2.1 Introduction

Differential models either linear or nonlinear often describe physical problems in engineering science. There is also an abundance of transformations of various types that appear in the literature of engineering and mathematics that are generally aimed at obtaining some sort of simplification of a differential model. The transformations (Hansen, 1964; Ames, 1972) which essentially reduce the number of independent variables in partial differential systems have been widely used in fluid mechanics and heat transfer. Transformations have also been used to convert a boundary value problem to an initial value problem suitable for numerical procedures that employ forward marching schemes (Na, 1979). In other instances, transformations have been used to reduce the order of an ordinary differential equation. Mappings have also discovered which transform nonlinear partial/ordinary differential equations to linear forms. Underlying these seemingly unrelated transformations is a unified general principle, which is based on the theory of continuous group of transformations.

The power of group theory was first recognize and rightly applied extensively by a mathematician Sophus Lie from Norway in the latter part of the nineteenth century while finding first integral of first order ordinary differential equation (Bluman and Anco, 2002). In recent years, there has been a revival of interest in applying the principles of continuous group of transformations to differential models, linear as well as nonlinear. Birkhoff (1950) proposed a method based upon simple groups of transformations for obtaining invariant solutions for some problems in the general area of hydrodynamics. The method essentially involves algebraic manipulations, an aspect that makes the method attractive. Group theoretic methods are a powerful tool because they are not based on linear operators, superposition, or any other aspects of linear solution techniques. Therefore, these methods are applicable to non-linear differential models.
A majority of the books on the applications of continuous transformation groups (Bluman and Cole, 1974; Ovsiannikov, 1982) have approached from a mathematical standpoint. In dealing with different boundary value problems in engineering science, the physical aspects associated with the problem need to be properly addressed. The treatment of boundary conditions as an integral part of the differential model in group theoretic methods becomes relevant. The purpose of this thesis is to provide a comprehensive treatment of the subject from a standpoint of engineering science, with special reference to boundary value problems. Applications of the group theoretic principles involved are presented in a clear and systematic fashion. The contents and the treatment of this thesis are particularly suitable for senior undergraduate students, graduate students and analytical and numerical workers in the area of engineering, applied sciences, applied mathematics and pure mathematics also.

In the present chapter, the concepts of continuous transformation groups are presented. In addition, this chapter contains the theoretical background needed for all the subsequent chapters, and forms the basis for the entire thesis.

2.2 Classification of Methods of Transformations

In the literature the role of various transformation techniques have found for the solution of differential equations arising in engineering and sciences. These transformation techniques are dividing in two main headings:

I. Integral Transformation Techniques
II. Transformation of Variables Techniques

The former technique have some sort of limitations in the applications to non-linear partial differential equations arising in real word problems while the later technique provides remarkable results which are consistent with the physical nature of the problem. The entire methods of transformations of variables can be sub-divided in to the following categories.

1 Transformation for partial differential equation to the algebraic equation
2 Transformation for non-linear to linear differential equation
3 Transformation from differential equation to differential equation having known solution
4 Transformation of boundary value problem to initial value problem (BVP to IVP)
5 Reduction of order of the differential equation
6 Reduction of variable technique
7 Similarity technique

Further, the last technique, similarity technique, is most popular and has been widely used in the engineering and science, especially in fluid mechanics and heat transfer. It is well-known fact that the class of solutions known as similarity solutions is only the class of exact solution that works for many cases. Today there are number of similarity techniques available in literature (Hansen, 1964). Some of the well-known techniques are:

a. Variable separable method
b. Dimensional analysis
c. Method of free parameter
d. Group theoretic method

The group theoretic method is classify in to two types, one is Lie group technique, which involves the finite and infinitesimal group and its special version which is known as inspectional group theoretic method, are includes linear group of transformation, scaling group of transformation, spiral group of transformation etcetera. The general Lie group technique is very complicated in its nature and involves lot of algebraic simplifications and hence its application becomes tedious and lengthy. While on the other hand the second type of group theoretic method is more general and quite approachable in application side, and is known as deductive group symmetry technique. In addition, the use of deductive group symmetry procedure, which starts out with a general group of transformations, that leads to some similarity solutions that are not obtainable by inspectional group procedures.

The group invariance analysis imply that to search group of continuous transformations that produce the similarity solutions of a system of partial differential equations is equivalent to the determination of solutions of these equations invariant under a group of transformations. For boundary value problems, it follows that the auxiliary conditions also be invariant under the same group of transformations.

Throughout the entire thesis, we have endeavored to cater to the needs of a novice in the area of group theory, and the analytical worker seeking to use the more
rigorous deductive group symmetry procedures. It is hoped that the thesis will adequately address some of the needs of students and researchers in engineering science who are seeking to apply group theoretic methods to non-linear boundary value problems.

The foundation of the group invariance analysis is contained in the general theories of continuous transformation groups that were introduced and treated extensively by Lie (1875), Lie and Engel (1890) and Lie and Scheffers (1891) in the latter part of the nineteenth century. Subsequently, the books by Cohen (1911), Campbell (1863), Eisenhart (1961), Bluman and Cole (1974), Ovsjannikov (1982), have contributed greatly to the development and clarification of many of Lie's theories, particularly its applications to the invariant solutions of differential equations. In the literature of engineering and applied sciences, the works of Birkhoff (1950), Morgan (1952), Hansen (1964), Na, et. al. (1967), Ames (1972), and Seshadri and Na (1985) give quite extensively the general theories involved in the similarity solutions of partial differential equations as applied to engineering problems. It is assumed, however, that the average engineer may not be thoroughly acquainted with the concepts of that branch of modern algebra designated as group theory. For this reason as well as for clarifications of the terms and concepts involved, a brief review of some of the key aspects of the theory of transformation groups will be given in the present chapter.

2.3 Group Approach

An algebraic group is a set (collection of elements) which has some sort of operation defined among its elements, called binary operation. In addition, a certain set of rules and statements regarding the elements and the defined operation must be satisfied. The elements in a set can be anything: integers, complex numbers, vectors, matrices, transformations etc. One important criterion, however, is the definition of an operation of these elements. Typical operations are integer additions, complex number multiplications, vector additions, and successive transformations.

The formal definition of group is stated below,

A non-empty set $G$ is called a group under the binary operation *, if

2.3.1 Closer Property

The set is closed, that is, if $a$ and $b$ are two elements of the set, then $a*b \in G$
2.3.2 **Existence of Identity**

There exists an identity element $e$ such that $a * e = e * a = a$

2.3.3 **Existence of Inverse**

Every element in $G$ has an inverse, say $a^{-1}$ in $G$ for the operation such that $a * a^{-1} = a^{-1} * a = e$

2.3.4 **Associative Law**

The operation $*$ is associative. That is, $a * (b * c) = (a * b) * c$, $\forall a, b, c \in G$

For example, the set of integers $\mathbb{Z}$ is a group under the operation of addition in $\mathbb{Z}$, because addition of two integers is again an integer, zero is an additive identity, negative of any integer provides the additive inverse and usual addition is associative.

