CHAPTER 3

Finite Pulses and Correlations between Different Pulses

As discussed in chapter 2, a theory of Reactor Noise in ADS assuming a general non-Poisson periodically pulsed source of neutrons was constructed by Degweker (2003). In this chapter, we generalize the non-Poisson character of the source to include the possibility of correlations between pulses and derive formulae for Rossi alpha and Feynman alpha. We also take up the case of pulses of finite widths by considering rectangular and Gaussian pulse shapes. We present numerical results based on the derived formulae to illustrate the importance of correlations between pulses in typical experimental conditions. The effect of finite width of source pulses is also illustrated.

Pazsit et al. (2005) have also considered finite width pulses in the context of reactor noise in ADS but for Poisson sources. The formulation in the present chapter is different in that the source is assumed to be non-Poisson with exponential correlation between pulses. In section 3.2, we consider the case of correlated non-Poisson delta function source pulses. The case of finite width pulses is discussed in section 3.3. Our analysis is restricted to the experimental situation in which the counting interval is opened at a time point that is essentially random since this has been shown to be experimentally better for extracting parameters of interest (Pazsit et al., 2005). We do not consider the effect of delayed neutrons. Hence, the formulae are valid only for time scales which are short compared to the delayed neutron precursor lifetimes and all quantities (such as $k$, $\nu$, etc.) are to be regarded as prompt.
3.1 The doubly stochastic Poisson point process

If \( I(t) \) is the ion current, the probability of a neutron producing event (spallation, D-D or D-T reactions) in a short time \( dt \) can be written as

\[
I(t)E(t)dt
\]

where, \( E(t) \) is the deterministic variation of the current (periodically occurring pulses). Since this is the probability at any time and independent of any occurrences at other times, we can write for the probability of obtaining \( s \) source events at times \( t_1, t_2, \ldots, t_s \) as follows:

\[
Q_s (t_1, t_2, \ldots, t_s) = E(t_1)I(t_1) \cdots E(t_s)I(t_s) \exp\left[- \int_{-\infty}^{\infty} E(t)I(t)dt\right]
\]

(3.1)

As mentioned before, we assume \( I(t) \) as an exponentially correlated Gaussian stochastic process. Since, for averaging various functions we have to perform a double averaging; one over the variables \( s \) and \( t \) and the second over the stochastic variable \( I(t) \), we have a doubly stochastic Poisson point process (Saleh, 1978). We shall use such a description in a later section. The explicit form for the functions \( Q_s \) (Van Kampen, 1983) given above will not be required in the subsequent discussions as we shall see that it is only the averages of products of the beam current at various times that will be required.

3.2 Correlated Gaussian pulsed source

We assume that the pulses are short compared to all other time scales in the problem and may be represented as a sum of delta functions. Moreover, the neutron source pulses have an exponential correlation in intensity.
3.2.1 Rossi alpha formula

We assume that the first count occurs at $t = 0$ and the second one at $t = \tau$. The occurrence of the last source pulse at $t_0$ is randomly distributed between $-1/f$ and $0$. Following Degweker (2003), the expression for the Rossi alpha can be written as follows:

$$f_2(0, \tau) = f^2 \beta^2 \int_{-1/f}^{0} \left[ \frac{\partial^2 F_Q(x, t_0)}{\partial x^2} \frac{\partial G_1(x, \tau)}{\partial x} + \frac{\partial F_Q(x, t_0)}{\partial x} \frac{\partial F_R(x, \tau, t_0)}{\partial x} \right] dt_0$$  \hspace{1cm} (3.2)

where $F_Q$ and $F_R$ are the pgfs of the distributions $P$ and $R$ defined by Degweker (2003) while $G_1$ is the pgf of the distribution $P$ for $n = 1$ and by the independence of neutrons, clearly $G_1^n(x, \tau)$ is the corresponding pgf for arbitrary $n$. We write expressions for $F_Q$ and $F_R$ taking into account the fact that there are correlations between pulses, as follows:

$$F_Q(x, t_0) = \sum_{N_0, N_1, \ldots, N_n} P(N_0, N_1, \ldots, N_n) \prod_{n=0}^{\infty} \left( G_1(x, -t_0 + \frac{n}{f}) \right)^{N_n}$$  \hspace{1cm} (3.3)

$$F_R(x, t_0) = \sum_{N_1, N_2, \ldots, N_{[f/t]}} P(N_1, N_2, \ldots, N_{[f/t]}) \prod_{n=0}^{[f/t]} \left( G_1(x, \tau - t_0 - \frac{n}{f}) \right)^{N_n}$$  \hspace{1cm} (3.4)

where $P(N_0, N_1, \ldots,)$ is the joint probability distribution of the source leading to production of $N_0, N_1, \ldots$ etc. neutrons at the corresponding times. We have used the standard notation $[x]$ to denote the largest integer less than or equal to $x$. Using these equations, it is possible to write down the derivatives required in Eq. (3.2) in terms of the derivatives of $G_1$ evaluated at $x=1$.

The pgf $G_1$ is given by

$$G_1(x, t) = 1 - \frac{e^{-\alpha t}(1-x)}{1 + Y_t(1-e^{-\alpha t})(1-x)}$$  \hspace{1cm} (3.5)

where $Y_t = \lambda_f \nu(\nu-1)/2\alpha$

Using Eq. (3.5), we obtain the following expressions for the first two derivatives of $G_1$:
\[ G_1'(1,t) = e^{\alpha t} \] (3.5a)

\[ G_1''(1,t) = 2Y_1(1- e^{-\alpha t})e^{-\alpha t} \] (3.5b)

The number of neutrons in a source event is due to the compounding of the number of protons in a bunch and the number of spallation neutrons. The compounded pgf can be written as

\[ F_{N_jN_j}(x,y) = F_{p_jp_j}[f_{sp}(x), f_{sp}(y)] \] (3.6)

