{\alpha\beta\mu\nu} h_{\mu\nu} V(\phi_0)$$  \hspace{1cm} (5.22)

If we write down an effective linearized equation of motion for the graviton field from the quadratic part of the Lagrangian, we get an equation

$$\mathcal{T}^{\alpha\beta,\mu\nu} h_{\mu\nu} = \kappa T^{\alpha\beta},$$  \hspace{1cm} (5.23)

where $T^{\mu\nu}$ contains interaction terms containing appropriately contracted products of terms linear in the scalar and graviton fluctuation fields. The operator $\mathcal{T}^{\mu\nu,\alpha\beta}$ can be extracted from the bilinear effective Lagrangian. In Fourier space this takes the following form:

$$\mathcal{T}^{\mu\nu,\alpha\beta} = (-k^2 + \kappa^2 V) O^{\mu\nu,\alpha\beta}$$  \hspace{1cm} (5.24)
Now this operator can be inverted to get the propagator,

$$D_{\mu\nu\alpha\beta}(k) = \frac{\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}}{k^2 - \kappa^2 V(\phi_0)} $$ (5.25)

Clearly, the poles of the propagator are at $k_0 = \pm \sqrt{k^2 - \kappa^2 V}$ which give rise tachyon in the infrared limit for positive definite potential terms [96]!

This tachyonic mode in the propagator in the infrared limit can induce a Jeans Like instability. Recall that the dispersion relation in Jeans’ treatment of the gravitational instability of a homogeneous fluid is $\omega^2 = v_s^2 k^2 - 4\pi G \rho$. Hence

$$h_{\mu\nu}(k) = D_{\mu\nu,\alpha\beta}(k) T^{\alpha\beta}(k)$$

will produce a runway solution triggering a Jeans-like instability.

Thus in the infrared limit the constant scalar background induces an imaginary mass proportional to the potential of the field. If we choose $V(\phi)$ to be $\lambda \frac{\phi^4}{4!}$ then the induced mass is proportional to the fourth power of constant background $\phi_0$. It is perhaps not a coincidence that in (5.10) the induced tachyonic mass is proportional to the fourth power of the temperature [96].

Let us now proceed to investigate how this effect is manifested in the quantum effective potential for this theory.
5.3 One-loop effective potential for graviton-Higgs theory

Here, the one-loop effective potential for a theory where gravity is coupled to a Higgs field minimally is computed. Let us start with the the Euclidean path integral,

\[ Z = \int \mathcal{D}g \mathcal{D}\chi e^{-S_E} = e^{-W} \]  

(5.26)

where \( S_E \) is the Euclidean action for the full theory and \( \chi \) is any generic field. However, since the path integral has configurations which are identical under the diffeomorphic transformations we have to impose gauge condition to integrate over gauge inequivalent configurations.

The Lagrangian (for positive definite Euclidean metric) for gravity minimally coupled to scalar field is,

\[ \sqrt{g} L = -\frac{1}{\kappa^2} R + \sqrt{g} \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + \sqrt{g} V(\phi) \]  

(5.27)

The one-loop effective potential is obtained from (5.26) employing the loop expansion technique \[24\] and setting the background scalar field to be a constant. Remember, it is sufficient to expand the terms in the full Lagrangian upto quadratic in fluctuating fields to get the one-loop effective potential.

The full Lagrangian (up to quadratic order in fluctuating fields) is the sum of following terms,

\[ \mathcal{L}_q = \left\{ \mathcal{L}_m^{(0)} + \mathcal{L}_g^{(2)} + \mathcal{L}_{gf} + \mathcal{L}_{ghost} + \mathcal{L}_m^{(2)} \right\} \]  

(5.28)

The gauge fixing and ghost Lagrangians are

\[ \mathcal{L}_{gf} = \frac{1}{2} \left[ \partial_{\mu} h^{\mu\nu} - \frac{1}{2} \partial^\mu h \right]^2 \]  

(5.29)
and
\[ \mathcal{L}_{\text{ghost}} = \frac{1}{2} \partial_{\alpha} \zeta_{\mu} \partial^{\alpha} \bar{\zeta}^{\mu} \]  

(5.30)