2.4 **Transformations Group**

Let $f^i(x^1, x^2, ..., x^n, a^1, a^2, ..., a^r)$; $(i = 1, 2, 3, ..., n)$ be a set of functions continuous in both the variables $x^i$ and $a^j$ $(j = 1, 2, ..., r)$. We will also assume the continuity of derivatives as may be required in the following discussions. The variables $a^j$ are the parameters of the functions. The $a^j$ are assumed to be “essential parameters”, that is, it is not possible to find $(r - 1)$ functions of $a^j$: $\alpha^1(a), \alpha^2(a), \alpha^3(a), ..., \alpha^{r-1}(a)$ such that

$$f^i(x^1, x^2, ..., x^n, a^1, a^2, ..., a^r) = F^i(x^1, x^2, ..., x^n, \alpha^1, \alpha^2, ..., \alpha^{r-1})$$

If the $a^j$ are not essential parameters, it means that fewer parameters can be constructed from the $a^j$ to serve the same purpose in a function.

We now consider a set of functions $f^i$ as a set of transformations depending on the parameters $a^1, a^2, ..., a^r$ and transforming a point $(x^1, x^2, ..., x^n)$ into $(\overline{x}^1, \overline{x}^2, ..., \overline{x}^n)$, that is

$$\overline{x}^i = f^i(x^1, x^2, ..., x^n, a^1, a^2, ..., a^r)$$

Successive transformations employing various sets of functions are considering being the “operation” of the set. We will establish that the set of functions associated with
the "operation" of a transformation is a group. Clearly, for this to be valid, the four conditions listed previously must be met.

For this instance, corresponding to particular set of values of \( a^j \), say \( a_1^1, a_1^2, \ldots, a_1^r \), we denote the set of transformations of a point \( x = (x^1, x^2, \ldots, x^n) \) into \( x = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) \) as

\[
T_{a_1} x = \bar{x}
\]

Further on extending the concept of elementary calculus, for the existence of inverse function, we can notice that the set of functions \( f^i \left( x^1, x^2, \ldots, x^n, a^1, a^2, \ldots, a^r \right) \) with essential parameters \( a^j \) have an inverse if the Jacobian of the \( f^i \) with respect to \( x^i \) is non vanishes for a set of values of \( x^i \) in some neighborhood of \( x^i \) and we denote the inverse of \( f^i \) by \( f^{i-1} \). Hence follows that, if

\[
\frac{\partial \left( f^1, f^2, \ldots, f^n \right)}{\partial \left( x^1, x^2, \ldots, x^n \right)} \neq 0
\]

in the neighborhood of \( x = (x^1, x^2, \ldots, x^n) \), then the inverse set of transformations, \( f^{i-1} \left( \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n, a^1, a^2, \ldots, a^r \right) \), exists such that \( x^i = f^{i-1} \left( \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n, a^1, a^2, \ldots, a^r \right) \)

In addition, for future reference, we say that a set of functions \( f^i \) is functionally independent if the Jacobian of the set of functions does not vanish. We consider the functions \( f^i \) as a set of transformations depending on the parameters \( a^1, a^2, \ldots, a^r \) and transforming a point \( x = (x^1, x^2, \ldots, x^n) \) into \( x = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) \).

For a particular set of values of the \( a^j \), say \( a_1^1, a_1^2, \ldots, a_1^r \) and we write the transformations as \( T_{a_1} x = \bar{x} \), where \( x \) and \( \bar{x} \) are general points. The inverse functions \( f^{i-1} \) we denote by \( T^{-1}_{a_1} \) and carrying \( \bar{x} \) back into \( x \) as \( T^{-1}_{a_1} \bar{x} = x \)

Two different transformations are defined by different sets of values of the \( a^j \). Thus, if \( a_2^1, a_2^2, \ldots, a_2^r \) is a set of values distinct from \( a_1^1, a_1^2, \ldots, a_1^r \), we consider \( T_{a_1} \) and \( T_{a_2} \) to be different transformations.
Chapter 2: Group Symmetry Invariance Analysis

By the product of two transformations $T_{a_1}$ and $T_{a_2}$ we mean application of the transformations successively; that is, a point $x$ is taken into a point $x'$ as follows:

$$x' = T_{a_2}x = T_{a_2}T_{a_1}x$$

A set of transformations is said to be closed under the product definition if, given any set of parametric values $a_1$ and $a_2$, a set of parametric values $a_3$ (called the compound value of $a_1$ and $a_2$) can always be found such that $T_{a_3}$ is a unique member of the set of given transformations and

$$T_{a_2}T_{a_1} = T_{a_3}$$

Any transformation, which leaves a point unaltered, is defined as an identity transformation.

We now proceed to the concept of a transformation group. Quite simply, a set of transformations say $G$ constitutes a group if the transformations are considered as elements of a set and, as such, obey the group requirements under the operation of taking a product of two transformations. Specifically,

1. **Closer Property**
   
   The set is closed. That is, $T_{a_2}T_{a_1} = T_{a_3} \in G, \ \forall T_{a_2}, T_{a_1} \in G$

2. **Existence of Identity**
   
   There exist a transformation $I$ in $G$ such that $IT_{a_1}x = T_{a_1}Ix = T_{a_1}x, \ \forall T_{a_1} \in G$

3. **Existence of Inverse**
   
   Given any transformation $T_{a_1} \in G$ there exist an inverse transformation

   $$T_{a_1}^{-1} \in G$$

   such that $T_{a_1}^{-1}T_{a_1}x = T_{a_1}T_{a_1}^{-1}x = Ix = x$

4. **Associative Law**

   $$T_{a_1}(T_{a_2}T_{a_3})x = (T_{a_1}T_{a_2})T_{a_3}x, \ \forall T_{a_1}, T_{a_2}, T_{a_3} \in G$$

For the better understanding, Let us show that the set of one-parameter transformations defined as

$$T_{a}x = \begin{cases} 
  x' = (x^1 \cos a - x^2 \sin a)e^a \\
  x'' = (x^1 \sin a + x^2 \cos a)e^a 
\end{cases}$$
This constitutes a group of transformations where \( a \) is a real parameter.

1. **Closer Property**

For two real numbers \( a^1, a^2 \), we have

\[
T_{a_2}T_{a_1}x = \begin{cases} 
\bar{x}^{1} &= \left[ (x^1 \cos a^1 - x^2 \sin a^1) \cos a^2 - (x^1 \sin a^1 + x^2 \cos a^1) \sin a^2 \right] e^{a^1+a^2} \\
\bar{x}^{2} &= \left[ (x^1 \cos a^1 - x^2 \sin a^1) \sin a^2 + (x^1 \sin a^1 + x^2 \cos a^1) \cos a^2 \right] e^{a^1+a^2} 
\end{cases}
\]

\[
\therefore T_{a_2}T_{a_1}x = \begin{cases} 
\bar{x}^{1} = x^1 \cos (a^1 + a^2) - x^2 \sin (a^1 + a^2) e^{a^1+a^2} \\
\bar{x}^{2} = x^1 \sin (a^1 + a^2) + x^2 \cos (a^1 + a^2) e^{a^1+a^2} 
\end{cases}
\]

For the real number \( a^1 + a^2 = a^3 \), we have

\[
T_{a_2}T_{a_1}x = \begin{cases} 
\bar{x}^{1} &= x^1 \cos a^3 - x^2 \sin a^3 e^{a^3} \\
\bar{x}^{2} &= x^1 \sin a^3 + x^2 \cos a^3 e^{a^3} 
\end{cases}
\]

\[
\therefore T_{a_2}T_{a_1}x = T_{a_3}x \\
\therefore T_{a_2}T_{a_1} = T_{a_3} \in G
\]

Hence, \( G \) is closed.

