Chapter 3

Gluon-Gluon Contribution to Di-vector Boson + Jet Production

In this chapter, we will study within the SM, the production of a pair of electroweak vector bosons in association with a hard jet via gluon fusion. In particular, we will consider leading order $gg \rightarrow \gamma \gamma g$, $gg \rightarrow \gamma Z g$, $gg \rightarrow ZZ g$ and $gg \rightarrow W^+ W^- g$ processes at hadron colliders such as the LHC. These processes proceed via quark loop diagrams at the leading order itself. In fact, the process $gg \rightarrow VV' g$ contributes to the hadronic process $pp \rightarrow VV' j + X$ at the next-to-next-leading order (NNLO) in $\alpha_s$. Being leading order contributions, these are finite and therefore their contributions towards the hadronic cross sections can be calculated separately. We will compute the hadronic cross sections of these gluon initiated process at various collider centre-of-mass energies and compare them with the corresponding tree-level hadronic processes. We will also make a comparison with the corresponding $gg \rightarrow VV'$ processes. Some important kinematic distributions common to them, will also be given. We will present a detailed study of $gg \rightarrow \gamma Z g$ process which is central to all other processes. We will make comments on the issue of numerical instabilities which commonly arises in such calculations. This chapter is based on the work reported in [71–73].

The search for new physics at the LHC is in progress and the collider is delivering data presently at 8 TeV centre-of-mass energy. The discovery of a fundamental scalar particle (most probably a Higgs boson) of mass around 125 GeV has received a lot of world-wide attention [7–9]. We expect more good news from the experiments at the LHC before the
collider goes for a two year long pause. So far, the SM of particle physics seems to be in excellent agreement with the collected data (more than 10 fb$^{-1}$). There have been searches for the hints of physics beyond the SM such as supersymmetry, large extra dimensions, etc. But, as of now, there is no clear evidence [12–15]. The process of identifying the discovered fundamental scalar particle as the Higgs boson is also continuing. Due to the lack of signals for beyond the SM scenarios, there is a need to look for the SM processes that were not accessible earlier at the Tevatron. Most of such processes have several particles in the final state, and/or occur at the higher order. Calculations of multiparticle processes not only provide tests of the SM, they can also contribute to the background to new physics signals. One such class of processes is multi-vector boson production in association with one or more jets.

At the LHC centre-of-mass energy, the collider has another useful feature. In the proton-proton collisions, the gluon luminosity can be quite significant. It can even dominate over the quark luminosity in certain kinematic domains. Therefore, at the LHC, loop mediated gluon fusion processes can be important. Di-vector boson production via gluon fusion have been studied by many authors [74–80]. We consider another class of processes $gg \rightarrow VV'g$, where $V$ and $V'$ can be any allowed combination of electroweak vector bosons. These processes can be a background to the Higgs boson production as well as new physics scenarios such as the Technicolor. At the leading order, these processes receive contribution from quark loop diagrams. The prototype diagrams are displayed in Fig. 3.1. The calculations for the process $gg \rightarrow \gamma\gamma g$ have already been performed [81,82]. Preliminary results for $gg \rightarrow \gamma Z g$ were presented in [71]. Recently, Melia et al. have presented calculations for $gg \rightarrow W^+W^- g$ [83]. In the next section, we give details on the structure of the amplitudes. In section 3.2, we describe the method of calculation and important numerical checks. Numerical results are presented in section 3.3 and a discussion on the issue of numerical instability in our calculations follows in section 3.4.

3.1 The Amplitudes

The processes $\gamma\gamma g$, $\gamma Z g$ and $ZZ g$ receive contribution from two main classes of quark loop diagrams – pentagon and box types, as shown in (a) and (b) of Fig. 3.1. The box class of diagrams are due to the triple gluon vertices and they can be further divided into three
Figure 3.1: The prototype diagrams for the processes $gg \rightarrow VV'g$. The wavy lines represent the appropriate combination of the $\gamma$, $Z$, or $W$ boson. The last two classes (c) and (d) are relevant to $WWg$ production only. We do not consider diagrams involving Higgs boson for the $ZZg$ and $WWg$ cases.

subclasses. This subclassification has its own physical importance. These subclasses are separately gauge invariant with respect to the electroweak vector bosons. Other diagrams can be obtained by a suitable permutation of external legs. For each quark flavor, there are 24 pentagon-type and 18 box-type diagrams. However, due to Furry’s theorem, only half of the 42 diagrams are independent. The $\gamma\gamma g$ amplitude is purely vector type. In $\gamma Z g$ and $ZZg$ cases, the pentagon diagrams give both the vector as well as axial-vector contributions, while the box diagrams give only vector contribution. We work with five massless quark flavors and expect decoupling of the top quark.

