The main results established in this chapter have been presented/communicated as detailed below:

1. *On a generalized Mittag-Leffler type function with four parameters.*
   (Presented in “National Conference on Computational and Mathematical Sciences”, held during Dec. 08-10, 2011 at VIT, Jaipur) (communicated)
1.1 INTRODUCTION

In this chapter, we introduce and study a Mittag-Leffler type function $\gamma, \delta E_{\alpha, \beta}(z)$. This function includes the Mittag-Leffler function defined by Mittag-Leffler (1903) and its generalization given by Wiman (1905), as its special cases. Here, we first prove that $\gamma, \delta E_{\alpha, \beta}(z)$ is an entire function in the complex plane and obtain its order and type. Next, we obtain two integral representations and Mellin-Barnes contour integral representation of $\gamma, \delta E_{\alpha, \beta}(z)$. We further obtain two recurrence relations, differential formula and fractional integral and derivative of $\gamma, \delta E_{\alpha, \beta}(z)$. We also obtain Euler (Beta) transform, Laplace transform and Mellin transform of $\gamma, \delta E_{\alpha, \beta}(z)$. Finally, we define an integral operator with $\gamma, \delta E_{\alpha, \beta}(z)$ as kernel and show that it is bounded on the Lebesgue measurable space $L[a,b]$.

1.2 MITTAG-LEFFLER FUNCTION AND ITS GENERALIZATION

The Mittag-Leffler function was introduced by the Swedish mathematician Mittag-Leffler (1903) in terms of the following power series

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \text{Re}(\alpha) > 0. \quad (1.2.1)$$

It reduces to the exponential function when $\alpha = 1$. For $0 < \alpha < 1$, it
interpolates between the pure exponential $e^z$ and the geometric function

$$
\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z| < 1.
$$

A generalization of (1.2.1) was studied by Wiman (1905) in the form

$$
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an + \beta)}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0. \quad (1.2.2)
$$

A further generalization of (1.2.2) was studied by Prabhakar (1971) as

$$
E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(an + \beta) n!}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0. \quad (1.2.3)
$$

Kiryakova (2000) defined a multi-index Mittag-Leffler type function by means of the following power series

$$
E_{\left(\frac{1}{\rho_j}\right),\left(\mu\right)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})}, \quad (1.2.4)
$$

where $m > 1$ is an integer, $\rho_1, \rho_2, \ldots, \rho_m$ and $\mu_1, \mu_2, \ldots, \mu_m$ are arbitrary real parameters.

Shukla & Prajapati (2007) defined a generalization of Mittag-Leffler type function (1.2.3) in the form

$$
E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_q^n z^n}{\Gamma(an + \beta) n!}, \quad (1.2.5)
$$

where $\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0, \; q \in (0,1) \cup \mathbb{N}$.
A further generalization of the function defined by (1.2.5) has been introduced and studied by Srivastava & Tomovski (2009) wherein the parameter \( q \) is replaced by \( K \) with \( \text{Re}(K) > 0 \).

In 2010, Saxena & Nishimoto defined an extension of (1.2.4), in the following form

\[
E_{r,K} [(\alpha_j, \beta_j)_{1,m}; z] = E_{r,K} [(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m); z] = \sum_{r=0}^{\infty} \frac{(\gamma)_r z^r}{r!} \prod_{j=1}^{m} \Gamma(\alpha_j r + \beta_j)(r)!
\]  

(1.2.6)

where \( \gamma, \beta_j \in \mathbb{C}, \text{Re}(K) > 0, \sum_{j=1}^{m} \text{Re}(\alpha_j) > \text{Re}(K) - 1, j = 1, \ldots, m. \)

In 2011, Paneva-Konovska defined the following 3m-indices Mittag-Leffler type function, in the form

\[
E_{(\gamma_1, \ldots, \gamma_m)}^{(\rho_1, \ldots, \rho_m)} (z) = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k \cdots (\gamma_m)_k}{\Gamma(\alpha_1 k + \beta_1) \cdots \Gamma(\alpha_m k + \beta_m)} \frac{z^k}{(k!)^m}.
\]  

(1.2.7)