2. **Existence of Identity**

For the real numbers \( a^0 = 0 \), the transformations become

\[
T_{a^0}x = \begin{cases} 
\bar{x}^{1} &= x^1 \\
\bar{x}^{2} &= x^2 
\end{cases}
\]

This provides the identity element of \( G \)

3. **Existence of Inverse:**

For the real number \(-a^1\), corresponding to each real number \( a^1 \) the transformations become

\[
T_{a^1}^{-1}x = \begin{cases} 
\bar{x}^{-1}^{1} &= x^1 \cos (-a^1) - x^2 \sin (-a^1) e^{-a^1} \\
\bar{x}^{-1}^{2} &= x^1 \sin (-a^1) + x^2 \cos (-a^1) e^{-a^1} 
\end{cases}
\]

Further,
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\[ T_{a_1}^{-1}T_{a_1}^{-1}x = \begin{cases} \bar{x}^1 = \left[x^1 \cos(a^1 - a^1) - x^2 \sin(a^1 - a^1)\right]e^{a^1 - a^1} = x^1 \\ \bar{x}^2 = \left[x^1 \sin(a^1 - a^1) + x^2 \cos(a^1 - a^1)\right]e^{a^1 - a^1} = x^2 \end{cases} \]

Similarly,

\[ T_{a_1}^{-1}T_{a_1}^{-1}x = \begin{cases} \bar{x}^1 = \left[x^1 \cos(-a^1 + a^1) - x^2 \sin(-a^1 + a^1)\right]e^{-a^1 + a^1} = x^1 \\ \bar{x}^2 = \left[x^1 \sin(-a^1 + a^1) + x^2 \cos(-a^1 + a^1)\right]e^{-a^1 + a^1} = x^2 \end{cases} \]

This precisely defined \( T_{a_1}^{-1} \) is an inverse of \( T_{a_1} \) in \( G \).

4. Associative Law

For three real numbers \( a^1, a^2, a^3 \), we have

\[
T_{a_3} \left( T_{a_2}T_{a_1} \right) x = \begin{cases} \bar{x}^1 = \left[x^1 \cos\left(a^3 + \left(a^2 + a^1\right)\right) - x^2 \sin\left(a^3 + \left(a^2 + a^1\right)\right)\right]e^{a^3 + \left(a^2 + a^1\right)} \\ \bar{x}^2 = \left[x^1 \sin\left(a^3 + \left(a^2 + a^1\right)\right) + x^2 \cos\left(a^3 + \left(a^2 + a^1\right)\right)\right]e^{a^3 + \left(a^2 + a^1\right)} \end{cases}
\]

\[
T_{a_3} \left( T_{a_2}T_{a_1} \right) x = \begin{cases} \bar{x}^1 = \left[x^1 \cos\left(\left(a^3 + a^2\right) + a^1\right) - x^2 \sin\left(\left(a^3 + a^2\right) + a^1\right)\right]e^{\left(a^3 + a^2\right) + a^1} \\ \bar{x}^2 = \left[x^1 \sin\left(\left(a^3 + a^2\right) + a^1\right) + x^2 \cos\left(\left(a^3 + a^2\right) + a^1\right)\right]e^{\left(a^3 + a^2\right) + a^1} \end{cases}
\]

\[
\therefore T_{a_3} \left( T_{a_2}T_{a_1} \right) x = \left( T_{a_3}T_{a_2} \right)T_{a_1}
\]

Thus, all the group properties for transformations has been established.

2.5 Absolute Invariant

An important concept in the study of group of transformations is the concept of absolute invariant. (Hansen, 1964)

Let \( T_{a_1} \) be a continuous transformation group and let \( f(x) \) be a function of variable \( x \). Let \( \bar{x} \) be defined by

\[
\bar{x} = T_{a_1}x
\]

Now, if \( f(\bar{x}) = f(x) \)

for every transformation \( T_{a_1} \), then \( f(x) \) is called an absolute invariant of the group.

For example, \( f(x) = x^2 + x^{22} \) is an absolute invariant of the one-parameter transformation group defined by
Because, it can be easily to verify that \( \bar{x}^2 + \bar{x}^2 = x^2 + x^2 \) for any value of \( a \) whatsoever. In fact, any arbitrary function of the form \( F(x) = F\left(x^2 + x^2\right) \) is an absolute in variant of the group. Moreover, it might be well to point out that two absolute invariant such as \( f^1(x) = x^2 + x^2 \) and \( f^2(x) = \cos(x^2 + x^2) \) are not functionally independent because it is easily verify that the Jacobian \( \frac{\partial (f^1, f^2)}{\partial (x^2)} = 0 \).

In general, it is shown in general group theory that the \( r \) parameter transformation group defined by

\[
\bar{x}^i = f^i \left(x^1, x^2, ..., x^n, a^1, a^2, ..., a^r\right)
\]

has exactly \((r-1)\) functionally independent absolute invariants (Eisenhart, 1961) \( \xi_j \left(x^1, x^2, ..., x^n\right), (j = 1, 2, 3, ... r - 1) \). That is

\[
\xi_j \left(x^1, x^2, ..., x^n\right) = \xi_j \left(\bar{x}^1, \bar{x}^2, ..., \bar{x}^n\right)
\]  \( (2.1) \)

### 2.6 Partial Differential Equations and Group of Transformations

Applications of group of continuous transformations in the theory of differential equations open the new era in the subject of partial differential equations. The significant contributions to applied mathematics, especially for partial differential equations, are found in the form of the general theory of Morgan (1952) and Michal (1952). In this section we shall be present the basic definitions and theorems of Morgan (1952). This will form the foundation for discussion of the subsequent deductive theory.

Consider the one-parameter groups \( S, G \), and \( E_k \).
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\[
G: \begin{cases} 
S: \{ \bar{x}^i = f^i(x^1, x^2, \ldots, x^m, a), & i = 1, 2, \ldots, m, \quad m \geq 2, \\
\bar{y}_j = f_j(y_j, a), & j = 1, 2, \ldots, n, \quad n \geq 1 
\end{cases}
\]

\[
E_k : \frac{\partial^l \bar{y}_j}{\partial (\bar{x}^1)^l \ldots \partial (\bar{x}^m)^l_m} = f_{(j, l_1, \ldots, l_m)}^{(l)} \left[ \frac{\partial^l y_j}{\partial (x^1)^l} \frac{\partial^l y_j}{\partial (x^1)^{l-1}} \frac{\partial^l (x^2)}{\partial (x^1)^l} \ldots \frac{\partial^l (x^m)}{\partial (x^1)^l} \right], \\
\sum_i l_i = l \leq k,
\]

(2.2)

where the functions \( f \) are continuous in the parameter \( a \). And the value of parameter which yields identity element is denoted by \( a^0 \), thus \( x^i = f^i(x^1, x^2, \ldots, x^m, a^0) \). The value of the parameter \( a \), for the inverse transformation is denoted by \( \bar{a} \), that is if \( \bar{y}_j = f_j(y_j, a) \) then \( y_j = f_j(\bar{y}_j, \bar{a}) \). The transformations \( x^i \to \bar{x}^i \) form a subgroup \( S \) of \( G \) (Cohen, 1911; Eisenhart, 1961). Subsequently the \( x^i \) and \( y_j \) will be identified with the independent and dependent variables, respectively, of a system of partial differential equations.