In the case of $WWg$ process, instead of a single quark flavor, two quark flavors of a single generation contribute to the above discussed pentagon and box diagrams. To keep the matter simple, for this process, we work with the first two generations of massless quarks. It is expected that the contribution from the third generation will not be significant in low $p_T$ region [80]. There are additional box and triangle classes of diagrams due to $\gamma/Z WW$ vertex for each quark flavor. These are shown in (c) and (d) of Fig. 3.1. Due to Furry’s theorem, the triangle diagrams with $\gamma WW$ coupling do not contribute and only axial-vector part of the triangle diagrams with $ZWW$ coupling contribute to the amplitude. Since the axial-vector coupling of the $Z$ boson to a quark is proportional to the $T^a_q$ value, the axial-vector contributions from additional triangle and box diagrams, when summed
over a massless quark generation, vanish. The vector contribution from the additional box-type diagrams is separately gauge invariant. Because of its color structure, it interferes with the axial-vector part of the pentagon amplitude. We have explicitly checked that its contribution towards the cross section is very small; therefore we have dropped this contribution. Thus, effectively we are left with the $ZZg$-like contributions for $WWg$. The Higgs boson interference effects for the cases of $ZZg$ and $WWg$ are ignored in the present calculation. Our one-loop processes, being the leading order processes, are expected to be finite, i.e., free from ultraviolet (UV) and infrared (IR) divergences. The amplitudes of our processes has the following general structure:

$$M^{abc}(gg \rightarrow VV'g) = i \frac{f^{abc}}{2} M_V(VV'g) + \frac{d^{abc}}{2} M_A(VV'g),$$

$$M_V(VV'g) = -e^2 g_3^2 C_V(VV'g) \left( P_V - B_V \right),$$

$$M_A(VV'g) = -e^2 g_3^2 C_A(VV'g) \left( P_A \right).$$

This structure is explained in detail in [71]. Here $M_{V,A}$ are amplitudes for the vector and axial-vector parts of the full amplitude under consideration. Because of the Bose symmetry $M_V \rightarrow -M_V$ under the exchange of any two external gluons while $M_A$ remains same. $B_V$ and $P_{V,A}$ are the box and pentagon contributions from a single flavor (single generation for the $WWg$ case) of quarks. The structure of the amplitude suggests that the vector and axial-vector contributions, in the $\gamma Zg$, $ZZg$ and $WWg$ cases, should be separately gauge invariant. Moreover, due to the color structure when we square the amplitude, the interference between the vector part and the axial-vector part vanishes, i.e.,

$$|M(gg \rightarrow VV')|^2 = \left( 6|M_V|^2 + \frac{10}{3}|M_A|^2 \right).$$

Therefore the cross section of any of these processes, is an incoherent sum of the vector and axial-vector contributions. The couplings $C_{V,A}$ for various cases are listed below and
contributions from all the relevant quark flavors (described above) are included appropriately.

\[
C_V(\gamma\gamma g) = \frac{11}{9}, \quad C_A(\gamma\gamma g) = 0,
\]

\[
C_V(\gamma Z g) = \frac{1}{\sin\theta_w \cos\theta_w} \left( \frac{7}{12} - \frac{11}{9} \sin^2\theta_w \right),
\]

\[
C_A(\gamma Z g) = \frac{1}{\sin\theta_w \cos\theta_w} \left( -\frac{7}{12} \right),
\]

\[
C_V(Z Z g) = \frac{1}{\sin^2\theta_w \cos^2\theta_w} \left( \frac{5}{8} - \frac{7}{6} \sin^2\theta_w + \frac{11}{9} \sin^4\theta_w \right),
\]

\[
C_A(Z Z g) = \frac{1}{\sin^2\theta_w \cos^2\theta_w} \left( -\frac{5}{8} + \frac{7}{6} \sin^2\theta_w \right),
\]

\[
C_V(W W g) = \frac{1}{\sin^2\theta_w} \left( \frac{1}{2} \right),
\]

\[
C_A(W W g) = \frac{1}{\sin^2\theta_w} \left( -\frac{1}{2} \right). \tag{3.3}
\]

We would also like to see if the top quark loop contributes significantly in $\gamma\gamma g$ and $\gamma Z g$ processes. The top quark coupling in these amplitudes can be found in [73].