A multivariate analogue of generalized Mittag-leffler type function is defined by Saxena et al. (2011) in the form

\[
E_{(\rho_1), \ldots, (\rho_m)}^{(\gamma_1, \ldots, \gamma_m)} (z_1, \ldots, z_m) = E_{(\rho_1, \ldots, \rho_m), \ldots, (\rho_m), \ldots, (\rho_m)}^{(\gamma_1, \ldots, \gamma_m)} (z_1, \ldots, z_m)
\]

\[
= \sum_{k_1, \ldots, k_m=0}^{\infty} \frac{(\gamma_1)_{k_1} \cdots (\gamma_m)_{k_m}}{\Gamma(\lambda + \sum_{j=1}^{m} \rho_j k_j)(k_1)! \cdots (k_m)!} \frac{z_1^{k_1} \cdots z_m^{k_m}}{(k_1)! \cdots (k_m)!},
\]  

(1.2.8)

where \( \lambda, \gamma_j \in \mathbb{C}, \text{Re}(\rho_j) > 0, j = 1, 2, \ldots, m. \)
A generalized multiparameter function of Mittag-Leffler type is defined by Kalla et al. (2012) in the form

$$HE_{\mu_1, \mu_2, \ldots, \mu_r}^{\lambda_1, \lambda_2, \ldots, \lambda_r}(z) \equiv HE_{\mu}^{\lambda}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\prod_{i=1}^{r} \Gamma(1 + \mu_i + \lambda_i k)} \left( \frac{z}{\lambda} \right)^{\Lambda k + M}, \quad (1.2.9)$$

where $\mu_i \in \mathbb{C}, \lambda_i > 0, i = 1, 2, \ldots, r, \sum_{i=1}^{r} \mu_i = M$ and $\sum_{i=1}^{r} \lambda_i = \Lambda$.

Recently, Garg et al. (2013) studied a Mittag-Leffler type function of two variables in the form

$$E_1(x, y) = E_1\left(\begin{array}{c} \gamma_1; \alpha_1; \gamma_2, \beta_1 \\ \gamma_2; \alpha_2, \beta_2, \gamma_3, \beta_3 \end{array} \left| z_1, z_2 \right.\right)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{\gamma_1}{\alpha_1 m} \right)_{\beta_1 n} \frac{z_1^m}{\Gamma(\delta_1 + \alpha_2 m + \beta_2 n)} \frac{z_2^n}{\Gamma(\delta_2 + \alpha_3 m + \beta_3 n)} \Gamma(\delta_3 + \beta_3 n), \quad (1.2.10)$$

where $\gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3 \in \mathbb{C}$ and $\min\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} > 0$.

### 1.3 A NEW MITTAG-LEFFLER TYPE FUNCTION

Avoiding increasing number of variables and parameters, we define and study here the following Mittag-Leffler type function with four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ as

$$\gamma, \delta E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\beta n}}{\Gamma(\alpha n + \beta)} z^n, \quad \text{Re}(\alpha) > \text{Re}(\delta) > 0. \quad (1.3.1)$$
SPECIAL CASES

I. On taking $\delta = 1$ in (1.3.1), $\gamma \delta E_{\alpha, \beta}(z)$ reduces to a new **generalized Mittag-Leffler type function** given as follows

$$\gamma \delta E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)^n}{\Gamma(\alpha n + \beta)} z^n, \quad \text{Re}(\alpha) > 1. \quad (1.3.2)$$

II. On taking $\delta \to 0$ in (1.3.1), $\gamma \delta E_{\alpha, \beta}(z)$ reduces to the function $E_{\alpha, \beta}(z)$ defined by (1.2.2) as

$$\gamma \delta E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = E_{\alpha, \beta}(z). \quad (1.3.3)$$

III. On taking $\beta = 1$ in (1.3.3), $\gamma \delta E_{\alpha, \beta}(z)$ reduces to the Mittag-Leffler function $E_{\alpha}(z)$ defined by (1.2.1) as

$$\gamma \delta E_{\alpha, 1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} = E_{\alpha}(z). \quad (1.3.4)$$

1.3.1 ORDER AND TYPE

**THEOREM 1.1**

For $\text{Re}(\alpha) > \text{Re}(\delta) > 0$, the Mittag-Leffler type function $\gamma \delta E_{\alpha, \beta}(z)$ defined by (1.3.1) is an entire function in the complex plane and its order $(\rho)$ and type $(\sigma)$ are given by
\[ \rho = \frac{1}{\text{Re}(\alpha - \delta)} \]  

and

\[ \sigma = \frac{1}{\rho} \left( \frac{\text{Re}(\delta)}{\text{Re}(\alpha)} \right)^\rho . \]  

Further when \( \text{Re}(\alpha) = \text{Re}(\delta) > 0 \), the power series in (1.3.1) converges absolutely for \(|z| < 1\).