Let the set of functions \( \{ y_j \} \), \( y_j = y_j(x^1, x^2, \ldots, x^m) \) be differentiable in \( x^i \) up to order \( k \), and append to the transformations of \( G \) the transformations of the partial derivatives of the \( y_j \) with respect to the \( x^i \). That is to say, consider the set of functions \( \{ \bar{y}_j \} \) defined by

\[
\bar{y}_j = f_j \left[ y_j \left( f^1(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m, \bar{a}), f^2(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m, \bar{a}), \ldots, f^m(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m, \bar{a}) \right) ; a \right].
\]

\( E_k \) is a continuous group called the \( k^{th} \) enlargement of \( G \) for \( \{ y_j \} \), where the functions \( f_{(j, l_1, \ldots, l_m)}^{(l)} \) are defined in (2.2).
By elementary group theory (Eisenhart, 1961), $G$ has $(m+n-1)$ functionally independent (when considered as functions of the $(m+n)$ independent variables as $x^i, i = 1, 2, ..., m$; $y_j, i = 1, 2, ..., n$) absolute invariants, designated by

$$
\begin{align*}
&\eta_1(x^1, x^2, ..., x^m), \eta_2(x^1, x^2, ..., x^m), ..., \eta_{m-1}(x^1, x^2, ..., x^m); \\
g_j(y_1, y_2, ..., y_n; x^1, x^2, ..., x^m), & \quad j = 1, 2, ..., n.
\end{align*}
$$

(2.3)

The $g_j$ can be so chosen that the Jacobian $\frac{\partial (g_1, g_2, ..., g_n)}{\partial (y_1, y_2, ..., y_n)} \neq 0$ and the rank of the Jacobian matrix $\frac{\partial (\eta_1, \eta_2, ..., \eta_{m-1})}{\partial (x^1, x^2, ..., x^m)}$ is equal to $m-1$ and only such $g_j$ is considered here.

A differential form of the $k^{th}$ order in $m$ independent variables is a function, usually in class $C^1$ (consists of all differentiable functions whose derivative is continuous; such functions are called continuously differentiable) or greater, is of the type $\Xi(\zeta_1, \zeta_2, ..., \zeta_p)$ whose arguments $\zeta_1, \zeta_2, ..., \zeta_p$ are the variables $x^1, x^2, ..., x^m$, functions $y_1, y_2, ..., y_n$ and the derivative thereof up to the $k$th order. Following Morgan (1952), $\Xi(\zeta_1, \zeta_2, ..., \zeta_p)$ is said to be conformally invariant under the enlarged group $E_k$, if

$$
\Xi(\bar{\zeta}_1, \bar{\zeta}_2, ..., \bar{\zeta}_p) = F(\zeta_1, \zeta_2, ..., \zeta_p; a) \Xi(\zeta_1, \zeta_2, ..., \zeta_p)
$$

(2.4)

for some function $F$. In particular, if $F$ is a function of $a$ only, $\Xi$ is said to be constant conformally invariant (or invariant) under $E_k$ and in the event that $F = 1$, $\Xi$ is said to be absolute invariant under $E_k$.

And a system of partial differential equations
is said to be invariant under $G$ if each of the differential forms $\Phi_1, \Phi_2, \ldots, \Phi_N$ are conformally invariant under $E_k$.

By invariant solutions of a system of partial differential equations is meant that class of solutions of the system, which have the property, that the $y_j$ are exactly the same functions of the $x^i$ as the $\bar{y}_j$ are of the $\bar{x}^i$. The principal results obtained by Morgan (1952) are contained in the following theorems:

**Theorem 2.1**

If the differential $\Phi_\gamma$ are conformally invariant under the group $E_k$ and if $\{I_j\}$ is any set of functions such that, when $y_j = I_j \left(x^1, x^2, \ldots, x^m\right)$ then $\bar{y}_j = I_j \left(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m\right)$, then under a transformation from

$$(y_1, y_2, \ldots, y_n; x^1, x^2, \ldots, x^m)$$

to the functionally independent set

$$(g_1, g_2, \ldots, g_n; \eta_1, \eta_2, \ldots, \eta_{m-1}; \check{x})$$

consisting of one of the set of absolute invariants and one of the $x^i$, there exist differential form $\Omega_\mu$ and $\Lambda_\mu$ such that

$$\Phi_\gamma \left(x^1, \ldots, x^m, I_1, \ldots, I_n, \frac{\partial I_1}{\partial x^1}, \ldots, \frac{\partial^k I_n}{(\partial x^m)^k}\right) = \Lambda_\mu \left(\eta_1, \ldots, \eta_{m-1}, F_1, \ldots, F_n, \frac{\partial F_1}{\partial \eta_1}, \ldots, \frac{\partial^k F_n}{(\partial \eta_{m-1})^k}\right) \times \exp \left[\Omega_\mu \left(\check{x}, \eta_1, \ldots, \frac{\partial^k F_n}{(\partial \eta_{m-1})^k}\right)\right]$$

(2.6)

Where the functions $F_j$ are defined by

$$F_j(\eta_1, \ldots, \eta_{m-1}) = g_j \left(I_1, \ldots, I_n, x^1, \ldots, x^m\right); \ I, \eta \rightarrow (x^1, \ldots, x^m)$$

(2.7)
Theorem 2.2
If each of the differential $\Phi_j$ of the equation (2.5) is conformally invariant, if and only if the set of functions $\{I_j\}$ is a solution of the system of equation (2.5), then from the equation (2.6), with the arguments of $\Lambda_\mu$ given there,
$$\Lambda_\mu = 0, \quad \gamma = 1,2,\ldots,N \leq n.$$ (2.8)

Theorem 2.3
If the set of functions $\{F_j\}$ is any solution of the system of equation (2.8), the functions $\{I_j\}$ given by the inverse transformation of (2.7) is a solution of the equation (2.5). The resultant set $\{I_j\}$ is an invariant solution for each value of group parameter $a$. Conversely, any invariant solution $\{I_j\}$ of equation (2.5) yields a solution $\{F_j\}$ of equation (2.8) upon transforming the variables to a set of functionally independent set of invariants of $G$.

For the purposes of solving the system of equations (2.5), we can summarize the impact of the foregoing theorems as follows: Sufficient conditions for reducing equations (2.5) to equation (2.8), which has one less independent variable, are that equations (2.5) be invariant under a group of the form equation (2.2) and that invariant solutions exist. Solutions to equations (2.8) yield invariant solutions to equations (2.5). Equations (2.8) constitute a similarity representation of the system of differential equations (2.5); the underlying transformation from
$$(y_1,y_2,\ldots,y_n;x^1,x^2,\ldots,x^m)$$
to
$$(g_1,g_2,\ldots,g_n;\eta_1,\eta_2,\ldots,\eta_{m-1})$$
is called a similarity transformation, and the variables
$$(g_1,g_2,\ldots,g_n;\eta_1,\eta_2,\ldots,\eta_{m-1};\tilde{x})$$
are called similarity variables.

Theorems 2.2 and 2.3 consider systems consisting of partial differential equations alone, without regard for auxiliary conditions. The usual manner of application to systems that possess auxiliary conditions is to examine the equations by themselves. If a set of similarity variables is found, we test to see if the auxiliary conditions are also expressible, without inconsistency, in terms of these similarity variables. If so, these variables are termed the similarity variables for the composite
system of equations and auxiliary conditions. The resultant composite system, expressed in terms of these similarity variables, is called a similarity representation of the composite. All too often, however, the set of similarity variables for the equations alone are found to be inappropriate for the auxiliary conditions and the cycle must be repeated.