Differentiating with respect to \( x \) and \( y \) and setting \( x=y=1 \), we get

\[ \overline{N_i} = \langle \overline{N_{pi}} \rangle > = m_i \] (3.6a)

\[ \overline{N_iN_j} = \langle N_{pi}N_{pj} > = m_i^2 + \Gamma^{ij}\exp(-\beta |i-j|/f) \] (3.6b)

\[ \overline{N_i(N_i-1)} = \overline{N_j(N_j-1)} = \langle N_{pi}N_{pj} > + \overline{N_{pi}}\overline{N_{sp}}(\overline{N_{sp}}-1) = m_2 \] (3.6c)

where \( \Gamma^{ij} \) is the variance of the number of neutrons produced in a pulse and \( \beta \) is the decay constant of the source correlations. We obtain,

\[ f_2(0,\tau) = f_2 \lambda_2^2 \int_{-1/f}^{0} \sum_{k=0}^{\infty} \left[ m_2 \exp(2\alpha(t_0-\frac{k}{f})) + 2m_1Y_1 \left(1-\exp(\alpha(t_0-\frac{k}{f}))\right)\exp(t_0-\frac{k}{f})\right] e^{-\alpha \tau} dt_0 \]

\[ + f_2 \lambda_2^2 \int_{-1/f}^{0} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left\{ \left( m_1^2 + \Gamma^{ij}\exp(-\beta |l-k|/f)\exp(\alpha(t_0-\frac{k}{f})\exp(\alpha(t_0-\frac{l}{f})\right)\right\} e^{-\alpha \tau} dt_0 \]

\[ + f_2 \lambda_2^2 \int_{-1/f}^{0} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left\{ \left( m_1^2 + \Gamma^{ij}\exp(-\beta |l+k|/f)\exp(\alpha(t_0-\frac{k}{f})\exp(-\alpha(\tau-t_0-\frac{l}{f})\right)\right\} dt_0 \]

\[ + f_2 \lambda_2^2 \int_{-1/f}^{0} \sum_{k=0}^{\infty} \left\{ \left( m_1^2 + \Gamma^{ij}\exp(-\beta |f\tau+k+1|/f)\exp(\alpha(t_0-\frac{k}{f})\exp(-\alpha(\tau-t_0-\frac{1+[f\tau]}{f})\right)\right\} dt_0 \]

Integration and summation are again straight forward and we finally obtain,
The presence of the last term is due to correlations in the source fluctuations. For \( \beta \gg f \) i.e., for correlation times which are short compared to the time between successive neutron pulses, this term vanishes and the formula reduces to that derived by Degweker (2003). When the frequency \( f \) is large compared to alpha and beta, the uncorrelated term tends to its usual form for uncorrelated sources while the correlated term remains the same. Thus, for large frequencies we have,

\[
f_2(0, \tau) = \frac{f^2 \lambda_d^2}{\alpha^2} + \frac{f \lambda_d^2}{2\alpha} \left[ \left( m_2 - m_1^2 + 2m_1 Y_1 \right) \exp\left(-\alpha \tau\right) + \frac{2\beta f \Gamma^2}{\alpha^2 - \beta^2} \exp\left(-\beta \tau\right) \right]
\]

(3.9)

This shows that even for large frequencies, the above distribution does not reduce to the random source distribution.

### 3.2.2 The Variance to Mean Ratio

The expression for the variance to mean ratio is derived as usual by integration of the expression (3.8) as follows:

\[
\frac{\nu}{m} = 1 - m + \frac{2}{m_0} \int_0^\tau (T - \tau) f'_2(0, \tau) d\tau
\]

(3.10)

where \( m \) is the mean given by \( f_1(t)T \); \( f'_1(t) = m_1 \lambda_d f / \alpha \) being the mean count rate.

We obtain,
\[ \frac{\nu}{m} = 1 + \frac{\lambda_d m_1}{\alpha^2 T (1 - e^{-\alpha/T})} \left[ \exp \left( \alpha(T - \frac{[fT]}{f} + 1) \right) + \exp \left( -\alpha(T - \frac{[fT]}{f}) \right) + \exp \left( -\alpha(T + \frac{1}{f}) \right) - 2e^{-\alpha/f} - e^{-\alpha T} \right] \]

\[ -\frac{\lambda_d m_1}{\alpha T} \left[ fT - 2[fT] + \frac{[fT]([fT] + 1)}{fT} \right] + \frac{\lambda_d m_1}{m_1 \alpha} \left( m_2 + 2m_1 \gamma \right) + \Gamma^2 \left[ \frac{e^{-(\alpha + \beta)/f}}{1 - e^{-(\alpha + \beta)/f}} - \frac{1}{1 - e^{-(\alpha + \beta)/f}} \right] \]

\[ \left( 1 - \frac{1 - \exp(-\alpha T)}{\alpha T} \right) + \frac{\lambda_d \Gamma^2}{m_1 T (1 - e^{-(\alpha + \beta)/f})} \left[ \frac{(\alpha T - 1)(1 - e^{-\alpha T})}{\alpha^2} + \frac{e^{-\alpha T}}{\alpha f} - \frac{e^{-\beta [fT]/f}}{1 - e^{-\beta [fT]/f}} - \frac{1 - e^{-\alpha T}}{\alpha f} \right] \]

\[ + \frac{\lambda_d \Gamma^2}{m_1 T (1 - e^{-(\alpha + \beta)/f})} \left[ \frac{(e^{\alpha T} - 1)(1 - e^{\alpha T})}{\alpha^2} - \frac{e^{\alpha T}/f}{1 - e^{-\beta [fT]/f}} - \frac{e^{\alpha T}/f - 1}{\alpha f} \right] \]

\[ + \frac{\lambda_d \Gamma^2}{\alpha^2 m_1 T} \left[ \frac{\alpha (T - [fT]/f) - (1 - e^{-\alpha (T - [fT]/f)})}{(1 - e^{-(\alpha + \beta)/f})} + \frac{e^{-(\alpha + \beta)/f} \left[ e^{\alpha (T - [fT]/f)} - 1 - \alpha (T - [fT]/f) \right]}{(1 - e^{-(\alpha + \beta)/f})} \right] e^{-\beta [fT]/f} \]