**PROOF**

Here we follow the classical techniques used by Kriyakova (2000) to find the order and type of \( \gamma, \delta E_{\alpha, \beta}(z) \). The radius of convergence \( R \) of the power series \( \gamma, \delta E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n} z^n}{\Gamma(\alpha n + \beta)} = \sum_{n=0}^{\infty} \phi_n z^n \) is given by

\[ R = \limsup_{n \to \infty} \left| \frac{\phi_n}{\phi_{n+1}} \right|. \]  

Using the asymptotic formula [Erdélyi *et al.* (1953)]

\[ \frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{-a+b} \left[ 1 + \frac{1}{2z} (a-b)(a+b-1) + O\left( \frac{1}{z^2} \right) \right], \]  

where \( a \) and \( b \) are fixed arbitrary complex numbers and \( -\pi < \arg z < \pi \), we get from (1.3.1)

\[ R = \limsup_{n \to \infty} \left| \frac{\phi_n}{\phi_{n+1}} \right|. \]
Chapter 1

\[
= \limsup_{n \to \infty} \left| \frac{\Gamma(\gamma + \delta n) \cdot \Gamma(an + \alpha + \beta)}{\Gamma(\gamma + \delta + \delta n) \cdot \Gamma(an + \beta)} \right| \sim \left\{ \frac{\Re(\alpha)}{\Re(\delta)} \right\}^{\Re(\alpha)} n^{\Re(\alpha - \delta)} \]

\[
= \begin{cases} 
\infty, & \text{when } \Re(\alpha) > \Re(\delta) > 0, \\
1, & \text{when } \Re(\alpha) = \Re(\delta) > 0, 
\end{cases}
\]

which proves that

(i) \( \gamma, \delta E_{\alpha, \beta}(z) \) is an entire function for \( \Re(\alpha) > \Re(\delta) > 0 \) and

(ii) The series of \( \gamma, \delta E_{\alpha, \beta}(z) \) converges absolutely for \( \Re(\alpha) = \Re(\delta) > 0 \),

when \( |z| < 1 \).

Next, the order \( \rho \) of an entire function \( \gamma, \delta E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \phi_n z^n \) is given by

the formula

\[
\rho = \limsup_{n \to \infty} \frac{n \ln n}{\ln \left( \frac{1}{|\phi_n|} \right)}. \tag{1.3.10}
\]

The Stirling formula in the form [Erdélyi et al. (1953)]

\[
\Gamma(z + a) \sim \sqrt{2\pi z} e^{\frac{1}{2}z} e^{-z}, \quad |z| \to \infty, \tag{1.3.11}
\]

gives

\[
\frac{1}{|\phi_n|} = \left| \frac{\Gamma(\gamma) \Gamma(an + \beta)}{\Gamma(\gamma + \delta n)} \right| \sim |\Gamma(\gamma)| e^{\Re\left(\frac{an + \beta - 1}{2}\right)\ln(\gamma - \frac{1}{2}) - \frac{1}{2} \ln(\delta n) - (\alpha - \delta)n}, \quad n \to \infty. \tag{1.3.12}
\]

Hence by definition (1.3.10), we have
\[ \frac{1}{\rho} = \limsup_{n \to \infty} \frac{\ln \left| 1/\phi_n \right|}{n \ln n} \]

\[ \ln |\Gamma(\gamma)| + \text{Re} \left[ \left( \alpha n + \beta - \frac{1}{2} \right) \ln (\alpha n) - \left( \gamma + \delta n - \frac{1}{2} \right) \ln (\delta n) - (\alpha - \delta) n \right] \]

\[ = \lim_{n \to \infty} \frac{\ln |\Gamma(\gamma)| + \text{Re} \left[ \left( \alpha n + \beta - \frac{1}{2} \right) \ln (\alpha n) - \left( \gamma + \delta n - \frac{1}{2} \right) \ln (\delta n) - (\alpha - \delta) n \right]}{n \ln n} \]

\[ = \text{Re}(\alpha - \delta), \]

which is the required result (1.3.5).