The ghosts decouple in this gauge and don’t contribute to the effective action. Retaining terms upto quadratic in fluctuating field \( \Phi \) we get the Lagrangian relevant for one-loop EP effective potential

\[
\mathcal{L}^1 = \frac{1}{2} \Phi (\Box_E + V'') \Phi + \frac{1}{4} h_{\mu\nu} (-\Box_E - \kappa^2 V) h^{\mu\nu} \\
- \frac{\kappa}{2} h V' (\phi_0) \Phi - \frac{1}{8} h [(-\Box_E) - \kappa^2 V] h
\]  

(5.31)

Inserting the above Lagrangian into the path integral (5.26) and integrating over scalar fluctuations \( \Phi \) one gets an effective action which contains a propagator corresponding to the operator \( (-\Box_E + V'') \), coming from the bilinear interaction term proportional to \( h\Phi \). This effective action \( \Gamma \), is now quadratic in \( h_{\mu\nu} \).

\[
e^{-\Gamma[\phi_0]} = e^{-V(\phi_0) - \frac{1}{2} \text{Tr} \text{Log}[(-\Box_E + V'')] \int D h_{\mu\nu} \epsilon f - \frac{1}{2} h_{\mu\nu} M^{\alpha\beta} h_{\alpha\beta}}
\]  

(5.32)

The exponent inside the functional integral can be cast in a compact form as follows,

\[
\frac{1}{2} \Psi_i M_{ij} \Psi_j
\]

where \( \Psi_i \) (\( i = 1, 2, ..., 10 \)) represents ten independent components of \( h_{\mu\nu} \) [101, 102]. The components of the ten dimensional vector \( \Psi_i \) are related to the graviton field tensor components as follows:

\[
\Psi_1 = h_{ii}, \ i = 1, ... 4 \\
\Psi_5 = h_{12}, \ \Psi_6 = h_{13}, \ \Psi_7 = h_{14} \\
\Psi_8 = h_{23}, \ \Psi_9 = h_{24}, \ \Psi_{10} = h_{34}
\]  

(5.33)
The matrix elements of $M$ can now be easily obtained from (5.33). The matrix takes a simple block diagonal form. The lower $6 \times 6$ part of the matrix is diagonal, with each of them having same entry $k^2 - \kappa^2 V$. The upper $4 \times 4$ part is a symmetric matrix with diagonal entries $(-k^2 + \kappa^2 V)/4 + \frac{V''^2 \kappa^2}{(k^2 - V')}$ and off-diagonal entries are $-(k^2 + \kappa^2 V)/4 + \frac{V''^2 \kappa^2}{(k^2 - V')}$.

The eigenvalues for the matrix $M$ are,

\[
\begin{align*}
\lambda_i &= k^2 - \kappa^2 V ; (1 \leq i \leq 6) \\
\lambda_i &= \frac{1}{2}(k^2 - \kappa^2 V) ; (7 \leq i \leq 9) \\
\lambda_{10} &= -\frac{1}{2} \left[ \frac{(k^2 - \kappa^2 V)(k^2 + V'' + 2\kappa^2 V'')}{k^2 + V''} \right] 
\end{align*}
\]

(5.34)

The eigenvalue for the operator coming from the quadratic part of the scalar field is given by,

\[
\lambda_\phi = k^2 + V''
\]

The effective potential is given by

\[
V_{eff}^1 = V(\phi_0) + \frac{1}{2} \log \text{Det}[M] + \frac{1}{2} \text{Tr} \log \lambda_\phi
\]

(5.35)

In terms of the momentum integrals the effective potential is given by,

\[
V_{eff}^1 = V + \frac{9}{2} \int_0^\Lambda \frac{d^4k}{(2\pi)^4} \ln (k^2 - \kappa^2 V) \\
+ \frac{1}{2} \int_0^\Lambda \frac{d^4k}{(2\pi)^4} \ln \left[ k^4 + (V'' - \kappa^2 V)k^2 + \kappa^2(2V'^2 - VV'') \right]
\]

(5.36)