A second weakness of the classical method is the lack of a systematic procedure for establishing the required set of \((m+n-1)\) functionally independent absolute invariants. The invariants have determined by trial or inspection, which has been possible because of the simple groups used. Furthermore, rather than deducing groups under which the hypotheses of Theorems 2.2 and 2.3 are satisfied, applications of the theorems have been based on particular assumed transformation groups. Some of these studies have been very successful. In the next section, we prepare the foundation for developing a deductive theory, which while complicated, removes the above objections to the classical method.

### 2.7 Proposed Systematic Deductive Group Symmetry Method

In this section, we offer the systematic development of to deduce the classical group theoretic method of Morgan (1952) together with removal of objections of classical group theoretic method. We also discuss the incorporation of auxiliary conditions and systematically derivation of the absolute invariants of the underlying group.

In the general theory of Section 2.6, all possible transformation groups given by equation (2.2) are considering at the outset. Then those under which the system of differential equations does not transform conformally are eliminated from further consideration. That is, restrictions on the functions \(f\) of \(G\) are found to satisfy the conditions of Theorem 2.1 including group properties. There may exist many different groups satisfying all of these restrictions, and each predicts a similarity representation of the problem consisting of the differential equations alone. Generally, not all of these, and perhaps none, will generate a similarity representation of the problem with auxiliary conditions. Since a solution of a similarity representation of the equations alone is invariant under the group, any such solution can yield an invariant solution to the complete problem (equations with auxiliary conditions) only if the auxiliary conditions when transformed by the group can be satisfied by the invariant solution.
Thus, further restrictions on the functions \( f \) can be determined by the requirements that the auxiliary conditions be compatible with invariant solutions.

In Summary, if all the conditions placed on the \( f \)'s to satisfy Theorem 2.1 (Morgan theorem) are met, a similarity representation is predicted for the equations. This can yield a similarity representation for the problem provided the conditions placed on the \( f \)'s by the requirement of auxiliary condition compatibility with invariant solutions are also satisfied.

Hence, in the virtue of limitations of the classical group theoretic analysis and for the incorporation of auxiliary conditions, our deduced transformation group, so-called deductive group, is initiated with the class \( C_G \) of one-parameter continuous transformation of the form

\[
C_G : \begin{cases}
S : & \bar{x}^i = f^i(x^1,\ldots,x^m;a) \\
\bar{y}_j = f_j(y_1,\ldots,y_n;a) = C^{y_j}(a)y_j + K^{y_j}(a), & j = 1,2,\ldots,n. \quad n \geq 1,
\end{cases}
\]

(2.9)

Where \( C^{x^i}(a), K^{x^i}(a), C^{y_j}(a), K^{y_j}(a) \) are real valued continuous functions of parameter ‘\( a \)’ and at least differentiable in argument ‘\( a \)’.

Inclusion of auxiliary conditions and development of the resulting deductive similarity theory was pioneered by Gaggioli and Moran (1966), Moran (1967), Moran and Gaggioli (1967, 1968a, 1968b, 1968). Employing the notation of Section 2.6, their basic result is embodied in the following:

**Theorem 2.4**

Let the auxiliary conditions be

\[
\beta_{\delta}\left[ \frac{\partial^t y_1}{\partial x^1}, \ldots, \frac{\partial^t y_n}{\partial x^m}, y_1, \ldots, y_n, x^1, \ldots, x^m \right] = B_{\delta}\left( \sigma^1, \sigma^2, \ldots, \sigma^t \right)
\]

(2.10)

on \( \Sigma_{\delta} \) defined by

\[
\Sigma_{\delta} : \left\{ x^i = b^i_{\delta}\left( \sigma^1, \sigma^2, \ldots, \sigma^t \right), t \leq m \text{ for } \sigma^q \in \{S^q_{\delta} \} \}
\]

If \( \bar{y} = I_j(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m) \) for all \( a \), is an invariant solution then
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\[ \beta_\delta \left( \frac{\partial^s y_1}{(\partial x^1)^s}, \ldots, \frac{\partial y_n}{(\partial x^m)^n}, y_1, \ldots, y_n, x^1, \ldots, x^m, \bar{a} \right) \]

\[ \equiv \beta_\delta \left[ \left( f_{(1:2:0,0)}^{(x)} \left( \frac{\partial^s y_1}{(\partial x^1)^s}, \ldots, \frac{\partial y_n}{(\partial x^m)^n}, y_1, \ldots, y_n, x^1, \ldots, x^m, \bar{a} \right) \right) \right. \]

\[ \ldots, f_{n} \left( y_n; \bar{a} \right), \ldots, f_{m} \left( x^1, \ldots, x^m \right); \bar{a} \right] \]

\[ = B_\delta \left( \sigma^1, \sigma^2, \ldots, \sigma^t \right) \]

When

\[ x^i = f^i \left( b^1_\delta \left( \sigma^1, \sigma^2, \ldots, \sigma^t \right), \ldots, b^n_\delta \left( \sigma^1, \sigma^2, \ldots, \sigma^t \right); a \right). \]

[An alternate form and proof of above theorem are found in Moran and Gaggioli (1968).]

Thus with equation (2.11), the single auxiliary condition \( \beta_\delta (\ldots) = B_\delta (\ldots) \) on \( \Sigma_\delta \) of equation (2.10) leads to a family of auxiliary conditions \( \left\{ \beta_\delta^a (\ldots) = B_\delta (\ldots) \right\} \) on the family

\[ \Sigma_\delta : x^i = f^i \left( b^1_\delta \left( \sigma^1, \sigma^2, \ldots, \sigma^t \right), \ldots, b^n_\delta \left( \sigma^1, \sigma^2, \ldots, \sigma^t \right); a \right). \]

With requirement that equation (2.11) be satisfied for each auxiliary boundary condition \( \beta_\delta (\ldots) = B_\delta (\ldots) \) on \( \Sigma_\delta \), further restrictions on \( f' \)'s will be imposed. These types of restrictions can be observed according to the similarity requirement which yields the subgroup \( G \) of the class \( C_G \) of equation (2.9).

2.8 Determination of Absolute Invariants

Transformation groups (if any) have now been determined whose \( f' \)'s [Eq. (2.2)] are consistent with the twin requirements of equation invariance and auxiliary condition compatibility. It now remains to establish a set of functionally independent invariants for each group in order to complete the construction of the similarity representation.

Determination of the absolute invariants proceeds in a manner exactly analogous to that of the classical Lie theory (Cohen, 1911; Ames, 1960; Eisenhart, 1961). It will be convenient to use the symbol of a group in our subsequent discussion. In order to facilitate the determination of invariants, a basic theorem from
group theory is invoked. The theorem refers to a symbol \( \Theta_{\Gamma} \) of the one-parameter group of the form \( \{ \xi = f^i(\xi^1, \xi^2, ..., \xi^p; a) \} \), \( i = 1, 2, ..., p \). By definition, the symbol is given by

\[
\Theta_{\Gamma} = \sum_{i=1}^{p} \left[ \frac{\partial f^i}{\partial a} \left( \xi^1, \xi^2, ..., \xi^p; a^0 \right) \right] \frac{\partial}{\partial \xi^i}
\]

and \( a^0 \) is the value of \( a \) generating the identity element of group.