The type \( \sigma \) of the function \( \gamma, \delta E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \phi_n z^n \) of order \( \rho \) is determined by the relation

\[ \sigma e \rho = \limsup_{n \to \infty} \left[ n |\phi_n|^{\rho/n} \right]. \quad (1.3.13) \]

Proceeding as above for finding \( \rho \), we can easily obtain

\[ \sigma = \frac{1}{\rho} \left( \frac{\text{Re}(\delta)}{\text{Re}(\alpha)} \right)^{\rho} \]

\[ = \frac{1}{\rho} \left( \frac{\text{Re}(\delta)}{\text{Re}(\alpha)} \right)^{\rho}, \quad (1.3.14) \]

which is as given by (1.3.6).

1.3.2 INTEGRAL REPRESENTATIONS

RESULT 1.2

(a) If \( \text{Re}(\alpha) > \text{Re}(\delta) > 0, \text{Re}(\gamma) > 0 \), we have the following integral representation of \( \gamma, \delta E_{\alpha, \beta}(z) \)
Chapter 1

\[ \gamma, \delta E_{a, \beta}(z) = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} E_{a, \beta}(zt^\delta) dt. \quad (1.3.15) \]

(b) If \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > \text{Re}(\gamma) > 0 \), we have following beta integral representation for \( \gamma, \delta E_{a, \beta}(z) \)

\[ \gamma, \delta E_{a, \beta}(z) = \frac{1}{\Gamma(\gamma)\Gamma(\beta - \gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\beta-\gamma-1}(1-zt^\alpha)^{-1} dt. \quad (1.3.16) \]

**PROOF**

(a) For the integral representation (1.3.15) of \( \gamma, \delta E_{a, \beta}(z) \), we write

\[ \gamma, \delta E_{a, \beta}(z) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^\infty \frac{\Gamma(\gamma + \delta n)}{\Gamma(\alpha n + \beta)} z^n. \]

For \( \text{Re}(\gamma) > 0 \) we write \( \Gamma(\gamma + \delta n) \) in the integral form, using the definition

\[ \Gamma(z) = \int_0^\infty t^{z-1} dt, \quad \text{Re}(z) > 0. \quad (1.3.17) \]

Next, we interchange the order of summation and integration, permissible under the conditions stated with the result, and express the power series thus obtained, in terms of the Mittag-Leffler function (1.2.2), and easily arrive at the integral representation (1.3.15).

(b) For the integral representation (1.3.16) of \( \gamma, \delta E_{a, \beta}(z) \), we write

\[ \gamma, \delta E_{a, \beta}(z) = \sum_{n=0}^\infty \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} z^n \]
\[=rac{1}{\Gamma(\gamma)\Gamma(\beta-\gamma)}\sum_{n=0}^{\infty} \frac{\Gamma(\gamma+\alpha n)\Gamma(\beta-\gamma)}{\Gamma(\alpha n + \beta)} z^n \]

\[=rac{1}{\Gamma(\gamma)\Gamma(\beta-\gamma)}\sum_{n=0}^{\infty} B(\gamma+\alpha n, \beta-\gamma) z^n, \quad (1.3.18)\]

where \( B(x, y) \) is the beta function defined as

\[B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_{0}^{1} t^{x-1}(1-t)^{y-1} dt, \text{ Re}(x)>0, \text{Re}(y)>0. \quad (1.3.19)\]

Now, for \( \text{Re}(\beta)>\text{Re}(\gamma)>0 \), we use the integral form of beta function, interchange the order of summation and integration, express the power series thus obtained in terms of binomial function and easily arrive at the integral representation (1.3.16).