Evaluating the momentum integrals with a cut-off $\Lambda$ the one loop effective potential is
given by,

\[
V^1_{eff}(\phi_0) = V + \frac{9}{32\pi^2} \left[ \frac{\kappa^4 V^2}{2} \left( \ln \frac{\kappa^2 V}{\Lambda^2} - \frac{1}{2} \right) - \kappa^2 V \Lambda^2 \right] - \frac{i9\kappa^4 V^2}{64\pi} \\
+ \frac{1}{32\pi^2} \left[ (V'' - \kappa^2 V) \Lambda^2 + \frac{a^2 - 2b}{4} \left( \ln \frac{b}{\Lambda^4} - 1 \right) \right] \\
+ \frac{a\sqrt{a^2 - 2b}}{64\pi^2} \ln \left[ \frac{a + \sqrt{a^2 - 4b}}{a - \sqrt{a^2 - 4b}} \right]
\]

(5.37)

where

\[
\begin{align*}
    a &= V'' - \kappa^2 V \\
    b &= \kappa^2 (2V'' - V V'')
\end{align*}
\]

The source of the imaginary part in the last term of the first line of eq. (5.37) is due to the non-linear nature of graviton-Higgs theory. It is clear from 5.36 in the infrared limit the momentum integrals (functional traces) become non-analytic due to negative logarithms.

One may think that the logarithmic terms in the second and third lines of the eq. (5.37) may give rise imaginary contributions also. Indeed this could happen in some cases. However, this is not possible for any monomial potential with positive coefficient. The most obvious example of this kind is \(\lambda \phi^4\). It is easy to see that \(b\) is positive for this case and since the terms which are Planck-suppressed always dominated by unsuppressed ones both \(a\) and \(a^2 - 4b\) are positive so long as \(\phi_0 < M_{\text{Planck}}\). These conditions appropriately rule out the possibility of any imaginary contribution from other terms of eq. (5.37). However, if we don’t restrict the potential to be of this particular form then for positive \(V\) the sufficient condition for not getting any additional imaginary part from the logarithmic terms in the effective potential reduces to \(a, b > 0\) [96].

Similar results have been obtained in [45, 46] etc. In a related work, Fradkin et al [42] have shown that for a gauged supergravity theory one of the modes in the spectral decomposed one-loop operators in de Sitter background contains negative modes. The appearance of imaginary part in one loop effective action was also reported by Odintsov for \(SU(5)\) GUT theory in de
Sitter background [103]. In some higher derivative gravity theories, with non-minimal coupling to scalar fields, similar imaginary terms in the effective potential have also been observed [47,104].

An interesting feature of the one-loop effective potential is that the effect completely disappears if the classical Higgs potential is set to zero. This is in contrast to the flat spacetime gauge field theories where a minimally coupled Higgs field generates an effective potential perturbatively, even if the classical potential vanishes. This is because in Higgs-graviton theory the absence of a classical Higgs potential, the Higgs field has no other coupling to the graviton field when expanded around a constant vacuum value. In standard electroweak theory in flat spacetime, in contrast, the classical Lagrangian has Higgs-gauge field seagull terms which lead to the one loop effective potential [87,105] even in the absence of a classical potential. This does not happen in perturbative quantum gravity since there is no such interaction for a constant Higgs background. In fact, this feature of scalar-graviton theory appears to persist to higher orders of perturbation theory for spacetime independent Higgs backgrounds. In the next section the finite temperature counterpart of the effective potential for graviton-Higgs theory has been taken into consideration to see the effect of nonzero temperature on the instability obtained in the zero temperature case.

5.4 Effect of finite temperature

In this section the effective potential for graviton-Higgs system is computed for finite temperature. The thermal contribution of the one loop effective potential is important to investigate, to ascertain whether the zero temperature instability is reinforced or weakened. Doubtlessly, the result of this assay will have implications for inflation and perhaps also for the electroweak phase transition in the early Universe. The recent discovery of a 125 Gev scalar boson at CERN lends special credence to theories with gravitons interacting with Higgs fields vis-a-vis their implication for various instabilities in the early Universe [106,107]. The appearance of an imaginary part in the zero temperature one-loop effective potential prompts one to investigate the situation for the finite temperature counterpart of the theory. I have already cited the literature where a tachyonic pole in the one loop graviton self-energy has been discerned, leading to an instability.
in the theory. The issues addressed in this section are: (a) how this instability manifests in the one-loop effective potential, and (b) if there are additional imaginary temperature-dependent contributions at one loop, whether these contributions neutralize the zero-temperature imaginary part of the effective potential found in the last section, or enhance it. It is found that the effect of constant scalar background is being amplified in the high temperature limit of Higgs-graviton effective potential. The low temperature limit, on the other hand, shows a rather interesting behaviour: in the physically relevant region the temperature dependent imaginary part oscillates with a damping amplitude. This oscillation may be a reminiscence of the instability of flat background under perturbation in presence of interaction between gravitons and thermally excited matter fields [96].