(3.11)

For high frequencies, the above formula reduces to

\[ \frac{\nu}{m} = 1 + \frac{\lambda_d (m_2 - m_1^2 + 2m_1 \gamma) + 2\beta f \Gamma^2}{m_1 \alpha} \left( 1 - \frac{1 - \exp(-\alpha T)}{\alpha T} \right) + \frac{2 \lambda_d \alpha f \Gamma^2}{(\alpha^2 - \beta^2) m_1 \beta} \left( 1 - \frac{1 - \exp(-\beta T)}{\beta T} \right) \]

(3.12)

The reduction of the variance due to the regularity of the pulses should be noted. The other point worth noting here is the appearance of an enhancement of the variance due to correlations in the number of source neutrons from successive pulses.

### 3.2.3 The ACF and PSD

We visualize the times of absorption of neutrons in a detector as a stochastic point process (Van Kampen, 1983) described by the functions \( Q_s(t_1, t_2, \ldots, t_s) \). Suppose absorption of a neutron in the detector produces a total charge \( q \) which may be a random number and results in a time response given by \( h(\tau) \) (the response function of the detector operating in the...
The mean current and the ACF are then given by

\[
\overline{i(t)} = \sum_{s=0}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} \langle [q_s h(t-t_1) + \ldots + q_s h(t-t_s)] > Q_s(t_1, \ldots, t_s) dt_1 \ldots dt_s = \langle q \rangle f(t) \tag{3.13}
\]

\[
\overline{i(t)(t')} = \sum_{s=0}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} \langle [q_s h(t-t_1) + \ldots + q_s h(t-t_s)][q_s h(t'-t_1) + \ldots + q_s h(t'-t_s)] > Q_s(t_1, \ldots, t_s) dt_1 \ldots dt_s
\]

\[
= \langle q^2 \rangle f_1 h_c'(t'-t) + \langle q \rangle^2 \int_{-\infty}^{\infty} h_c'(t'-t-\tau)f_2(\tau)d\tau \tag{3.14}
\]

where the bar indicates overall averaging while \(< >\) indicates averaging over the distribution of the number of charges per neutron detection and

\[
h_c(t-t') = \int_{-\infty}^{\infty} h(t-t_1)h(t'-t_1)dt_1 = \int_{0}^{\infty} h(T)h(t'-t+T)dT \tag{3.15}
\]

In writing Eqs. (3.13) and (3.14), we have used the fact that \(f_1(t_1)\) is independent of \(t_1\) while \(f_2(t_2-t_1)\) depends only on \(\tau=t_2-t_1\) due to stationarity of the process. If we use the expression for \(f_2\) [Eq. (3.8)] in Eq. (3.14) and the expression for \(f_1\) (see section 3.2.2 above) in Eq. (3.13), we can write the following formula for the auto covariance:

\[
\frac{(\overline{i(t)} - \overline{i(t)})(\overline{i(t)} - \overline{i(t)})}{\overline{i(t)}(\overline{i(t)} - \overline{i(t)})} = \frac{\langle q^2 \rangle f m_\lambda d h_c(t'-t) - \langle q^2 \rangle f^2 \lambda_d^2 m_i^2}{\overline{h_c(t'-t)} - \langle q^2 \rangle f^2 \lambda_d^2 m_i^2} \tag{3.16}
\]

\[
+ \frac{< q^2 > \Gamma^2 f \lambda_d^2}{2\alpha} \int_{-\infty}^{\infty} h_c(t'-t-\tau) \left[ e^{a\tau + \frac{\tau}{f}} + e^{-a(\tau-\frac{f}{a} \tau)} \frac{1}{1-e^{-(a+\beta)f}} \right] d\tau
\]

\[
+ \frac{< q^2 > \Gamma^2 f \lambda_d^2}{2\alpha} \left[ (m_z-m_i^2 + 2m_i Y_1) + \Gamma^2 \left( \frac{e^{-a(\alpha+\beta)(t'-t)}}{1-e^{-(a+\beta)(t'-t)}} - \frac{1}{1-e^{-(a+\beta)(t'-t)}} \right) \right] \int_{-\infty}^{\infty} h_c(t'-t-\tau)e^{-a\tau} d\tau
\]
Taking the Fourier transform of the above equation it is clear that the PSD can be written as follows:

\[
G(\omega) = H(\omega) \left[ \frac{\langle q^2 \rangle f m_i \lambda_d}{\alpha} + \langle q \rangle^2 \varphi(\omega) \right]
\]  

(3.17)

where \( H(\omega) \) is the PSD of the detector response and \( \varphi(\omega) \) is the Fourier transform of \( f_z(0, \tau) - i^2 \). Before taking the Fourier transform of Eq. (3.8) we note that the two terms in the second line of Eq. (3.16) represent a periodic and even function of \( \tau \) having period \( 1/f \) and can therefore be expanded into a Fourier cosine series.

\[
e^{-\alpha/f} \exp \left( \alpha \left( \tau - \frac{[f \tau]}{f} \right) \right) + \exp \left( -\alpha \left( \tau - \frac{[f \tau]}{f} \right) \right) = \left( 1 - e^{-\alpha/f} \right) \left( \frac{2f}{\alpha} \sum_{n=1}^{\infty} \frac{4\alpha f}{\alpha^2 + (2n\pi f)^2} \cos 2n\pi f \tau \right)
\]

(3.18)

The constant term cancels with \(-i^2 \). The other terms give a discrete spectrum i.e. a sum of delta functions at frequencies \( \pm 2n\pi f \) whose strength is given by

\[
\frac{4\pi f^2 m_i^2 \lambda_d^2}{\alpha^2 + (2n\pi f)^2}
\]