**RESULT 1.3**

**Mellin-Barnes integral representation**

For \( \delta, \alpha > 0, \alpha < \delta + 2 \) and \( |\arg z| < \frac{\pi}{2}(\delta - \alpha + 2) \), we have

\[\gamma, \delta E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{L} \frac{\Gamma(-s)\Gamma(1+s)\Gamma(\gamma + \delta s)}{\Gamma(\gamma)\Gamma(\beta + \alpha s)} z^s ds, \quad (1.3.20)\]

where, for details of contour \( L \) and convergence of (1.3.20), we refer the book by Srivastava *et al.* (1982).
On comparing the series definition of $E_{\alpha,\beta}(x)$ with the series representation of $H$-function given by (0.4.2) and using Mellin-Barnes contour integral representation of $H$-function given by (0.4.1) we obtain the following result

$$
\gamma,\delta E_{\alpha,\beta}(z) = H_{2,2}^{1,2}\left[ \begin{array}{c}
(1-\gamma,\delta),(0,1) \\
(0,1),(1-\beta,\alpha)
\end{array} \right]
$$

$$
= \frac{1}{2\pi i} \int_{\gamma} \frac{\Gamma(-s)\Gamma(1+s)\Gamma(\gamma+\delta s)}{\Gamma(\beta+\alpha s)} z^s ds,
$$

(1.3.21)

which holds true for conditions stated with result (1.3.20).

### 1.3.3 Recurrence Relations

#### Result 1.4

If $\text{Re}(\alpha) > \text{Re}(\delta) > 0$, we have

(a) $\gamma,\delta E_{\alpha,\beta}(z) = \beta_{\gamma,\delta} E_{\alpha,\beta+1}(z) + \alpha z_{\gamma,\delta} E'_{\alpha,\beta+1}(z)$,

(1.3.22)

(b) $\beta(\beta+1)_{\gamma,\delta} E_{\alpha,\beta+3}(z) + \alpha z_{\gamma,\delta} E'_{\alpha,\beta+3}(z) + \alpha^2 x^2 \gamma,\delta E''_{\alpha,\beta+3}(z)$

$$
= \gamma,\delta E'_{\alpha,\beta+1}(z) - \gamma,\delta E_{\alpha,\beta+2}(z),
$$

(1.3.23)

where $\gamma,\delta E'_{\alpha,\beta}(z)$ denotes the differentiation of $\gamma,\delta E_{\alpha,\beta}(z)$ with respect to $z$. 
PROOF

(a) The recurrence relation (1.3.22) can easily be obtained on using definition (1.3.1), for the function in the R.H.S. adding the two series to form \( \gamma, \delta E_{\alpha, \beta}(z) \).

(b) To establish recurrence relation (1.3.23), we begin with \( \gamma, \delta E_{\alpha, \beta + 2}(z) \) and write it as

\[
\gamma, \delta E_{\alpha, \beta + 2}(z) = \gamma, \delta E_{\alpha, \beta + 1}(z) - S, \tag{1.3.24}
\]

where

\[
S = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}}{(an + \beta + 1)\Gamma(an + \beta)} z^n. \tag{1.3.25}
\]

Using simple identity \( \frac{1}{u} = \frac{1}{u(u+1)} + \frac{1}{u+1} \) and the result

\[
\Gamma(n+1) = n\Gamma(n), \text{ we can write}
\]

\[
S = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n} (an + \beta)}{\Gamma(an + \beta + 3)} z^n + \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n} (an + \beta + 1)(an + \beta)}{\Gamma(an + \beta + 3)} z^n
\]

\[
= p \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n} z^n}{\Gamma(an + \beta + 3)} + q \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n} nz^n}{\Gamma(an + \beta + 3)} + r \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n} n^2 z^n}{\Gamma(an + \beta + 3)}, \tag{1.3.26}
\]

where \( p = \beta(\beta+2), \ q = 2\alpha(\beta+1) \) and \( r = \alpha^2 \).

Next, we observe that

\[
\frac{d}{dz}[z, \gamma, \delta E_{\alpha, \beta + 3}(z)] = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n} (n+1)}{\Gamma(an + \beta + 3)} z^n \tag{1.3.27}
\]

and
\[
\frac{d^2}{dz^2} \left[ z^2 E_{\alpha,\beta+3}(z) \right] = \sum_{n=0}^{\infty} \frac{(\gamma)^{\delta_n} (n^2 + 3n + 2)}{\Gamma(\alpha n + \beta + 3)} z^n.
\]

(1.3.28)

Using these results, we can write

\[
\sum_{n=0}^{\infty} \frac{(\gamma)^{\delta_n} n}{\Gamma(\alpha n + \beta + 3)} z^n = z_{\gamma,\delta} E'_{\alpha,\beta+3}(z)
\]

(1.3.29)

and

\[
\sum_{n=0}^{\infty} \frac{(\gamma)^{\delta_n} n^2}{\Gamma(\alpha n + \beta + 3)} z^n = z_{\gamma,\delta} E'_{\alpha,\beta+3}(z) + z^2 \gamma_{\gamma,\delta} E''_{\alpha,\beta+3}(z).
\]

(1.3.30)

Using (1.3.29) and (1.3.30) in (1.3.26), we obtain the value of \( S \) which being substituted in (1.3.24), provides the recurrence relation (1.3.23).