### 3.2 Calculation and Numerical Checks

For each class of diagrams, we write down the prototype amplitudes using the SM Feynman rules [52]. The amplitude of all other diagrams are generated by appropriately permuting the external momenta and polarizations in our code. The quark loop traces without $\gamma_5$ are calculated in $n$ dimensions, while those with $\gamma_5$ are calculated in 4 dimensions using FORM [64]. We do not need any $n$-dimensional prescription for $\gamma_5$ as the pentagon diagrams, which are the only contributions to the axial-vector part of the amplitude, are UV finite and free of anomaly. The amplitude contains one-loop tensor integrals. In the case of pentagon-type diagrams, the most complicated integral is rank-5 tensor integral $(E_{\mu\nu\rho\sigma\delta}^\mu)$, while for the box-type diagrams, rank-4 tensor integral $(D_{\mu\nu\rho\sigma})$ is the most complicated one. Five point tensor and scalar integrals are written in terms of box tensor and scalar integrals using 4-dimensional Schouten Identity. The box tensor integrals are reduced, in $n = 4 - 2\epsilon$ dimensions, into the standard scalar integrals – $A_0, B_0, C_0$ and $D_0$ using FORTRAN routines that follows from the reduction scheme developed by Oldenborgh and Vermaseren [63]. We require box scalar integrals with two massive external legs at the
most. The scalar integrals with massive internal lines are calculated using OneLOop library [84]. Because of a very large and complicated expression of the amplitude, we calculate the amplitude numerically before squaring it. This requires numerical evaluation of the polarization vectors of the gauge bosons. We choose the real basis, instead of the helicity basis, for the polarization vectors to calculate the amplitude. This is to reduce the size of the compiled program and the time taken in running the code. We use RAMBO in our Monte Carlo integration subroutine to generate three particle phase space for our processes [85].

Figure 3.2: Ward identities for $gg \rightarrow \gamma\gamma g$ pentagon and box diagrams. All the momenta are taken incoming. The dotted lines take care of momentum insertion at relevant vertices.

Since the processes $gg \rightarrow VV'g$ are leading order one-loop processes, the general amplitude in Eq. 3.1 should be both UV as well as IR finite. However, individual diagrams may be UV and/or IR divergent. The IR divergence is relevant to only light quark cases. All these singularities are encoded in various scalar integrals. To make UV and IR finiteness checks on our amplitude we have derived all the required scalar integrals and they are listed in the appendix B.2. Following are the details of various checks made on our amplitude given in Eq. 3.1.

1. **UV Finiteness:** The tadpole and bubble scalar integrals ($A_0$ and $B_0$) are the only
sources of UV singularity in any one-loop amplitude. For the case of massless internal lines, $A_0$s do not appear in the tensor reduction. For both the massive and massless quark contributions, we have verified that the amplitude is UV finite. The amplitude of each pentagon diagram has only UV finite tensor integrals. Therefore, each pentagon diagram is UV finite by itself as expected from a naive power counting. The box diagrams individually are not UV finite. Therefore, the cancellation of the divergence in the sum of the box diagrams is an important check. We find that the three classes of box diagrams are separately UV finite.

2. **IR Finiteness:** The diagrams with massless internal quarks have mass singularities. Even in the case of a quark of small mass, like the bottom quark, these diagrams may have large logarithms which should cancel for the finiteness of the amplitude. There can be $\ln^2(m_q^2)$ and $\ln(m_q^2)$ types of mass singular terms. We have checked explicitly that such terms are absent from the amplitude. Moreover, we have verified that the IR finiteness holds for each fermion loop diagram, confirming the general result proved in Sec. 2.3.1.

3. **Ward Identities:** Certain mathematical identities can be obtained by replacing a polarization vector by its 4-momentum in any of the pentagon/box amplitudes. This way a pentagon amplitude can be written as a difference of two (reduced) box amplitudes and also a box amplitude can be written as a difference of two (reduced) triangle amplitudes. For example, if we replace the polarization vector of one of the photons by its 4-momentum in the prototype pentagon and box amplitudes for $gg \rightarrow \gamma \gamma g$, we get

$$M_B(p_1, p_2, p_3; e_1, e_2 = p_2, e_3, e_4, e_5) = M_B(p_1, p_2 + p_3, p_4; e_1, e_3, e_4, e_5) - M_B(p_1 + p_2, p_3, p_4; e_1, e_3, e_4, e_5),$$

(3.4)

$$M_B(p_1, p_2, p_3; e_1, e_2 = p_2, e_3, e_{45}) = M_T(p_1, p_2 + p_3; e_1, e_3, e_{45}) - M_T(p_1 + p_2, p_3; e_1, e_3, e_{45}).$$

(3.5)

Here $e_i$s are polarization vectors of the gauge bosons and $e_{45}$ denotes the triple gluon
vertex attached to the box/triangle diagram. These identities are shown diagrammatically in Fig. 3.2. We have verified them numerically. These Ward identities are important checks on individual diagrams and a set of these identities can also be utilized for a systematic study of numerical instabilities in the tensor reduction, near exceptional phase space points. An exceptional phase space point corresponds to the vanishing of a partial sum of the external momenta.