(3.19)

Fourier transformation of the other terms is as usual and finally we get the following expression for the PSD

\[
G(\omega) = H(\omega) \left[ \frac{\langle q^2 \rangle f m_i \lambda_d}{\alpha} + \langle q \rangle^2 f \lambda_d^2 \right. \\
\left. + \sum_{n=0}^{\infty} \frac{4\pi f^2 m_i^2 \delta(\omega - 2n\pi f)}{\alpha^2 + (2n\pi f)^2} \left( m_i - m_i^2 + 2m_i Y_i \right) + \Gamma \omega^2 \left( \frac{e^{-(\alpha + \beta)/f}}{1 - e^{-(\alpha + \beta)/f}} - \frac{1}{1 - e^{-(\alpha - \beta)/f}} \right) \right]
\]

(3.20)

The first term is the usual detector white noise. The second term is a series of discrete lines due to the periodic nature of the uncorrelated terms in the Rossi alpha and autocorrelation
function and is caused by the periodic source pulses. Such a term is not present in the usual PSD method. The third term is the usual response of the reactor to a white noise source and is usually sought to determine alpha. The fourth and fifth terms have the same functional form as the third but are obtained due to the non-Poisson character of the source. The last term is also due to the non-Poisson character of the source and shows the effect of correlations between different source pulses. The magnitude of various terms in the above expression follows the same pattern as that of the Rossi alpha function.

Similarly, expressions for cross correlation function and cross power spectral density can be derived (Degweker and Rana, 2007).

3.3 Effect of finite spread of source pulse

3.3.1 The Rossi alpha formula

Degweker (2003) has shown that the pulse widths of a source bunch are very short for typical RF proton accelerators producing spallation neutrons to be of any consequence in the noise characteristics of interest. Nevertheless, there could be situations where this might be important. One case is that of an experimental deuteron beam source such as the MUSE facility where the width is significant compared to the die away time. For such situations, Pazsit et al. (2005) have considered Gaussian and rectangular pulse shapes and used the Laplace transform approach to carry out the rather complicated mathematics. For the case of Poisson processes, one can use the Bartlett formula with a source intensity which is time varying in a periodically pulsed fashion. For non Poisson source events, it is not immediately clear how to generalize our approach for delta function pulses to the case of finite pulse widths. We treat the neutron source (spallation) events as a doubly stochastic Poisson point process and derive an expression for $f_2$. As regards detailed calculations for obtaining $f_2$, and
for Gaussian and rectangular pulses, we use a variant of the approach taken by Pazsit et al. (2005).

The pgf for getting one count in a small interval $d\tau$, at time 0 and another $d\tau$ at time $\tau$ due to one neutron at time $t$ in a source free medium is denoted by $G(z_1, z_2, t)$. Then the required pgf is given by

$$\mathcal{Z} = f \int dt_0 \left( \sum_{s} \frac{1}{s!} \int_{-\infty}^{\infty} Q_s(t_1, \ldots, t_s) f_{sp}[G(z_1, z_2, t_1)] \ldots f_{sp}[G(z_1, z_2, t_s)] dt_1 \ldots dt_s \right)$$

(3.21)

The required Rossi alpha function can now be written by differentiating successively with respect to $z_1$ and $z_2$

$$f_1 d\tau_0 = \frac{\partial \mathcal{Z}}{\partial z_1} = f \int dt_0 \left( \sum_{s} \frac{1}{s!} \int_{-\infty}^{\infty} dt_1 \ldots dt_s Q_s(t_1, \ldots, t_s) \sum_{i=1}^{s} V_{sp} \frac{\partial G(z_1, z_2, t_i)}{\partial z_1} \mid_{z_1=1} \right)$$

(3.22)

$$f_2 d\tau_0 d\tau = \frac{\partial^2 \mathcal{Z}}{\partial z_1 \partial z_2} = f \int dt_0 \left\{ \sum_{s} \frac{1}{s!} \int_{-\infty}^{\infty} dt_1 \ldots dt_s Q_s(t_1, \ldots, t_s) \left[ \sum_{i=1}^{s} \sum_{j=1}^{s} V_{sp} \frac{\partial^2 G(z_1, z_2, t_i)}{\partial z_1^2} \mid_{z_1=1} + \sum_{i=1}^{s} V_{sp} \frac{\partial^2 G(z_1, z_2, t_i)}{\partial z_1 \partial z_2} \mid_{z_1=1} \right] \right\}$$

(3.23)

Since $Q$ is symmetric in the interchange of any of its arguments, we can write the above functions as follows:

$$f_1 d\tau_0 = \frac{\partial \mathcal{Z}}{\partial z_1} = v_{sp} f \int_{-\infty}^{\infty} \phi_1(t) dt \frac{\partial G(z_1, z_2, t)}{\partial z_1} \mid_{z_1=1}$$

(3.24)

$$f_2 d\tau_0 d\tau = \frac{\partial^2 \mathcal{Z}}{\partial z_1 \partial z_2} = f \int_{-\infty}^{\infty} \phi_2(t_1, t_2) v_{sp}^2 \frac{\partial^2 G(z_1, z_2, t_i)}{\partial z_1^2} \mid_{z_1=1} + v_{sp} (v_{sp} - 1) \frac{\partial G(z_1, z_2, t_i)}{\partial z_1} \mid_{z_1=1} \frac{\partial G(z_1, z_2, t_i)}{\partial z_2} \mid_{z_2=1} dt_1 dt_2$$

(3.25)

where $\phi_1$ and $\phi_2$ are the average source event rate and the two point source event density respectively given as usual by.
\[
\varphi_1(t_1) = \left( \sum_s \frac{1}{(s-1)!} \int_{-\infty}^{\infty} dt_2 \ldots dt_s Q_s(t_1, \ldots, t_s) \right) (3.26)
\]

\[
\varphi_2(t_1, t_2) = \left( \sum_s \frac{1}{(s-2)!} \int_{-\infty}^{\infty} dt_3 \ldots dt_s Q_s(t_1, \ldots, t_s) \right) (3.27)
\]