From (5.37) one can write down the expression for the one-loop effective potential in momentum space in a slightly modified form,

\[
V_{\text{eff}} = V + \frac{9}{2} \int \frac{d^4k}{(2\pi)^4} \ln(k^2 - \kappa^2 V) + \frac{1}{2} \sum_{i=1}^{2} \int \frac{d^4k}{(2\pi)^4} \ln[k^2 + A_i]
\]  

(5.38)

with \(A_i\)'s are root of the quartic equation \(k^4 + ak^2 + b = 0\) where \(a = V'' - \kappa^2 V\) and \(b = \kappa^2(2V''2 - VV'')\).

To obtain finite temperature effective potential one has to shift the momentum integrals of (5.38) by

\[
\int d^4k \rightarrow T \sum_n \int d^3k
\]

\[
k \rightarrow (2\pi nT, \mathbf{k})
\]

Thus now, the finite temperature counterpart of the effective potential, in terms of Euclidean momentum integrals becomes [108, 109],

\[
V_{\text{eff}} = V_0 + \frac{1}{2} T \sum_{i=1}^{2} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \ln(k^2 + 4\pi^2 n^2 T^2 + A_i) + \frac{9}{2} T \int \frac{d^4k}{(2\pi)^4} \ln(k^2 + 4\pi^2 n^2 T^2 - \kappa^2 V)
\]

(5.39)
The above integrals can be represented in a general form,

\[ I(t, u) = \frac{t^{\frac{1}{2}}}{2\pi} \sum_{n=-\infty}^{n=\infty} \int \frac{d^3k}{(2\pi)^3} \ln(k^2 + tn^2 + u) \]  

(5.40)

Here \( t = 4\pi^2T^2 \).

Since \( I(t, u) \) is a divergent quantity one has to regularize this integral. Dimensional regularization is most convenient to evaluate such integrals. Let us perform an integral transform to tackle the infinite sum in the expression. The basic integral is,

\[ I(t, u, d) = \frac{t^{\frac{1}{2}}}{2\pi} \sum_{n=-\infty}^{n=\infty} \int \frac{d^3k}{(2\pi)^3} \ln(k^2 + tn^2 + u) = -\frac{t^{\frac{1}{2}}}{2\pi} \sum_{n=-\infty}^{n=\infty} \frac{1}{(4\pi)^{d/2}} \int_0^\infty d\tau \tau^{-d/2-1} e^{-\tau(tn^2+u)} , \]

(5.41)

where I have used the relation

\[ \int \frac{d^d k}{(2\pi)^d} \ln(k^2 + tn^2 + u) = -\frac{\partial}{\partial \alpha} \int \frac{d^d k}{(2\pi)^d} (k^2 + tn^2 + u)^\alpha \ \bigg|_{\alpha=0} = -\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} (tn^2 + u)^{\frac{d}{2}} , \]

(5.42)

and also assumed that the \( \tau \) integration has no singularities.

To evaluate the integral (5.41) a large temperature expansion of the integrand is performed. At high temperature limit i.e. for \( \frac{n}{T} << 1 \) one can write the sum over \( n \) as a binomial expansion in \( \frac{n}{T} \)

\[ \sum_{n=1}^{\infty} (tn^2 + u)^d = \sum_{n=1}^{\infty} t^d \left[ n^d + \left( \frac{d}{2} \right) \left( \frac{u}{T} \right) \frac{1}{n^{2-d}} + \frac{1}{2} \left( \frac{d}{2} \right) \left( \frac{d}{2} - 1 \right) \left( \frac{u}{T} \right)^2 \frac{1}{n^{4-d}} + O \left( \frac{u}{T} \right)^3 \right] \]

\[ = t^d \left[ \zeta(-d) + \left( \frac{d}{2} \right) \zeta(2-d) \left( \frac{u}{T} \right) + \zeta(4-d) \left( \frac{d}{4} \right) \left( \frac{d}{2} - 1 \right) + O \left( \frac{u}{T} \right)^3 \right] \]