4. **Gauge Invariance:** As we have seen, because of the color structure, the vector and axial-vector parts of the amplitude do not interfere and they are separately gauge invariant. The vector part of the amplitude has gauge invariance with respect to the three gluons and the electroweak bosons. This has been checked by replacing the polarization vector of any of these gauge particles by its momentum \( \varepsilon^\mu(p_i) \rightarrow p_i^\mu \) which makes the amplitude vanish. As one would expect the pentagon and the three classes of box contributions are separately gauge invariant with respect to the electroweak bosons. For each gluon, one of the three classes of box amplitudes is separately gauge invariant and further cancellation takes place among the pentagon and the other two box contributions. The axial-vector part of the amplitude is separately gauge invariant with respect to all the three gluons and the photon. We verify that due to explicit breaking of the chiral symmetry in the presence of a quark mass, the axial-vector part of the amplitude, in \( \gamma Zg, ZZg \) and \( WWg \) cases, vanishes on replacing the \( Z/W \) boson polarization by its 4-momentum only in the \( m_q \rightarrow 0 \) limit.

5. **Decoupling of heavy quarks:** As a consistency check, we have also verified that the vector and axial-vector parts of the amplitude vanish in the large quark mass limit [70]. This feature of the amplitude is very closely related to its UV structure. The decoupling theorem holds for each pentagon amplitude and also for each class of box amplitudes. For the process \( gg \rightarrow \gamma Zg \), in Fig. 3.3, we have plotted the ratio of the squared-amplitudes for five and six quark flavor contributions as a function of the top quark mass. The phase space point corresponds to a fixed partonic centre-of-mass energy, \( \sqrt{s} = 8M_Z \). The vector and axial-vector contributions are plotted separately. The \( m_t = 4M_Z \) corresponds to the scale above which the top quarks in the loop cannot go on-shell. Various slope changes shown in this plot correspond to the possibilities of producing one or more final state particles via on-shell \( t\bar{t} \) annihilation.
3.3 Numerical Results

Based on the procedure outlined above, we can now compute the hadronic cross sections and examine various features of our processes. As we have already mentioned, we compute amplitudes numerically using the real polarization vectors of the gauge bosons. There are 32 polarized amplitudes for the case of $\gamma\gamma g$, 48 for the case of $\gamma Z g$ and 72 for the cases of $ZZ g$ and $WW g$. Given the number of diagrams, the number of polarization combination and the length of the amplitude, the computation becomes very time consuming. Each phase space point evaluation takes about 1.3 seconds on a single machine that we use. We use ifort compiler on Intel Xeon CPU 3.20GHz machines. We, therefore, run the code in a parallel environment using AMCI package, a PVM implementation of the VEGAS algorithm [86, 87]. We have used more than 30 cores to run the code in the parallel environment. Still it takes more than 12 hours to get suitable cross section which includes both the massive and massless quark contributions. We will now divide our numerical results into two parts. In the first part, we will do a comparative study of the $\gamma Z g$, $ZZ g$ and $WW g$ processes. Also we will update the results for $\gamma\gamma g$ at 8 TeV LHC. The second part will deal with the numerical results for the $\gamma Z g$ process in greater detail.
3.3.1 Numerical results for $VV'g$

Since the processes $\gamma Zg$, $ZZg$ and $WWg$ have both the vector and the axial-vector contributions, we can put them in one class and study their relative behavior. Their results do not include the top quark-loop contribution. The comparative study, presented in this section, uses following kinematic cuts:

$$p_T^{\gamma,Z,W,j} > 30 \text{ GeV}, \ |\eta^{\gamma,Z,W,j}| < 2.5, \ R(\gamma,j) > 0.6,$$

where $R(\gamma,j) = \sqrt{\Delta \eta_{\gamma j} + \Delta \phi_{\gamma j}}$, is the isolation cut between the photon and the jet. In addition to this, we have chosen the factorization and the renormalization scales as $\mu_f = \mu_r = p_T^{\gamma/Z/W}$, as appropriate. In this section, we have used CTEQ6M parton distribution functions to obtain the results [88]. In Fig. 3.4 we present the results of the cross section calculation for the three processes. We note that at typical LHC energy, the cross sections are of the order of 100 fb. For example, at the centre-of-mass energy of the 8 TeV, the cross sections are 46.7 fb, 95.5 fb, and 225.2 fb, respectively for the $\gamma Zg, ZZg$ and $WWg$ production processes. Therefore, one may expect a few thousand of such events at the end of the present run. But a $W/Z$ boson is observed through its decay channels. If we consider the case when all the $W/Z$ bosons are seen through their decays to the electron/muon only, then, with 20 fb$^{-1}$ integrated luminosity, the number of events for these processes will be approximately 60, 10, and 220 respectively. However, if we allow one of the $W/Z$ boson to decay hadronically, then the number of events would increase significantly. At the 14
TeV centre-of-mass energy, the numbers will be about a factor of three larger. The relative behavior of cross sections of the three processes, as the centre-of-mass energy varies, is quite similar to the case of di-vector boson production via gluon fusion as shown in Fig. 3.5. This common behavior is mainly due to the relative couplings of the processes listed above in Eq. 3.3. We find that at 14 TeV the cross sections for our processes are $20 - 30\%$ of those for the corresponding di-vector boson production (without jet) processes. We can also compare the contributions of these loop processes with those of the corresponding tree-level processes. We find that the processes $gg \rightarrow \gamma Z g, WW g$ make about $4 - 5\%$ contribution to the processes $pp \rightarrow \gamma Z j, WW j$, while $gg \rightarrow ZZ g$ makes a contribution of about $10 - 15\%$ to the $pp \rightarrow ZZ j$ process. Here ‘$j$’ stands for a jet. This is quite similar to the case of the di-vector boson production. Tree-level estimates are obtained using MadGraph [89].