It is fairly easy to show that the first and second derivatives of G are given by (Kitamura et al., 2006)

\[
G_{z_i}(1,1,t) = \lambda d e^{\alpha t} d \tau_0, \text{ for } t < 0 \text{ and zero otherwise} \quad (3.27a)
\]

\[
G_{z_z}(1,1,t) = \lambda d e^{\alpha(t-\tau)} d \tau, \text{ for } t < \tau \text{ and zero otherwise} \quad (3.27b)
\]

\[
G_{z;z_i}(1,1,t) = 2Y \lambda d^2 e^{\alpha t} (1-e^{\alpha \tau}) e^{-\alpha t} d \tau_0 d \tau \text{ for } t < 0 \text{ and zero otherwise.} \quad (3.27c)
\]

Introducing (3.1) in (3.26) and (3.27) we can write

\[
\varphi_1(t) = E(t) < I(t) > = \sum_n \varepsilon(t - t_0 + n/ f) < I > (3.28)
\]

\[
\varphi_2(t_1, t_2) = E(t_1)E(t_2) < I(t_1)I(t_2) >
\]

\[
= \sum_n \sum_m \varepsilon(t_1 - t_0 + n/ f) \varepsilon(t_2 - t_0 + m/ f) < I(t_1)I(t_2) > (3.29)
\]

where we have written the (deterministic) variation of the current E(t) as a periodic sum of narrow pulse shape functions \( \varepsilon(t - t_n) \) around \( t_n \). The latter represents the pulse shape (obtained by chopping or bunching of the ion current). With the assumption of an exponentially correlated process for the current fluctuations, we can rewrite the above expressions for \( f_1 \) and \( f_2 \) as follows:

\[
f_1 = < I > \int_{-1/ f}^{0} dt_0 \sum_n \varepsilon(t - t_0 + n/ f) e^{\alpha t} dt (3.30)
\]
\[ f_2 = \frac{v_{sp}^2}{\lambda_0^2} \int_{-1/f}^{1/f} dt_0 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varepsilon(t_1 - t_0 + n/f) \varepsilon(t_2 - t_0 + m/f) \left( <I^2> + \Gamma^2 e^{-\beta n t_i} e^{\alpha t_i} e^{\alpha (t_2 - t_1)} dt_i dt_2 \right) \]  

\[ + <I^2> \lambda_0^2 \int_{-1/f}^{1/f} dt_0 \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \varepsilon(t - t_0 + n/f) \left( 2Y v_{sp} e^{\alpha t} (1 - e^{\alpha t}) e^{-\alpha t} + v_{sp} (v_{sp} - 1) e^{\alpha (t_2 - t_1)} \right) dt \] 

(3.31)

where, \( \Gamma^2 = <I^2> - <I^2> \).

The main difference between non-Poisson bunches and Poisson bunches had been noted by Degweker (2003) and was due to the absence of the \( n = m \) term in the first line of the second of the above two equations whereas here we seem to be summing over all values of \( n \) and \( m \).

To understand this effect, we look at the term representing the correlations between fluctuations in current. If \( 1/\beta \) is small compared to time differences within a bunch, we have perfectly uncorrelated (i.e. Poisson) statistics of the protons and as such we get the same results as those by Pazsit et al. (2005). However, for the opposite case, there is an additional contribution to the variance. If we further assume that \( 1/\beta \) is small compared to \( 1/f \), then the term is zero for all \( m \neq n \) but is non zero and equal to \( \Gamma^2 \) for \( m = n \). This corresponds to the situation wherein there is no correlation between successive pulses and was considered by Degweker (2003) assuming delta function pulses. Finally, we may have the situation in which \( 1/\beta \) is not small compared to \( 1/f \).

Assuming a finite spread in the protons within a bunch and that the bunches are non-Poisson, we evaluate the above integrals for the case of correlation time \( 1/\beta \) being much larger than the pulse width but much smaller than \( 1/f \). Evaluations for the other two cases (correlation time much smaller than the pulse width and correlation time much larger than the pulse width) have also been carried out by us (Degweker and Rana, 2007).
Rectangular pulses

The rectangular pulses are defined by

\[ \varepsilon(t) = \frac{g}{\sigma} \text{ for } 0 < t < \sigma ; \quad \varepsilon(t) = 0 \text{ otherwise} \]

We assume that the square pulse triggers at time \( t_0 \) and has a width of \( \sigma \). The one time probability or the count rate is simply given by

\[
f_1 = \langle I > g \lambda_0 \nu_p f \left[ \int_{-1/f}^{0} \sum_{n=1}^{\infty} \int_{t_n - n/f}^{t_n + n/f + \sigma} e^{\sigma dt_0} + \int_{-\sigma}^{-1/f} \int_{t_n}^{t_n + \sigma} e^{\sigma dt_0} + \int_{-\sigma}^{0} \int_{0}^{t_n} e^{\sigma dt_0} \right] \quad (3.32)
\]

Thus the integration over one period gives a factor of \( \alpha \) in the denominator:

\[
f_1 = \frac{\langle I > g \lambda_0 \nu_p f}{\alpha} = \frac{\lambda_0 f m_1}{\alpha} \quad (3.33)
\]

where \( m_1 = \langle I > g \nu_p \) is the mean number of neutrons per pulse. For finding the two time probability density or Rossi alpha, the integration and summation involved in second line of the expression for \( f_2 \) [Eq.(3.31)] is similar to the one for \( f_1 \) and we get the following contribution to \( f_2 \).

\[
\langle I > g \lambda_0^2 f \left[ \nu_p (\nu - 1) \lambda_f \frac{1}{2 \alpha} + \nu_p (\nu_p - 1) \right] e^{-\alpha \tau} \quad (3.34)
\]