(5.43)

where the definition of Riemann zeta function and its analytic continuation to the region \( n < 1 \) is used.

\[ \zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} , \ n > 1 \]

One can now easily extract the pole part of the integral. Defining \( \epsilon = 3 - d \) the high
temperature part of (5.41) becomes,

\[
I(t, u, d - 3) = -\frac{u^2}{16\pi^2} \left(1 + \frac{1}{\epsilon}\right) - \frac{1}{6\pi^2} \zeta(-3) t^2 - \frac{1}{4\pi^2} \zeta(-1) tu + \frac{u^2}{32\pi^2} \ln \frac{u^2}{M^2}
\]

\[
= \frac{1}{12\pi^2} u^2 t^2 + \frac{1}{32\pi^2} u^2 \ln u/t
\]

\[-\frac{u^2}{16\pi^2} \left(\gamma - \frac{3}{4} + \frac{1}{2} \psi(3) - \frac{1}{2} \ln \frac{M^2}{\pi}\right) + O(t^{-1})
\]

(5.44)

with

\[
\psi(x) = \frac{d}{dx} \Gamma[x]
\]

Here we have also introduced \(M\) as an arbitrary scale of renormalization. From the above expression we can easily see that there will be imaginary contributions from some of the terms involved. If we closely inspect the possible \(u\)'s from eq. (5.38) this becomes clear. Apart from the irrelevant constants and after getting rid of the pole term by a suitable counter term one can write the effective potential at high temperatures as,

\[
V_{\text{eff}} = V + \frac{1}{64\pi^2} \sum_{i=1}^{\infty} |A_i|^2 \ln \left(\frac{|A_i|}{M^2}\right) + \frac{9\kappa^4 V^2}{64\pi^2} \ln \left(\frac{\kappa^2 V}{M^2}\right)
\]

\[+ V_{\text{eff,im}} + V_{\text{eff,T}}
\]

(5.45)

where, \(V_{\text{eff,im}}\) consists of zero-temperature imaginary terms and \(V_{\text{eff,T}}\) is the temperature-dependent part of \(V_{\text{eff}}\).

It is easy to see that the imaginary part of effective potential in this limit is

\[
V_{\text{Im}} = \frac{\kappa^4 V^2}{16\pi} + \frac{\kappa^3 V^{3/2}}{6\pi} T
\]

(5.46)

This indicates, the temperature-dependent contribution to the imaginary part in fact reinforces the zero-temperature piece, thereby exacerbating the instability discussed in the last section. The plot (Fig.[5.1]) of temperature dependent imaginary contribution is simple and shows that it grows with the temperature. where \(x = \frac{\kappa v^{3/2}}{T}\) It is clear that since the dimensional pole term is proportional to \(V^2\) instead of \(V\) the theory is non-renormalizable. However, the main focus of this calculation is not on the ultraviolet behaviour of the theory, but rather its infrared
instabilities at zero and finite temperature. Thus, even if the ultraviolet divergences are tamed as conventional with appropriate counter-terms, it is obvious that the infrared instabilities will persist.

In order to ensure that the finite temperature treatment has the correct limit to vanishing temperature, one needs to consider the low temperature limit of the temperature-dependent part of $V_{\text{eff}}$. To obtain the low temperature limit of eq. (5.41) we now have to use the following identity

$$ \sum_{n=-\infty}^{\infty} e^{-\tau n^2} = \left( \frac{\pi}{\tau t} \right)^{1/2} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2 / t \tau} $$

With the help of this we can write eq. (5.41) as

$$ I(t, u, d) = -\frac{1}{2\pi^{1/2}} \sum_{n=-\infty}^{\infty} \frac{1}{(4\pi)^{d/2}} \int_0^{\infty} d\tau \tau^{-(d+3)/2} e^{-\tau t} e^{-\pi^2 n^2 / t \tau} $$