![Figure 3.6: Invariant mass distributions of the pair of vector boson at 8 TeV, for $gg \rightarrow VV'g$.](image)

![Figure 3.7: Transverse momentum distributions of the gluon jet at 8 TeV, for $gg \rightarrow VV'g$.](image)

We have also compared our results for the $WW j$ production case with those of Melia et al. [83]. Though they have considered the leptonic decays of the $W$ bosons and the kinematic cuts, choice of scales and parton distributions etc. are quite different, the percentage contribution of gluon-gluon channel as compared to the LO cross section is same within the allowed range of uncertainty, i.e., $4 - 5\%$. The values of the cross section are more strongly dependent on the values of parameters and kinematic cuts; still two results are similar if we take into account quoted uncertainties and branching ratios. The contribution of these gluon fusion processes can be even larger in appropriate kinematic regime.
We now discuss few kinematic distributions at 8 TeV collider centre-of-mass energy. These distributions remain same, characteristically, at 14 TeV centre-of-mass energy. The invariant mass distributions for the pair of vector boson are shown in Fig. 3.6. The positions of the peaks are related to the masses of the vector bosons in each case. Of course, the $WW$ invariant mass cannot be measured experimentally. In Fig. 3.7, the transverse momentum distributions for gluon jet is given for the three processes. The major contribution to the cross section comes from low $p_T$ region, as we expect. The cross section is very sensitive to the $p_T$ cut on gluon as it is radiated from one of the incoming gluons in the box diagrams.

![Figure 3.8: The collider energy dependence of the cross section for $gg \rightarrow \gamma\gamma g$.](image1)

![Figure 3.9: Transverse momentum distributions of the photon in $gg \rightarrow \gamma\gamma g$.](image2)

![Figure 3.10: Transverse momentum distribution of the gluon jet in $gg \rightarrow \gamma\gamma g$.](image3)

![Figure 3.11: Invariant mass distribution of the two photons in $gg \rightarrow \gamma\gamma g$.](image4)

The variation of the hadronic cross section with the collider centre-of-mass energy, for $\gamma\gamma g$ production via gluon fusion, is given in Fig. 3.8. The result is with the above kinematic
cuts and choices except that, $p_T > 20$ GeV. We reconfirm the importance of $gg \rightarrow \gamma\gamma g$ process at the LHC. We have checked that the top quark contribution to the cross section is very small. Therefore the results include light quark loop contributions only. The cross section at the 8 TeV (14 TeV) centre-of-mass energy is about 0.78 pb (1.86 pb), leading to several hundred (thousand) events with even 1 fb$^{-1}$ of integrated luminosity. Few important kinematic distributions related to this process are given at 8 TeV LHC in Figs. 3.9, 3.10 and 3.11.

### 3.3.2 Numerical results for $\gamma Zg$

Now we will have a detailed discussion on the results for the $gg \rightarrow \gamma Zg$ process. We first study the importance of the diagrams with the top quark in the loop. In Fig. 3.12 we see that the contribution of the top quark ($m_t = 175$ GeV) to the hadronic cross section is negligible. We also see a knee in the plot at $m_t = \frac{M_Z}{2}$. This corresponds to the $Z$ boson production via $t\bar{t}$ annihilation. The top quark decouples at around 100 GeV. We have, therefore, not included its contribution in our results presented below. The run time of our code is also reduced by 50%. It is not surprising that the top quark decouples at such a low value for our processes. This is simply because there are four/five quark propagators in each box/pentagon diagram, leading to a large power of the top quark mass in the denominator.