In terms of the symbol, the invariants are determined from the following result:

**Theorem 2.5**

The function \( h(\xi^1, \xi^2, ..., \xi^p) \) is an absolute invariant of the transformation group with symbol \( \Theta_{\Gamma} \) if and only if \( \Theta_{\Gamma} h \equiv 0 \). Furthermore, if \( h_1, h_2, ..., h_{p-1} \) are functionally independent solutions of \( \Theta_{\Gamma} h_j = 0 \), \( j = 1, 2, ..., p-1 \), then any solution of \( \Theta_{\Gamma} h \equiv 0 \) can be expressed as

\[
h(\xi^1, \xi^2, ..., \xi^p) = \Delta(h_1, h_2, ..., h_{p-1})
\]

where \( \Delta \) is a differentiable function.

Now we illustrate the systematic application of deductive group symmetry technique (deductive group theoretic method) by an example.

### 2.9 An Illustration on Deductive Group Symmetry Technique

The example of this section is the classical one of steady two-dimensional laminar incompressible boundary layer flow over an infinite flat plate. Moran and Gaggioli (1968) first treated it in the deductive format. Our main interest herein is to demonstrate how the auxiliary conditions are introduced.

With \( u(x, y), v(x, y), U(x) \) representing velocity components parallel and normal to the plate and the limit of \( u \) as \( y \to \infty \), respectively, the governing equation in terms of stream function \( \psi \), such that \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \), is given by

\[
\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} + U \frac{dU}{dx}; \quad (\nu \text{ constant})
\]

(2.13)
Subject to the auxiliary conditions

\[ \frac{\partial \psi}{\partial y} = 0 \quad \text{if } x > 0 \text{ and } y = 0, \quad (2.14) \]

\[ \frac{\partial \psi}{\partial x} = 0 \quad \text{if } x > 0 \text{ and } y = 0, \quad (2.15) \]

\[ \frac{\partial \psi}{\partial y} \to U(x) \quad \text{if } x > 0 \text{ and } y \to \infty, \quad (2.16) \]

To begin, consider the class \( C_G \) of one-parameter continuous transformation groups of the form

\[
\begin{align*}
S : \{ & \tilde{x} = C^x(a)x + K^x(a), \tilde{y} = C^y(a)y + K^y(a) \\
\psi & = C^\psi(a)\psi + K^\psi(a), \\
U & = C^U(a)U + K^U(a)
\end{align*}
\]  

(2.17)

Each member of \( C_G \) has sets of three functionally independent absolute invariants \( \{ \eta(x,y); g_j(x,y,\psi,U), j = 1, 2 \} \)

First, we search the subgroup of class \( C_G \) that invariantly transforms the partial differential equation (2.13) together with auxiliary conditions.

As per discussion in section (2.6), partial differential equation (2.13) is said to be invariant under the group (2.17) for some function \( A(a) \) whenever:

\[
\begin{align*}
\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial \tilde{x}} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \psi \frac{\partial^3 \psi}{\partial y^3} - U \frac{dU}{dx} \\
&= A(a) \left[ \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial \tilde{x}} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \psi \frac{\partial^3 \psi}{\partial y^3} - U \frac{dU}{dx} \right]
\end{align*}
\]

(2.18)

Substituting the values of partial derivatives from (2.17) using chain rule;

\[
\begin{align*}
\tilde{s}_i & = \left( \frac{C^s_i}{C^i} \right) s_i \\
\tilde{s}_{ij} & = \left( \frac{C^s_i}{C^iC^j} \right) s_{ij}
\end{align*}
\]

\( s = x, y, \psi, U; \quad i, j = x, y \)

where suffixes denote partial derivatives.
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The invariance of equation (2.19), implies that

\[
\frac{(C^\nu)^2}{C^x(C^y)^2} \left[ \frac{\partial C^\nu}{\partial y} \frac{\partial^2 C^\nu}{\partial y \partial x} - \frac{\partial C^\nu}{\partial x} \frac{\partial^2 C^\nu}{\partial y^2} \right] - \nu \frac{C^\nu}{(C^y)^3} \frac{\partial^3 C^\nu}{\partial y^3} - \left[ \frac{C^U U + K^U}{C^x} \frac{dU}{dx} \right] = A(a) \left[ \frac{\partial \nu}{\partial y} \frac{\partial^2 \nu}{\partial y \partial x} - \frac{\partial \nu}{\partial x} \frac{\partial^2 \nu}{\partial y^2} - \nu \frac{\partial^3 \nu}{\partial y^3} - U \frac{dU}{dx} \right]
\]

(2.19)

The invariance of equation (2.19), implies that

\[
\frac{(C^\nu)^2}{C^x(C^y)^2} = \frac{C^U}{(C^y)^3} = \frac{(C^U)^2}{C^x} = A(a); \quad K^U = 0
\]

Also for the absolute invariance of the auxiliary conditions, implies that \( K^y = 0 \)

These shows the restrictions on assumed transformations group, for the invariance of differential equation together with the auxiliary boundary conditions, as we pointed in last session of previous section and one can easily determine such restrictions by inspecting the conditions. These yields,

\[
C^\nu = (C^y)^2, \quad C^x = (C^y)^3, \quad C^U = C^y
\]

Finally, we get the one-parameter group \( G \), which is the subgroup of the class \( C_G \) and transforms invariantly the differential equation (2.13) and the auxiliary conditions (2.14) to (2.16). The subgroup \( G \) is of the form,

\[
G : \begin{cases}
S : \begin{cases}
\bar{x} = (C^y)^3 x + K^x, \quad \bar{y} = C^y y \\
\bar{\nu} = (C^y)^2 \nu + K^\nu, \quad \bar{U} = C^y U
\end{cases}
\end{cases}
\]

(2.20)

Now we shall obtain the absolute invariants of the group \( G \). To determine absolute invariants of groups with the form of equation (2.20) we employ the results of Section 2.8. Three functionally independent absolute invariants of any group satisfying equation (2.20) are required. One of these say \( \eta \), must be an invariant of the subgroup

\[
S : \begin{cases}
\bar{x} = (C^y)^3 x + K^x, \quad \bar{y} = C^y y
\end{cases}
\]

(2.21)

Then by Theorem 2.5, \( \eta(x, y) \) is an absolute invariant if and only if \( \Theta_\xi \eta = 0 \), where \( \Theta_\xi \) is the symbol of the group \( S \). Thus

\[
\Theta_\xi \eta = \left( \alpha_1 x + \beta_1 \right) \frac{\partial \eta}{\partial x} + \left( \alpha_2 y + \beta_2 \right) \frac{\partial \eta}{\partial y} = 0
\]

(2.22)
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Where \( \alpha_1 = \left( \frac{\partial C^x}{\partial a} \right)_{a=a_0}, \alpha_2 = \left( \frac{\partial C^y}{\partial a} \right)_{a=a_0}, \beta_1 = \left( \frac{\partial K^x}{\partial a} \right)_{a=a_0}, \beta_2 = \left( \frac{\partial K^y}{\partial a} \right)_{a=a_0} \)

and \(' a^0'\) denotes the value of \(' a'\) which yields the identity element of the group \( S \).

The \( \alpha' \)'s and \( \beta' \)'s from \( S \) can be derived as \( \alpha_1 = 3\alpha_2, \beta_2 = 0 \)

Substituting the values of \( \alpha' \)'s and \( \beta' \)'s in equation (2.22), we get

\[
3(x + \lambda) \frac{\partial \eta}{\partial x} + y \frac{\partial \eta}{\partial y} = 0, \quad \text{where } \lambda = \frac{\beta_1}{\alpha_1} \tag{2.23}
\]

The general solution of equation (2.23) is given by \( \eta = g \left( \omega \right) \) where \( g \) is an arbitrary function and \( \omega(x, y) \) is non-trivial solution equation (2.23).