This decomposes into two parts, one being temperature dependent and other, zero temperature. Once again the pole term is independent of temperature. After separating out the $n = 0$ piece from the above expression one gets,

$$ -\frac{1}{2\pi^{1/2}} \frac{1}{(4\pi)^{d/2}} u^{(d+1)/2} \Gamma\left( -\frac{d}{2} - \frac{1}{2} \right) - \frac{1}{\pi^{1/2}} \sum_{n=1}^{n=\infty} \frac{1}{(4\pi)^{d/2}} \int_0^{\infty} d\tau \tau^{-(d+3)/2} e^{-\tau t} e^{-\pi^2 n^2 / t \tau} \, . $$

Figure 5.1: Plot of temperature dependent imaginary part of EP versus $x$, for $x << 1$
From the first term we easily extract the pole term

\[- \frac{u^2}{16\pi^2} \left( \frac{1}{\epsilon} \right) - \frac{u^2}{32\pi^2} \left( \psi(3) + \ln \frac{4\pi}{u} \right) + O(\epsilon) \]  

(5.50)

One can perform the \( \tau \) integration to get the temperature dependent part. The result is given in terms of a modified Bessel function [110],

\[
\int_0^\infty d\tau \tau^{-(d+3)/2} e^{-\tau t} e^{-\pi^2 n^2 / t\tau} = \left( \frac{tu}{\pi n^2} \right) K_2(2\sqrt{\pi^2 n^2 u/t})
\]  

(5.51)

The low-temperature behaviour of the integral eq. (5.48)

\[
I(t, u, 3 - \epsilon) = - \frac{u^2}{16\pi^2} \left( \frac{1}{\epsilon} \right) - \frac{u^2}{32\pi^2} \left( \psi(3) + \ln \frac{4\pi}{u} - \ln \frac{u}{M^2} \right)
\]

\[
- \frac{u^2}{2\pi^2} \sum_{n=1}^\infty \left( \frac{t}{4\pi^2 n^2 u} \right) \frac{1}{n} e^{-4\pi^2 n^2 u/t} \right)^{1/2},
\]

(5.52)

where for large value of the argument I have taken an asymptotic expansion for the modified Bessel function.

To analyze the low temperature behaviour of the potential I have ignored any imaginary part coming from the \( A_i \)'s and will only concentrate on \( u = -\kappa^2 V \) part. Then at Low temperature imaginary contribution for effective potential has the following form [96]

\[
\frac{\kappa^4 V^2}{32\pi} + \frac{\kappa^4 V^2}{2} \left( \frac{T^2}{\kappa^2 V} \right)^{\frac{1}{2}} \sum_{n=1}^\infty \frac{1}{n} \left( \cos nx - \sin nx \right)
\]  

(5.53)

The second term above is temperature dependent. One can approximate the sum as a integral over \( n \) as \( n \) goes upto infinity or we can compute the sum exactly. Performing both using MATHEMATICA we have found the behaviour of the second term of eq. (5.53) is oscillatory with damping amplitude (Fig.[5.2]) for values of \( x > 0.6 \) approximately and for large value of \( x \) i.e. for \( T \to 0 \) the oscillations die away and temperature dependent imaginary part vanishes [96].

The analysis above is based on a one-loop effective action evaluated in a certain gauge. However as has mentioned earlier in this thesis, there is a gauge invariant way of calculating the one-loop effective potential due to Vilkovisky and De-Witt [?,41]. If one calculates the effective
Figure 5.2: Plot of temperature dependent imaginary part of EP versus \( x, \ x >> 1 \)

potential using this method, one finds that there is hardly any qualitative change in the effective potential particularly the imaginary part doesn’t disappear. However, the numbers do change. I state below the important factors here which undergo a change, without going into details of the calculative scheme. Details can be found in [46].

The structure of the potential is almost the same for VD approach except the prefactor of the momentum integral changes to 5 in place of 9 in eq. (5.36). The constants \( a \) and \( b \) in this case are

\[
a = V'' - \frac{3}{2} \kappa^2 V
\]

and

\[
b = \frac{1}{2} \kappa^4 V^2 - \kappa^2 VV'' + \frac{3}{4} \kappa^2 V'^2
\]

The above analysis may be redone with these minor changes which clearly do not affect the qualitative nature of the solution.