![Decoupling of the top quark in the cross section calculation of $gg \rightarrow \gamma Zg$.](image)

**Figure 3.12: Decoupling of the top quark in the cross section calculation of $gg \rightarrow \gamma Zg$.**
We can divide our numerical results that are presented in this section into two categories. We first discuss theoretical results, related to the structure of the amplitude, keeping the $Z$ boson on-shell. Theoretical results presented below include following kinematic cuts:

$$\begin{align*}
    p_{T}^{j} &> 30 \text{ GeV}, \quad p_{T}^{\gamma,Z} > 20 \text{ GeV}, \quad |\eta^{\gamma,Z,j}| < 2.5, \quad R(\gamma, j) > 0.6.
\end{align*}$$

We have chosen the factorization and the renormalization scales as $\mu = \mu_f = \mu_r = E_Z^T (= \sqrt{M_Z^2 + (p_{T}^{Z})^2})$, the transverse energy of the $Z$ boson. Results are obtained using CTEQ6M PDFs [88]. In Fig. 3.13, we give the dependence of the cross section on the collider centre-of-mass energy to see the effect of a large gluon luminosity at higher energies. We have already seen that the vector and axial-vector parts of the amplitude are separately gauge invariant. Their contributions towards the cross section are also included in the figure. The axial-vector contribution is only about $10\%$ of the total cross section; this contribution comes from the pentagon class of diagrams. Although the box-contribution to the cross section is not separately gauge invariant with respect to the gluons, it is gauge invariant with respect to the $\gamma$ and the $Z$ boson. We find that more than $70\%$ of the total cross section is due to the box-amplitude only, see Fig. 3.14. The scale variation of the cross section about the central value $\mu_0 = E_Z^Z$, is shown in Fig. 3.15. On increasing the scale by a factor of 2, the cross section decreases by about $25\%$; it increases by about $40\%$ on decreasing the scale by a factor of 2. These large variations are expected because our calculation is effectively LO as far as the $\mu$ dependence is concerned. We see that the cross section falls as we increase the scale $\mu$. This is because an increase in the factorization scale increases the cross section due to the increase in the gluon luminosity; but an increase in the renormalization scale decreases the cross section because of the decrease in the value of $\alpha_s(\mu)$. When we increase both the scales at the same time, the effect of the change in the renormalization scale is stronger. It leads to an overall decrease in the cross section with the increase in the scale $\mu$.

Next, we come to the discussion on our phenomenological results. These results include various kinematic distributions related to the final state particles. For phenomenological results, we work in the narrow width approximation. We allow the $Z$ boson to decay into two leptons in the phase space. In this case, the kinematic cuts are

$$\begin{align*}
    p_{T}^{l} &> 30 \text{ GeV}, \quad p_{T}^{\gamma} > 15 \text{ GeV}, \quad p_{T}^{l} > 10 \text{ GeV}, \quad |\eta^{\gamma,l,j}| < 2.5, \quad R(i_1, i_2) > 0.4.
\end{align*}$$

54
Here $i_1$ and $i_2$ may represent any of the $\gamma/l/j$. For convenience, we have chosen the scale $\mu = \mu_f = \mu_r = M_Z$. In Fig. 3.16 we give the cross section variation in the range of 8 TeV to 14 TeV centre-of-mass energies using both the CTEQ6l1 and CTEQ6M PDFs. These numbers do not include the branching ratio of $Z \rightarrow l^+l^-$. In particular, the cross sections with CTEQ6l1 (CTEQ6M) parton distributions, are 65.4 (53.0) fb and 202.4 (154.3) fb at 8 TeV and 14 TeV centre-of-mass energies respectively. With these cross sections, number of $gg \rightarrow \gamma Z g$ events can be as large as 20000 at the 14 TeV LHC, with 100 fb$^{-1}$ integrated luminosity. However, to observe these events, one may have to look at $Z \rightarrow l^+l^-$ decay channel; here $l$ can be an electron/muon. So including the branching ratios, one may expect more than 1000 events for $gg \rightarrow \gamma Z (\rightarrow l^+l^-)g$ process. The transverse momentum and rapidity distributions for the final state particles are shown in Figs. 3.17–3.22 at the 8 TeV centre-of-mass energy. We have given normalized distributions as they remain same for different choices of parton distributions and/or scales. These distributions are characteristically similar at different collider centre-of-mass energies, but at higher energies contribution coming from high $p_T$/rapidity region grows, while low $p_T$/rapidity region contribution goes down. We note that $p_T^j$ is softer as compared to $p_T^g$. It is because the cross section is dominated by the box class of diagrams and in these diagrams, the gluon is emitted as a bremsstrahlung radiation, see Fig. 3.14. Due to the same reason, i.e., the gluon is emitted more collinearly, the rapidity distribution of the gluon jet is broader as compared to that of the photon. The lepton-$p_T$ distribution peaks around $M_Z/2$. On the other hand the rapidity distribution of the lepton is more central compared to the $\eta^\gamma$ distribution.