Using Lagrange’s method, we get \( \omega(x, y) = y (x + \lambda)^{-1/3} \). So that.

\[
\eta = g \left[ y (x + \lambda)^{-1/3} \right] \tag{2.24}
\]

Next, two additional absolute invariants will be determined for groups of the form \( G \).

If \( \eta = \eta(x, y) \) is the absolute invariant of independent variables then

\[
g_j(x, y, \psi, U) = F_j(\eta), \quad j = 1, 2 \text{ be the absolute invariants of the dependent variables and will satisfy the first-order linear partial differential equation, (Theorem 2.5)}
\]

\[
\Theta_j g_j = \sum_{i=1}^4 (\alpha_i s_i + \beta_i) \frac{\partial g_j}{\partial s_i}; \quad j = 1, 2 \tag{2.25}
\]

where \( \alpha_i = \left( \frac{\partial C^i}{\partial a} \right)_{a=a_0^i}, \beta_i = \left( \frac{\partial K^i}{\partial a} \right)_{a=a_0^i} \).

Hence by definitions of \( \alpha' \)'s and \( \beta' \)'s, equation (2.25) reduces to

\[
3(x + \lambda) \frac{\partial g_j}{\partial x} + y \frac{\partial g_j}{\partial y} + 2(\psi + \lambda') \frac{\partial g_j}{\partial \psi} + U \frac{\partial g_j}{\partial U} = 0, \quad \lambda' = \frac{\beta_1}{\alpha_3} \tag{2.26}
\]

Further, if \( \omega_1, \omega_2, \omega_3 \) are linearly independent solutions of equation (2.26) then the general solution is given by \( g_j = \Pi_j (\omega_1, \omega_2, \omega_3) \), where \( \Pi_j \) is arbitrary function.

The characteristic equation of (2.26) is defined as

\[
\frac{dx}{3(x + \lambda)} = \frac{dy}{y} = \frac{d\psi}{2(\psi + \lambda')} = \frac{dU}{U} \tag{2.27}
\]

By elementary separation of variables, it is easily demonstrated that
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\[
\omega_1 = y(x + \lambda)^{-1/3}, \quad \omega_2 = (x + \lambda)^{-2/3}(\psi + \lambda'), \quad \omega_3 = U(x + \lambda)^{-1/3}
\]

are independent solutions of \( \Theta G_j g = 0 \). Thus for \( j = 1, 2, 3 \)

\[
g_j = \Pi_j \left[ y(x + \lambda)^{-1/3}, (x + \lambda)^{-2/3}(\psi + \lambda'), U(x + \lambda)^{-1/3} \right]
\]

(2.28)

Of these, many choices in equation (2.28) are possible. As a specific we select functionally independent quantities be

\[
\eta = g(x, y) = \omega_1 = y(x + \lambda)^{-1/3}
\]

\[
g_1(x, y, \psi, U) = \omega_2 = (x + \lambda)^{-2/3}(\psi + \lambda')
\]

\[
g_2(x, y, \psi, U) = \omega_3 = U(x + \lambda)^{-1/3}
\]

Now these, together with the reduction from Morgan’s theorem

\[
g_1(x, y, \psi, U) = (x + \lambda)^{-2/3}(\psi + \lambda') = F_1(\eta) \]

\[
g_2(x, y, \psi, U) = U(x + \lambda)^{-1/3} = F_2(\eta)
\]

(2.29)

Permit the differential equation (2.13) to be transformed into ordinary differential equation.

Since \( U = U(x) \), it follows that \( U(x + \lambda)^{-1/3} \) is a function of \( x \) only where as \( F_2(\eta) \) depends on both \( x \) and \( y \), it follows that it must be a constant say \( U_0 \). Consequently,

\[
U(x) = U_0(x + \lambda)^{1/3}
\]

(2.30)

\[
(\psi + \lambda') = (x + \lambda)^{2/3}F(\eta)
\]

where \( \eta = y(x + \lambda)^{-1/3}, \quad F_1(\eta) = F(\eta) \)

(2.31)

Using equation (2.30) and (2.31), the partial derivative of equation will transform to ordinary derivatives as

\[
\frac{\partial \psi}{\partial y} = (x + \lambda)^{1/3}F'(\eta),
\]

\[
\frac{\partial^2 \psi}{\partial y^2} = F''(\eta),
\]

\[
\frac{\partial^3 \psi}{\partial y^3} = (x + \lambda)^{-1/3}F'''(\eta)
\]

\[
\frac{\partial \psi}{\partial x} = \frac{1}{3}(x + \lambda)^{-1/3}\left[2F(\eta) - \eta F'(\eta)\right],
\]

\[
\frac{\partial^2 \psi}{\partial y \partial x} = \frac{1}{3}(x + \lambda)^{-2/3}\left[F'(\eta) - \eta F''(\eta)\right]
\]
Substituting the values of derivatives in equation (2.13), we derived the following ordinary differential equation in one similarity variable $\eta$,

$$3\nu F^m(\eta) - \left[ F'(\eta) \right]^2 + 2F(\eta)F'(\eta) + U_0^2 = 0$$  \hspace{1cm} (2.32)

Together with the boundary conditions,

$$F(0) = 0, \quad F'(0) = 0, \quad F'(\infty) = U_0$$  \hspace{1cm} (2.33)

### 2.10 Similarity Formalism with Multi-Parameter Groups

In many problems of engineering science, a formulism of multi-parameter group is useful. In this section we shall provide the basic theorems for multiparameter deductive group invariance analysis with theorems from Eisenhart (1961) as discussed by Morgan (1952).

Consider the $r$-parameter continuous transformations group of the form

$$G: \left\{ Z^i = f^i(z^1, z^2, ..., z^m; a_1, a_2, ..., a_r), \quad r \leq m, \quad i = 1, 2, ..., m, \quad m \geq 2 \right\}$$  \hspace{1cm} (2.34)

The symbols of group $G$ are the operator $\Theta_\ell$ defined by

$$\Theta_\ell \equiv \sum_{i=1}^{m} \xi^i_{\ell} \left( z^1, z^2, ..., z^m \right) \frac{\partial}{\partial z^i}, \quad \ell = 1, 2, ..., r.$$  \hspace{1cm} (2.35)

The elements of the matrix $\xi = \left[ \xi^i_{\ell} \right]$ are defined by

$$\xi^i_{\ell} = \frac{\partial f^i}{\partial a^\ell} \left( z^1, z^2, ..., z^m; a^0 \right), \quad \ell = 1, 2, ..., r.$$  \hspace{1cm} (2.36)

where $a^0 = \left( a^0_1, a^0_2, ..., a^0_r \right)$, the particular value of parameters that signifies the group identity.

In terms of these definitions, pertinent results are stated by the following theorem.

**Theorem 2.6**

A. A function $F \left( z^1, z^2, ..., z^m \right)$ is an absolute invariant of group $G$, equation (2.34), if and only if

$$\Theta_\ell F = 0$$  \hspace{1cm} (2.37)
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B. The group possesses exactly \((m - r)\) functionally independent absolute invariants, where \(r \leq m\) is the rank of the matrix \(\xi = \left[\xi^j_i\right]\).