### 5.5 Discussions

In this chapter, it is shown that for a theory in which the graviton field is minimally coupled to a Higgs scalar field, flat Minkowski spacetime is unstable. This instability is exhibited as a tachyonic mode in the one-loop propagator. A constant scalar field background resembles a thermal bath which backreacts to the gravitons to produce the instability in the system. This
infrared instability in the effective propagator may be regarded as a graviton induced Jeans-like instability. This infrared instability is also manifested in the one-loop effective potential as an imaginary term, independent of the ultraviolet cut off. This term arises also from the infrared limit of the loop integrals.

I have also computed the effect of finite temperature for the graviton-Higgs theory and compared it with the zero temperature result. The high temperature sector involves temperature dependent terms which adds to the imaginary contribution obtained in the zero temperature case. The infrared sector exhibits an instability because of imaginary contribution from both zero temperature and temperature dependent part. Moreover it exhibits an oscillatory behaviour which eventually gets damped out as we lower the temperature, thereby ensuring that the temperature-dependent calculation smoothly interpolates to the previous zero-temperature one-loop effective potential.

The existence of a non-vanishing constant Higgs field background itself is known to signify a vacuum instability which, for the standard electroweak theory, resolves itself by producing masses for the gauge bosons via the Higgs mechanism, as also for fermionic fields Yukawa-coupled to the Higgs field, and the Higgs field itself. The additional instability of such fields coupled to gravitons may have originated from that vacuum instability itself, although it is obvious that this does not resolve itself by generating a mass for the gravitons. To this extent, this latter instability found in this study may be of a more serious nature than the electroweak vacuum instability, since it is far from clear what a system bound by this sort of behaviour will decay into. While the possibility of the role this instability in structure formation (à la the Jeans instability) is intriguing, I do not have a viable cosmological picture so far to claim that this is what this instability must do.

One possible explanation of the zero-temperature instability has been given by Smolin [45] where it has been claimed that it might disappear if one begins with a de Sitter background spacetime instead of a Minkowski spacetime. However, in a recent work by Polyakov [112], it has been pointed that even de Sitter space possesses various quantum instabilities. In any case, it is necessary to estimate the lifetime of any system subject to such an instability and compare that with the age of the Universe to ensure that it does play a significant role.
It is worthwhile to note here that what is being computed in this paper entails fluctuating gravitons coupled to fluctuating Higgs fields, in the absence of a cosmological constant term in the classical action. When one perturbs this theory around a constant scalar as well as a flat spacetime background and integrate out fluctuations around that, an instability develops in the infrared regime of the fluctuations, manifesting in the one loop effective potential. One might construe that in the one-loop approximation a cosmological term has been induced by the constant scalar background. This observation may provoke one to interpret the source of this instability as being due to the wrong choice of vacuum since flat space is not a solution of Einstein equation with a cosmological constant. However this argument has two possible pitfalls: first, there are quantum fluctuations around the constant classical potential in the full perturbative expansion which will contribute to the Higgs propagator and also at higher loops. Secondly, interpreting the one loop vacuum energy as a possible cosmological constant is incorrect since it is unacceptably large phenomenologically for any reasonable minimum of the one loop effective potential.

Although in chapter 3, gauge-free formulation for free massless gravitons has been discussed but calculations carried out in this chapter don’t follow that way. Since free graviton has more than one unphysical degrees of freedom extracting the gauge-free part in an interacting theory is nontrivial. One possible way to achieve this might be to perform a Hodge-de Rham decomposition of gravitons and try to construct gauge-free couplings of Higgs with gravitons but due to highly non-linear nature of the interactions this procedure doesn’t work. Therefore, this is still an open issue which is to be investigated with care.

5.6 Summary

In this chapter, the appearance of the tachyonic pole in the classical graviton propagator in minimally-coupled graviton-Higgs theory is explicitly exhibited. The one-loop effective potential of the graviton-Higgs theory also develops an instability. The presence of this instability is traced to the tachyonic pole in the graviton propagator. A comparison of the nature of the one-loop effect between gauge-Higgs and graviton-Higgs theory has been made. Then a detailed study
on the one-loop effective potential at finite temperature is done. The infrared limit describes an interesting situation exhibiting an instability due to the temperature dependent contribution to the effective potential developing an additional imaginary part over and above the one in the $T = 0$ limit. Various aspects of this instability are discussed. In the next chapter couplings of massless scalar and pseudoscalar with higher derivative gravity are considered.