### 1.3.4 Differential Formula

#### RESULT 1.5

If \( \text{Re}(\alpha) > \text{Re}(\delta) > 0, \ m \in \mathbb{N} \), we have

\[
\left( \frac{d}{dz} \right)^m \left[ z^{\beta-1}_{\gamma,\delta} E_{\alpha,\beta} (\omega z^\alpha) \right] = z^{\beta-m-1}_{\gamma,\delta} E_{\alpha,\beta-m} (\omega z^\alpha).
\]

(1.3.31)

**PROOF**

Applying ordinary differentiation on the definition of \( z_{\gamma,\delta} E_{\alpha,\beta} (\omega z^\alpha) \) given by (1.3.1), in the left hand side of (1.3.31) and after doing some mathematical simplifications we arrive at the required result (1.3.31).
1.3.5 FRACTIONAL CALCULUS

RESULT 1.6

Let \( a \in \mathbb{R}^+ \) and \( \omega, \lambda, \alpha, \beta, \gamma, \delta \in \mathbb{C}, \) \( m - 1 < \text{Re}(\lambda) \leq m, \) \( m \in \mathbb{N} \)

\( \text{Re}(\alpha) > \text{Re}(\delta) > 0 \) then for \( x > a, \) we have following results

(a) \[ aI_x^\lambda \left( (x-a)^{\beta-1}_{\gamma,\delta} E_{\alpha,\beta} \left[ \omega(x-a)^\alpha \right] \right) = (x-a)^{\beta+\lambda-1}_{\gamma,\delta} E_{\alpha,\beta+\lambda} \left[ \omega(x-a)^\alpha \right], \]  
(1.3.32)

(b) \[ aD_x^\lambda \left( (x-a)^{\beta-1}_{\gamma,\delta} E_{\alpha,\beta} \left[ \omega(x-a)^\alpha \right] \right) = (x-a)^{\beta-\lambda-1}_{\gamma,\delta} E_{\alpha,\beta-\lambda} \left[ \omega(x-a)^\alpha \right]. \]  
(1.3.33)

where \( aI_x^\lambda \) and \( aD_x^\lambda \) are Riemann-Liouville fractional integral and derivative of order \( \lambda, \) defined by (0.5.1) and (0.5.3) respectively.

PROOF

(a) Result (1.3.32) can be established on using the definitions (1.3.1), (0.5.1) and the known result (0.5.2).

(b) Using the definition of fractional derivative (0.5.3) and the result (1.3.32), we can easily establish the result (1.3.33).
1.3.6 INTEGRAL TRANSFORMS

RESULT 1.7

Euler (beta) transform

If \( \text{Re}(a) > 0, \text{Re}(b) > 0, \text{Re}(\sigma) > 0, \text{Re}(\alpha) > \text{Re}(\delta) > 0 \), we have

\[
\int_0^1 u^{a-1} (1-u)^{b-1} \gamma, \delta E_{\alpha,\beta} \left( z u^\sigma \right) du = \frac{\Gamma(b)}{\Gamma(\gamma)} \psi_2 \left[ (\gamma, \delta), (a, \sigma), (1,1); z \right] \psi(\beta, \alpha)(a+b, \sigma),
\]

(1.3.34)

where \( \psi_{q} \) is the Fox-Wright function defined by (0.3.5).

Particularly, for \( a = \beta, \sigma = \alpha \), (1.3.34) reduces to the integral

\[
\int_0^1 u^{\beta-1} (1-u)^{b-1} \gamma, \delta E_{\alpha,\beta} \left( z u^\alpha \right) du = \Gamma(b) \gamma, \delta E_{\alpha,\beta+b} \left( z \right).
\]

(1.3.35)

PROOF

Using the definition of \( \gamma, \delta E_{\alpha,\beta} \left( z \right) \) given by (1.3.1), changing the order of summation and integration, which is permissible under the given conditions, evaluating the beta integral and writing the power series thus obtained in terms of the Fox-Wright function \( \psi_{q} \), we arrive at the required result (1.3.34).