In the first line of expression for \( f_2 \)[Eq. (3.31)], we have uncorrelated and correlated terms. For the first term (uncorrelated), the integrations over \( t_1 \), and \( t_2 \) factorize. Moreover each of these factors which are functions of \( t_0 \), are periodic with period \( 1/f \) having a phase difference of \( \delta = ([f \tau] + 1)/f - \tau \). We can therefore expand each of the factors in a Fourier series. The product is also a Fourier series related to the Fourier series of the individual factors. Integration over \( t_0 \) can then be carried out term by term. We write the two factors as \( G(t_0) \) and \( G(t_0 + \delta) \) where \( G(t_0) \) is obtained as follows:
\[
G(t_0) = \sum_{n=0}^{\infty} \int_{t_0-n/f}^{t_0} \frac{1}{\sigma} e^{\alpha t} dt = \frac{e^{\alpha t_0} (e^{\alpha \sigma} - 1)}{\alpha \sigma (1 - e^{-\alpha/f})} \quad \text{for } t_0 < -\sigma \tag{3.35a}
\]

\[
= \sum_{n=1}^{\infty} \int_{t_0-n/f}^{t_0} \frac{1}{\sigma} e^{\alpha t} dt + \int_{t_0}^{0} \frac{1}{\sigma} e^{\alpha t} dt = \frac{e^{\alpha t_0} (e^{\alpha \sigma} - 1)e^{-\alpha/f}}{\alpha \sigma (1 - e^{-\alpha/f})} + \frac{1}{\alpha \sigma} (1 - e^{\alpha t_0}) \quad \text{for } t_0 > -\sigma \tag{3.35b}
\]

Expanding each of these factors (which are functions of \( t_0 \)) in Fourier series we can integrate over \( t_0 \) as follows

\[
\int_{-1/f}^{0} G(t_0)G(t_0 + \delta)dt_0 = \int_{-1/f}^{0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n a_m e^{2\pi f \delta\tau} dt_0
\]

\[
= \int_{-1/f}^{0} \sum_{m,n} a_n a_m e^{2\pi f \delta} e^{2\pi f (n+m)\delta} dt_0 \tag{3.36}
\]

Only terms for \( m = -n \) have a non zero value, and hence,

\[
\int_{-1/f}^{0} G(t_0)G(t_0 + \delta)dt_0 = 2 \int_{-1/f}^{0} \sum_{n=1}^{\infty} a_n a_{-n} \cos(2\pi f \tau) + \frac{a_0^2}{f} \tag{3.37}
\]

We can compute the coefficients \( a_n, a_0 \) using Fourier formula & the above expressions for \( G(t_0) \)

\[
a_n = f \int_{-1/f}^{0} e^{-2\pi f \delta} G(t_0)dt_0 \tag{3.38}
\]

Integration is straight forward and we get

\[
a_n a_{-n} = \frac{f^2}{\sigma^2} \frac{2(1 - \cos(\omega_n \sigma))}{\omega_n^2(\alpha^2 + \omega_n^2)}, \quad \text{where } \omega_n = 2\pi f \text{ and} \tag{3.39a}
\]

\[
a_0^2 = \frac{f^2}{\alpha^2} \tag{3.39b}
\]

Substituting the values of \( a_n a_{-n} \) and \( a_0^2 \) in Eq. (3.37) and using Eq. (3.31), we get the following contribution from the uncorrelated term:

\[
\frac{< I > ^2}{2 \sigma^2 \vec{\lambda}_d \vec{\lambda}_d} = \frac{f}{\alpha^2 \sigma^2} \left[ \sum_{n=1}^{\infty} \frac{4\alpha^2 (1 - \cos(\omega_n \sigma)) \cos(\omega_n \tau)}{\omega_n^2 (\omega_n^2 + \alpha^2)} + \sigma^2 \right] \tag{3.40}
\]
As far as the second term (correlated) is concerned, our assumption about the correlation time $1/\beta$ being much smaller than $1/f$ implies that terms for which $n \neq m$ will give zero contribution. For $n=m$ we can set $\exp[-|t_1 - t_2|] = 1$. [Since this case shows no correlations between the number of protons (ions) in successive pulses, it will correspond to the results obtained earlier by Degweker (2003)]. The contribution of this term can then be written as

$$
\bar{v}_{sp}^2 \lambda^2 \Gamma^2 g^2 e^{-\alpha t} \int_{-1/f}^{0} dt_0 \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \epsilon(t_1 - t_0 + n/f) \epsilon(t_2 - t_0 + n/f) e^{i(\alpha t_1 - \gamma)} dt_1 dt
$$

(3.41)

$$
= \bar{v}_{sp}^2 \lambda^2 \Gamma^2 g^2 \left[ \int_{-\infty}^{0} dt_0 \sum_{n=1}^{\infty} \frac{1}{\sigma} e^{i(\alpha t_1 - \gamma)} \int_{-1/f}^{0} dt_1 \frac{1}{\sigma} e^{i(\alpha t_1 - \gamma)} \int_{-1/f}^{0} dt_1 + \int_{-\infty}^{0} dt_0 \frac{1}{\sigma} e^{i(\alpha t_1 - \gamma)} \int_{-1/f}^{0} dt_1 \right]
$$

$$
= \bar{v}_{sp}^2 \lambda^2 \Gamma^2 g^2 f \left[ 1 - \frac{1 - e^{-\alpha t}}{\alpha \sigma} \right] e^{-\alpha t}
$$

(3.42)