C. If \(F_j(z^1, z^2, \ldots, z^m)\) is the set of functionally independent solutions of equation (2.37) and if \(F(z^1, z^2, \ldots, z^m)\) any other solution of equation (2.37), then

\[
F = \Delta(F_1, F_2, \ldots, F_{m-r}), \quad \text{where } \Delta \text{ is a differentiable function.}
\]

Consequently, the deductive group

\[
G:\begin{cases}
S: \bar{x}^i = C^i x^i (a_1, a_2, \ldots, a_r) x^i + K^i x^i (a_1, a_2, \ldots, a_r), i = 1, 2, \ldots, m. \\
\bar{y}_j = C^j y^j (a_1, a_2, \ldots, a_r) y^j + K^j y^j (a_1, a_2, \ldots, a_r), \quad j = 1, 2, \ldots, n.
\end{cases}
\]

(2.38)

possesses \((m + n - r)\) functionally independent absolute invariants. We shall discuss only that system \(G\) such that the subgroup \(S\) possesses \((m - r)\) functionally independent absolute invariants denoted by

\[
\eta_j(x^1, x^2, \ldots, x^m) = \eta_j(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m), \quad j = 1, 2, \ldots, m - r
\]

In addition, additionally there are \(n\) absolute invariants

\[
g_k(y_1, \ldots, y_n, x^1, \ldots, x^m) = g_k(\bar{y}_1, \ldots, \bar{y}_n, \bar{x}^1, \ldots, \bar{x}^m), \quad k = 1, 2, \ldots, n
\]

are so selected that the rank of the Jacobian \(\frac{\partial (g_1, g_2, \ldots, g_n)}{\partial (y_1, y_2, \ldots, y_n)}\) is \(n\).

At present, we conclude this section with the note that, the multi-parameter group method is very powerful technique for reducing more than one variable from the system of partial differential equations, in one attempt. In the subsequent chapter, the applications of all theories presented in this chapter, will be discussed in detail along with all possible conditions for the existence of similarity solutions for the system of non-linear partial differential equations governing various flow geometries.
2.11 MATLAB Coding for Numerical Solutions

In this section, we shall discuss about the application of MATLAB coding, that we are going to use in order to obtain the numerical solutions of the derived similarity ordinary equations. The derived similarity equations, by applying the deductive group symmetry technique are highly non-linear, two point boundary value problems. In the present work, we are using the bvp4c solver in MATLAB.

* The description of bvp4c is given below:

**bvp4c**  Solve boundary value problems for ordinary differential equations

**Syntax**

\[
\text{sol} = \text{bvp4c(odefun,bcfun,solinit)}
\]

\[
\text{sol} = \text{bvp4c(odefun,bcfun,solinit,options)}
\]

\[
\text{solinit} = \text{bvpinit(x, yinit, params)}
\]

* http://www.mathworks.in/help/techdoc/ref/bvp4c.html
### Arguments:

| Odefun       | A function handle that evaluates the differential equations \( f(x,y) \). It can have the form  
|             | \[
|             | \quad dydx = odefun(x,y)  
|             | \[
|             | \quad dydx = odefun(x,y,parameters)  
|             | where \( x \) is a scalar corresponding to \( x \), and \( y \) is a column vector corresponding to \( y \). parameters is a vector of unknown parameters. The output \( dydx \) is a column vector. |
| Bcfun       | A function handle that computes the residual in the boundary conditions. For two-point boundary value conditions of the form \( bc(y(a),y(b)) \), bcfun can have the form  
|             | \[
|             | \quad res = bcfun(ya,yb)  
|             | \[
|             | \quad res = bcfun(ya,yb,parameters)  
|             | where \( ya \) and \( yb \) are column vectors corresponding to \( y(a) \) and \( y(b) \). parameters is a vector of unknown parameters. The output \( res \) is a column vector. |
| Solinit     | A structure containing the initial guess for a solution. We create \( solinit \) using the function \( bvpinit \). \( solinit \) has the following fields.  
| X           | Ordered nodes of the initial mesh. Boundary conditions are imposed at \( a = solinit.x(1) \) and \( b = solinit.x(end) \).  
| Y           | Initial guess for the solution such that \( solinit.y(:,i) \) is a guess for the solution at the node \( solinit.x(i) \).  
| Parameters  | Optional. A vector that provides an initial guess for unknown parameters.  
|             | The structure can have any name, but the fields must be named \( x \), \( y \), and \( \text{parameters} \). We can form \( solinit \) with the helper function \( bvpinit \).  

Description

`sol = bvp4c(odefun,bcfun,solinit)` integrates a system of ordinary differential equations of the form

\[ y' = f(x, y) \]

on the interval \([a, b]\) subject to two-point boundary value conditions \(bc(y(a), y(b)) = 0\).

`odefun` and `bcfun` are function handles.

`bvp4c` produces a solution that is continuous on \([a, b]\) and has a continuous first derivative there. Use the function `deval` and the output `sol` of `bvp4c` to evaluate the solution at specific points `xint` in the interval \([a, b]\).

\[ sxint = deval(sol,xint) \]

The structure `sol` returned by `bvp4c` has the following fields:

- `sol.x` Mesh selected by `bvp4c`
- `sol.y` Approximation to \(y(x)\) at the mesh points of `sol.x`
- `sol.yp` Approximation to \(y'(x)\) at the mesh points of `sol.x`
- `sol.parameters` Values returned by `bvp4c` for the unknown parameters, if any
- `sol.solver` 'bvp4c'

The structure `sol` can have any name, and `bvp4c` creates the fields `x`, `y`, `yp`, `parameters`, and `solver`.

`sol = bvp4c(odefun,bcfun,solinit,options)` solves as above with default integration properties replaced by the values in `options`, a structure created with the `bvpset` function.

Example

Boundary value problems can have multiple solutions and one purpose of the initial guess is to indicate which solution you want. The second-order differential equation
$y'' + |y| = 0$

has exactly two solutions that satisfy the boundary conditions $y(0) = 0, y(4) = -2$. 

Prior to solving this problem with \texttt{bvp4c}, we must write the differential equation as a system of two first-order ODEs

\begin{align*}
y_1' &= y_2 \\
y_2' &= -|y_1|.
\end{align*}

Here $y_1 = y$ and $y_2 = y'$. This system has the required form

\begin{align*}
y' &= f(x, y) \\
bc(y(a), y(b)) &= 0
\end{align*}

The function $f$ and the boundary conditions $bc$ are coded in MATLAB software as functions \texttt{twoode} and \texttt{twobc}.

\begin{verbatim}
function dydx = twoode(x,y)
    dydx = [ y(2) \\
             -abs(y(1))];
end

function res = twobc(ya,yb)
    res = [ ya(1) \\
            yb(1) + 2];
end
\end{verbatim}

Form a guess structure consisting of an initial mesh of five equally spaced points in $[0, 4]$ and a guess of constant values

\begin{align*}
y_1(x) &\equiv 0 \\
y_2(x) &\equiv 0
\end{align*}

with the command
solinit = bvpinit(linspace(0,4,5),[1 0]);

Now solve the problem with

sol = bvp4c(@twoode,@twobc,solinit);

Evaluate the numerical solution at 100 equally spaced points and plot $y(x)$ with

\[
x = \text{linspace}(0,4);
\]

\[
y = \text{deval}(sol,x);
\]

\[
\text{plot}(x,y(1,:));
\]

We can obtain the other solution of this problem with the initial guess

solinit = bvpinit(linspace(0,4,5),[-1 0]);