We have also compared results of this NNLO calculation with the LO and NLO predictions for $pp \rightarrow \gamma(Z \rightarrow \nu\bar{\nu})j + X$ [90]. The LO and NLO results are obtained using parton-level next-to-leading order program MCFM\(^1\). The comparison is presented after removing the branching ratios in Table 3.1. The table includes results at three different centre-of-mass energies and for two values of the $p_T^\gamma_{min}$. We have included the centre-of-mass energy of 35 TeV, as it is proposed for the HE-LHC collider. The other kinematic cuts are: $p_T^j > 30$ GeV, $p_T^{miss} > 30$ GeV, $|\eta^{\gamma,j}| < 2.5$, $R(\gamma,j) > 0.4$. This table illustrates two facts – 1) the fraction of NNLO events increases with the increase in $p_T^\gamma_{min}$, 2) the NNLO process becomes more important as we increase the centre-of-mass energy. There is an increase in the NNLO fraction with an increase in $p_T^\gamma_{min}$ because, in the NLO events, photon

\(^1\)http://mcfm.fnal.gov/
Table 3.1: Cross sections for the production of $pp \rightarrow \gamma Z_j + X$ at various collider centre-of-mass energies. We use CTEQ6l1 PDF set at the LO and CTEQ6M PDF set at the NLO. The NNLO predictions are with CTEQ6l1(CTEQ6M) parton distribution. The factorization and renormalization scales are set to, $\mu_f = \mu_r = \mu_0 = M_Z$.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ (TeV)</th>
<th>$p_T^{\gamma,min}$ (GeV)</th>
<th>$\sigma^{LO}$ (pb)</th>
<th>$\sigma^{NLO}$ (pb)</th>
<th>$\sigma_{gg}^{\text{NNLO}}$ (fb)</th>
<th>$\sigma_{gg}^{\text{NNLO}} / (\sigma^{NLO} - \sigma^{LO})$ (%)</th>
<th>$\sigma_{gg}^{\text{NNLO}} / \sigma^{NLO}$ (%)</th>
</tr>
</thead>
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<tr>
<td>8</td>
<td>30</td>
<td>2.202</td>
<td>3.391</td>
<td>46.05 (38.25)</td>
<td>3.87 (3.22)</td>
<td>1.36 (1.13)</td>
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<td></td>
<td>50</td>
<td>1.144</td>
<td>1.744</td>
<td>30.49 (25.61)</td>
<td>5.08 (4.27)</td>
<td>1.75 (1.47)</td>
</tr>
<tr>
<td>14</td>
<td>30</td>
<td>4.868</td>
<td>7.722</td>
<td>158.72 (124.48)</td>
<td>5.56 (4.36)</td>
<td>2.06 (1.61)</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>2.608</td>
<td>4.158</td>
<td>109.92 (86.61)</td>
<td>7.09 (5.59)</td>
<td>2.64 (2.08)</td>
</tr>
<tr>
<td>35</td>
<td>30</td>
<td>14.973</td>
<td>23.548</td>
<td>854.09 (606.07)</td>
<td>9.96 (7.07)</td>
<td>3.63 (2.57)</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>8.220</td>
<td>13.514</td>
<td>607.35 (438.88)</td>
<td>11.47 (8.29)</td>
<td>4.49 (3.25)</td>
</tr>
</tbody>
</table>

is emitted from a quark line; a larger $p_T^{\gamma,min}$ suppresses the NLO contribution more than the NNLO contribution. In Fig. 3.23, we have compared the normalized $p_T^{\gamma}$-distributions at the NLO and NNLO, leading to the same conclusion. The importance of the NNLO process is more at higher centre-of-mass energy simply because of the increase in the gluon-gluon luminosity. At 8 TeV, the scale uncertainties in the NLO calculations, on changing the scale by a factor of two in both the directions of the central value, are in the range $7 - 8\%$, while the same in the NNLO calculations is $30 - 50\%$.