Particularly, for \( a = \beta, \sigma = \alpha \), we have the R.H.S. of (1.3.34)
\[ \frac{\Gamma(b)}{\Gamma(\gamma)} \frac{\psi_2'\left(\gamma, \delta, (\beta, \alpha), (1,1); z\right)}{\Gamma(\gamma)} = \frac{\Gamma(b)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + \delta n)}{\Gamma(\beta + \alpha n)} z^n \]

\[ = \Gamma(b) \gamma \delta E_{a, \beta+b}(z), \]

which proves (1.3.35).

**RESULT 1.8**

Laplace transform

(a) If \( \text{Re}(\alpha) > \text{Re}(\sigma + \delta), \text{Re}(\delta) > 0, \text{Re}(\sigma) > 0, \text{Re}(s) > 0 \), we have

\[ L\left\{ u^{-1}_{\gamma, \delta} E_{a, \beta} \left( zu^\sigma \right); s \right\} = \int_0^\infty u^{-1} e^{-zu^\sigma} E_{a, \beta} \left( zu^\sigma \right) du \]

\[ = \frac{s^{-\alpha}}{\Gamma(\gamma)} \frac{\psi_1\left((\gamma, \delta), (a, \sigma), (1,1); z\right)}{s^\alpha} \]

(b) \( L^{-1}\left\{ s^{-1}_{\gamma, \delta} E_{a, \beta} \left( s^{-1} \right); z \right\} = E_{a, \beta}^{\gamma, \delta}(z), \) (1.3.37)

where \( L \) and \( L^{-1} \) denote the well known Laplace and inverse Laplace transforms of a function respectively and the function \( E_{a, \beta}^{\gamma, \delta}(z) \) is defined by (1.2.5).
PROOF

(a) Using the series definition of $E_{\alpha,\beta}(z)$ given by (1.3.1), the well known result of Laplace transform of power function and the definition (0.3.5), we arrive at the result (1.3.36).

(b) On taking the Laplace transform of the generalized Mittag-Leffler type function $E_{\alpha,\beta}(z)$ given by (1.2.5), we arrive at (1.3.37).

RESULT 1.9

Mellin transform

If $\alpha > \delta > 0$, $0 < s < \frac{\gamma}{\delta}$, we have

$$M\left\{E_{\alpha,\beta}(wx); s\right\} = \int_{0}^{\infty} x^{s-1} E_{\alpha,\beta}(wx) dx = \frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma - \delta s)}{w^\gamma \Gamma(\gamma)\Gamma(\beta - \alpha s)}. \quad (1.3.38)$$

where $M$ denotes the well known Mellin transform of the function.

PROOF

Using the Mellin Barnes integral representation of $E_{\alpha,\beta}(z)$, given by (1.3.20) and the Mellin inversion theorem [Sneddon (1972)], we obtain

$$M\left\{E_{\alpha,\beta}(wx); s\right\} = \frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma - \delta s)}{w^\gamma \Gamma(\gamma)\Gamma(\beta - \alpha s)},$$

which proves the result (1.3.38).
1.3.7 AN INTEGRAL OPERATOR INVOLVING THE FUNCTION \( \gamma, \delta E_{\alpha, \beta}(z) \)

We consider the following integral operator

\[
\left( \mathfrak{S}_{a+\alpha, \beta}^{\omega, \gamma, \delta} \phi \right)(x) = \int_{a}^{x} (x-t)^{\beta-1} \gamma, \delta E_{\alpha, \beta} \left[ \omega (x-t)^{\alpha} \right] \phi(t) \, dt, \quad x > a,
\]

(1.4.1)

where \( \gamma, \omega \in \mathbb{C}, \quad \text{Re}(\alpha) > \text{Re}(\delta) > 0, \quad \text{Re}(\beta) > 0. \)

Particularly, for \( \omega = 0, \) \( \mathfrak{S}_{a+\alpha, \beta}^{\omega, \gamma, \delta} \) corresponds to the Riemann-Liouville fractional integral defined by (0.5.1).