Adding (3.34), (3.40) and (3.42), the final expression for $f_2$ becomes

$$
f_2 = \frac{1 > g^2 \lambda^2 \bar{v}_{sp}^2 f}{\alpha^2 \sigma^2} \left[ \sum_{n=1}^{\infty} 4\alpha^2 (1 - \cos(\omega_n \sigma)) \cos(\omega_n \tau) \frac{1}{\omega_n^2 (\omega_n^2 + \alpha^2)} + \sigma^2 \right]
$$

$$
+ \frac{1 > g^2 \lambda^2 \bar{v}_{sp}^2 f}{\alpha} \left[ \frac{1}{2\alpha} + \frac{1}{2} \right] e^{-\alpha t} + \frac{\bar{v}_{sp}^2 \lambda^2 \Gamma^2 g^2 f}{\alpha} \sigma \left[ 1 - \frac{1 - e^{-\alpha t}}{\alpha \sigma} \right] e^{-\alpha t}
$$

(3.43)

The Feynman Y function can be obtained (Degweker and Rana, 2007) from the expression for $f_2$ using the relation in Eq. (3.10) and we get:

$$
Y = \frac{4}{\sigma^2 T f} \left[ \sum_{n=1}^{\infty} \frac{1 - \cos(\omega_n \sigma)) \sin^2(\omega_n T / 2)}{n^2 \pi^2 \omega_n^4 (\omega_n^2 + \alpha^2)} \right]
$$

$$
+ \left( \frac{\lambda_f \lambda \bar{v}_{sp} (v-1)}{\alpha^2} + \frac{\lambda_f \bar{v}_{sp} (v-1)}{\alpha} \frac{2g^2 \lambda^2 \bar{v}_{sp}}{\alpha} \left[ 1 - \frac{1 - e^{-\alpha T}}{\alpha T} \right] \right) (1 - \alpha \sigma / 3)
$$

(3.44)
Gaussian pulses

The Gaussian pulses are defined by

\[ e(t) = \frac{g}{\sqrt{2\pi}\sigma} \exp\left(-\frac{t^2}{2\sigma^2}\right) \]

The pulses are assumed to be centered at \( t_0 + n/f \) and having a width (S.D.) \( \sigma \). The summation is over all integral values of \( n \) since even pulses which appear later than 0 (or \( \tau \)) will have leading edges which lie before these times.

\[ f_1 = \frac{< I > g\lambda_d\varphi_{sp}f}{\sqrt{2\pi}\sigma} \int_{-\infty}^{0} dt_0 \sum_{n=-\infty}^{0} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(t-t_0+nf)^2}{2\sigma^2}} e^{\alpha t} dt \]  

(3.45)

Substituting \( u = t-t_0+nf \)

\[ f_1 = \frac{< I > g\lambda_d\varphi_{sp}f}{\sqrt{2\pi}\sigma} \int_{-\infty}^{0} dt \sum_{n=-\infty}^{0} \int_{t+n/f}^{\infty} e^{\frac{u^2}{2\sigma^2}} du e^{\alpha t} = \frac{< I > g\lambda_d\varphi_{sp}f}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\frac{u^2}{2\sigma^2}} du e^{\alpha t} \]  

(3.46)

Thus the integration over one period gives a factor of \( \alpha \) in the denominator.

For finding Rossi alpha formula we note that the integration involved in second part of the expression for \( f_2 \) [Eq. (3.31)] is similar to the one for \( f_1 \) and gives us the following term

\[ \frac{< I > g\lambda_d^2 f}{\alpha} \left[ \frac{\lambda_j\varphi_{sp}v(v-1)}{2\alpha} + \frac{\varphi_{sp}(v_{sp}-1)}{2} \right] e^{-\alpha t} \]  

(3.47)

The uncorrelated component in the first line of the expression for \( f_2 \) [Eq. (3.31)] can be written as

\[ < I > g^2 \lambda_d^2 \varphi_{sp}^2 f \int_{-\infty}^{0} dt_0 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{0} e^{\frac{(t-t_0+nf)^2}{2\sigma^2}} e^{\alpha t_0} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(t_2-t_0+nf)^2}{2\sigma^2}} e^{\alpha (t_2-t)} dt_1 dt_2 \]  

(3.48)
For solving Eq. (3.48), we again use the Fourier series expansion technique described in detail for rectangular pulses. In the present case

\[
G(t_0) = \frac{1}{\sqrt{2\pi\sigma}} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{0} e^{-\frac{(t-t_0-m)^2}{2\sigma^2}} e^{i\omega t} dt
\]

and hence we get for \(a_n\), the Fourier coefficient

\[
a_n = f \int_{-1/f}^{0} e^{-2\pi \beta_b} G(t_0) dt_0 = f \int_{-\infty}^{0} dt \int_{-1/f}^{0} dt_0 \sum_{m=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(t-t_0-m)^2}{2\sigma^2}} e^{i\omega t} e^{-2\pi \beta_b} \tag{3.50}
\]

Setting \(u=t-t_0+m/f\), we can write

\[
a_n = f \int_{-\infty}^{0} dt \int_{-\infty}^{\infty} du \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{u^2}{2\sigma^2}} e^{i\omega (t-t_0+m)} = f \frac{e^{\frac{(\omega \sigma)^2}{2}}}{(\alpha + i\omega_n)} \tag{3.51}
\]

Thus,

\[
a_n a_{-n} = \frac{f^2 e^{-(\omega_n \sigma)^2}}{(\omega_n^2 + \alpha^2)}, \quad \text{and} \quad a_0^2 = \frac{f^2}{\alpha^2} \tag{3.52}
\]

Substituting \(a_n a_{-n}\), and \(a_0^2\) in Eq. (3.37) and using Eq. (3.31), we get the following contribution:

\[
\frac{<I>^2 g^2 \lambda_2 \lambda_1^2 \nu_p^2 f^2}{\alpha^2} \left[ \sum_{n=1}^{\infty} 2\alpha^2 e^{-(\omega_n \sigma)^2} \cos(\omega_n \tau) \right] + 1 \tag{3.53}
\]