### 3.4 The Issue of Numerical Instability

Like other calculations of our types, we have also faced the issue of numerical instability in our calculations for certain phase space points. This is a well known issue in the reduction of one-loop tensor integrals of higher rank and higher points. The issue of numerical instability may also occur in the evaluation of the scalar integrals. We have taken care of this by using the OneLOop implementation of the scalar integrals. We face numerical instabilities
primarily in the evaluation of pentagon tensor integrals. This is related to the inaccurate evaluation of the Gram determinants in those phase space regions where the linear independence of external momenta (modulo 4-momentum conservation) is compromised, i.e., near the exceptional phase space points. The inverse Gram determinants appear in the reduction of tensor integrals. Near exceptional phase space points, the Gram determinants become very small and give rise to numerical problems. These numerical problems are result of a loss of precision due to the large cancellations. This problem can be handled in several ways. One way is to use higher precision for the tensor reduction and for the evaluation of scalar integrals. This certainly reduces the number of exceptional phase space points but the code becomes enormously slow. Another approach could be to use special expressions for the tensor reduction, near such phase space points [91]. It is important to mention here that none of these two approaches cure the problem of numerical instability completely [92]. A more economic and convenient way to proceed in this situation is to judiciously ignore the contributions from such phase space points. This one can do because we are not doing precision calculations and exceptional phase space points are unlikely to give a significant contribution to the total cross section. We perform a gauge invariance (GI) test on the full amplitude for each phase space point. In practice, we introduce a small cut-parameter $\delta$, and under GI we check, if $|\mathcal{M}|^2 < \delta$ holds true for each phase space point. We ignore all those points which fail to pass this test. In our code we have set $\delta = 10^{-6}$. With this value of $\delta$ and other kinematic cuts, the fraction of points ignored is below 2%. However, with higher $p_T$ cut and/or less stringent cut on the $\delta$, the number of such points can drop to a level of 0.01%. Our Monte Carlo phase space integration subroutine is based on the VEGAS algorithm. We believe that the adaptive nature of the VEGAS algorithm also affects the percentage of exceptional phase space points that one may come across in such calculations. We sample about 0.4-0.5 million phase space points to obtain the numerical results. Given the volume of phase space, the number of exceptional phase space points is small and it is reasonable to assume that the cross section is not dominated by this region of phase space. We find that our result depends on this cut very weakly and remains quite stable over the range of $10^{-4} - 10^{-12}$ for the choice of the cut-parameter. This can be seen in Fig. [324]. This stability reflects that the exceptional phase space points are few and make small contribution at the level of total cross section.

We have also checked our results for the cross section calculation by implementing a set
of Ward identity tests and its sensitivity on $\delta$-like cut-parameter. Examples of these identities are given in Eqs. 3.4 and 3.5. However, these identities should not be implemented in a straightforward manner. For example, the pentagon ward identity in Eq. 3.4 can be implemented in the following form,

\[(1 - \frac{M_P}{M_B^1 - M_B^2}) < \delta.\]  \hspace{1cm} (3.9)

Numerically, due to the Ward identity, the ratio $M_P/(M_B^1 - M_B^2) \sim O(1)$. In Eq. 3.9 we compare two numbers of $O(1)$, which gives precise information on the order of cancellation within the working precision. Although the total cross section is quite stable, various distributions, specially the rapidity distributions near the edges, are quite sensitive to the variation in $\delta$. Edges of distributions define the region where exceptional phase space points may lie. Therefore, the inaccuracy of the distributions at the edges of the phase space is not surprising. We have seen that the exceptional phase space points may defy the GI and/or Ward identity tests sometimes. One can make these $\delta$-like cuts more stringent to get more reliable distributions. One can also identify exceptional phase space points at the level of Gram determinants which may be more economical. Phase space points corresponding to a large cancellation in the Gram determinants can be ignored without putting stringent cuts on $\delta$ and again leading to more reliable distributions. A method to implement this criterion is discussed in [92].
Figure 3.13: The vector and axial-vector contributions of the hadronic cross section for $gg \rightarrow \gamma Z g$.

Figure 3.14: Contribution of the box-amplitude towards the hadronic cross section for $gg \rightarrow \gamma Z g$.

Figure 3.15: Variation of the cross section for $gg \rightarrow \gamma Z g$ with the scale, $\mu = \mu_r = \mu_f$ at 14 TeV.

Figure 3.16: Dependence of the cross section on the collider energy, for $gg \rightarrow \gamma Z (\rightarrow l^+l^-) g$. 
Figure 3.17: Transverse momentum distribution of the photon at 8 TeV centre-of-mass energy in $gg \rightarrow \gamma Z(\rightarrow l^+ l^-)g$.

Figure 3.18: Rapidity distribution of the photon at 8 TeV centre-of-mass energy in $gg \rightarrow \gamma Z(\rightarrow l^+ l^-)g$.

Figure 3.19: Transverse momentum distribution of the gluon jet at 8 TeV centre-of-mass energy in $gg \rightarrow \gamma Z(\rightarrow l^+ l^-)g$.

Figure 3.20: Rapidity distribution of the gluon jet at 8 TeV centre-of-mass energy in $gg \rightarrow \gamma Z(\rightarrow l^+ l^-)g$. 
Figure 3.21: Transverse momentum distribution of lepton at 8 TeV centre-of-mass energy in $gg \rightarrow \gamma Z(\rightarrow l^+ l^-)g$.

Figure 3.22: Rapidity distribution of lepton at 8 TeV centre-of-mass energy in $gg \rightarrow \gamma Z(\rightarrow l^+ l^-)g$.

Figure 3.23: A comparison of the normalized $p_T$—distributions of $\gamma$ at NLO and NNLO. The NLO distribution is obtained using MCFM.

Figure 3.24: Dependence of the total cross section on GI test cut-parameter ($\delta$) for $gg \rightarrow \gamma\gamma g$ at 14 TeV. The error bars are shown explicitly.