On making the substitutions \(u = -t_0 + n/f\) \(t_1' = -t_1\) and \(t_2' = -t_2\) and substituting the Gaussian form for the pulse shape function, the correlated component of \(f_2\) [Eq. (3.41)] after slight algebraic manipulation becomes

\[
\frac{\nu_p^2 \lambda_2^2 \nu_p^2 f^2}{2\pi\sigma^2} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dt_1' \int_{-\infty}^{\infty} dt_2' \exp \left( -\frac{1}{\sigma^2} \left[ (u - t_1' + t_2')^2 + (t_1' - t_2')^2 \right] \right) e^{i\omega_n \sigma(t_2' - t_1')} \tag{3.54}
\]

Performing the integration over \(u\) gives
\[
\frac{v_{\text{sp}}^2 \lambda^2 \Gamma^2 g^2}{\sqrt{4\pi \sigma}} \int_0^\infty \int_{t'_2}^{t'_1} dt'_1 dt'_2 \exp \left\{ -\frac{(t'_1 - t'_2)^2}{4\sigma^2} + \alpha(t'_1 + t'_2) \right\} e^{-\alpha \tau} \tag{3.55}
\]

The integrations over \(t'_1, t'_2\) can be carried out using the substitutions \(x = t'_1 + t'_2, y = t'_1 - t'_2\) to give

\[
\frac{v_{\text{sp}}^2 \lambda^2 \Gamma^2 g^2}{2\sqrt{4\pi \sigma}} \int_{-\infty}^{\infty} \int_{-y}^{y} dy \, e^{y^2/4\sigma^2} + \int_{-\infty}^{\infty} \int_{y}^{\infty} dy \, e^{y^2/4\sigma^2} e^{-\alpha \tau} \tag{3.56}
\]

The expression can be evaluated in terms of the error function to give

\[
\frac{v_{\text{sp}}^2 \lambda^2 \Gamma^2 g^2}{4\alpha} \left[ 1 + \text{Erf} \left( \frac{\tau - 2\sigma^2 \alpha}{2\sigma} \right) + \left( 1 - \text{Erf} \left( \frac{\tau + 2\sigma^2 \alpha}{2\sigma} \right) \right) e^{2\alpha \tau} \right] e^{-\alpha \tau} \tag{3.57}
\]

The final expression for \(f_2\) can be obtained by adding (3.47), (3.53) and (3.57)

\[
f_2 = \frac{\lambda^2 \Gamma g}{\alpha^2} \sum_{n=1}^{\infty} \frac{\alpha^2 e^{-(\omega_n^2+\alpha^2)}}{\omega_n^2 + \alpha^2} + 1 + \frac{\lambda^2 \Gamma g}{\alpha^2} \left[ \frac{\lambda \bar{v}_{\text{sp}} (\nu - 1)}{2\alpha} + \frac{v_{\text{sp}} (v_{\text{sp}} - 1)}{2} \right] e^{-\alpha \tau}
\]

The corresponding Feynman Y function (Degweker and Rana, 2007) is given as

\[
Y(T) = \frac{2}{fT} \sum_{n=1}^{\infty} e^{-(\omega_n^2 + \alpha^2)} \sin^2 (\omega_n T / 2) + \frac{\lambda^2 \Gamma g}{\alpha^2} \left[ \frac{\lambda \bar{v}_{\text{sp}} (\nu - 1)}{2\alpha} + \frac{v_{\text{sp}} (v_{\text{sp}} - 1)}{2} \right] \left( 1 - e^{-\alpha T} \right)
\]

\[
\frac{v_{\text{sp}}^2 \lambda^2 \Gamma^2 g^2}{\alpha < I >} \left( 1 - \frac{1 - e^{-\alpha T}}{\alpha T} - \alpha^2 \sigma^2 \right) \tag{3.59}
\]

### 3.4 Numerical results

In Fig. 3.1, we show the variation with the delay time \(\tau\) of the Rossi alpha function of Eq. (3.8). The input parameters roughly correspond to the analysis of the MUSE experiment discussed in Ballester and Munoz-Cobo (2005). For spallation, we have taken the TARC...
experimental conditions as regards to fluctuations, but adjusted the pulsing rate and source strength to be the same as for the D–T MUSE experiment. Since measurements of the fluctuations of the D+ ion beams are not available, we have assumed for our calculations a $\sigma/m = 0.01$, i.e. 1% (and $\sigma/m = 0.1$, i.e. 10%) fluctuation in the beam current. The three sets of graphs in the figure correspond to the cases $\beta \ll \alpha$, $\beta \approx \alpha$ and $\beta >> \alpha$. It is obvious that if $\alpha$ and $\beta$ are of about the same magnitude or if $\beta \ll \alpha$, it is likely that noise experiments might yield $\beta$ which may be mistaken for $\alpha$! Only in the case $\beta >> \alpha$, i.e. where the source fluctuations can be treated as white, do we get a variation of the Rossi alpha which will give the correct value of $\alpha$.

Finiteness of the pulse width has a smoothening effect on the Rossi alpha function. This is illustrated in Fig. 3.2 which shows a comparison of the uncorrelated part of the Rossi alpha function.

3.5 Conclusion

The finiteness of the pulse width adds small corrections to the delta function based formulae. The correlations in the source fluctuations introduce additional terms which could confuse interpretation of alpha measurements by the variance method which is likely to suffer most from the presence of other sources of fluctuations. The Rossi alpha, correlation and spectral density methods might perform better in this case.
Figure 3.1: Variation of different terms of the Rossi alpha formula [Eq. (3.8)] with $\tau$. B, C, D, and E refer to the second, the third, the fourth and the sum of the second and fourth terms, respectively. The three sets of graphs (a)–(c) are for $\beta \ll \alpha$, $\beta \approx \alpha$ and $\beta \gg \alpha$ respectively.
Figure 3.2: Comparison of the (uncorrelated part) Rossi alpha formula for delta function and rectangular shaped pulses. The two are identical except close to an integral multiple of the pulse period. In the former we get a sharp cusp whereas for the latter we get a smooth curve in these regions.