**THEOREM 1.10**

Under the various parametric constraints stated with the definition (1.4.1), let the function \( \phi \) be in the space \( L[a,b] \) of Lebesgue measurable functions given by

\[
L[a,b] = \left\{ f : \|f\|_1 = \int_{a}^{b} |f(x)| \, dx < \infty \right\},
\]

(1.4.2)

then the integral operator \( \mathfrak{S}_{a+\alpha, \beta}^{\omega, \gamma, \delta} \) is bounded on \( L[a,b] \) and

\[
\| \mathfrak{S}_{a+\alpha, \beta}^{\omega, \gamma, \delta} \phi \| \leq B \| \phi \|_1,
\]

(1.4.3)

where

\[
B = (b-a)^{\text{Re}(\beta)} \sum_{n=0}^{\infty} \frac{\left| (\gamma)_{\delta n} \left[ \omega(b-a)^{\text{Re}(\alpha)} \right]^{n} \right|}{\Gamma \left( \text{Re}(\alpha)n + \text{Re}(\beta) \right) \Gamma \left( \alpha n + \beta \right)}.
\]

(1.4.4)
PROOF

First of all, we observe that if \( \phi_n \) denote the \( n^{th} \) term of the series given in (1.4.4) then

\[
\phi_{n+1} = \frac{\Gamma(\gamma + \delta + \delta n)}{\Gamma(\gamma + \delta n)} \left\{ \Re(\alpha)n + \Re(\beta) \right\} \frac{\Gamma(\alpha n + \beta)}{\Gamma(\alpha(n + \alpha + \beta))} \left| \phi(b-a)^{\Re(\alpha)} \right| \\
\sim \frac{\phi^{\Re(\beta)}\left| \phi(b-a)^{\Re(\alpha)} \right|}{a^{\Re(\alpha)}} n^{\Re(\delta) - \Re(\alpha)}
\]

\[
\rightarrow 0 \text{ as } n \rightarrow \infty, \text{ when } \Re(\alpha) > \Re(\delta) > 0.
\]

Hence the series in R.H.S. of (1.4.4) is bounded and the constant \( B \) is finite.

Using the definition of \( \| \cdot \|_1 \), given in (1.4.2) and the definition of integral operator, given in (1.4.1) and then interchanging the order of integrations, we have

\[
\|_w^{\omega, \gamma, \delta}_v \alpha, \beta \| \phi = \int_a^b \int_t^b (x-t)^{\beta-1} \cdot \gamma, \delta E_{\alpha, \beta} \left[ w(x-t)^{\alpha} \right] \phi(t) dt dx
\]

\[
\leq \int_a^b \int_t^b (x-t)^{\Re(\beta)-1} \cdot \gamma, \delta E_{\alpha, \beta} \left[ w(x-t)^{\alpha} \right] dx \| \phi(t) \| dt
\]

\[
= \int_a^b \int_t^b u^{\Re(\beta)-1} \cdot \gamma, \delta E_{\alpha, \beta} \left[ u^{\alpha} \right] du \| \phi(t) \| dt
\]

\[
\leq \int_a^b \int_t^b u^{\Re(\beta)-1} \cdot \gamma, \delta E_{\alpha, \beta} \left[ wu^{\alpha} \right] du \| \phi(t) \| dt.
\] (1.4.6)
For the inner integral, using (1.3.1), carrying out term by term integration and taking into account (1.4.4), we obtain

\[
\int_{0}^{b-a} u^{\Re(\beta)-1} \left|_{\gamma,\delta} E_{\alpha,\beta} \left[ wu^{\alpha} \right] \right| du \leq \sum_{n=0}^{\infty} \left| \left( \gamma \right)_{\delta} n \right| \left| \phi \right|^{n} \int_{0}^{b-a} u^{\Re(\beta) + \Re(\alpha)n-1} du = B. \tag{1.4.7}
\]

Now using (1.4.7) in (1.4.6), we arrive at (1.4.3).

**THEOREM 1.11**

Let \( \gamma, \omega \in \mathbb{C}, \Re(\alpha) > \Re(\delta) > 0, \Re(\beta) > 0, \Re(\lambda) > 0 \), then the relations

\[
a \lambda I_{x}^{\lambda} \Gamma^{a,\gamma,\delta} \phi = \lambda I_{u+a,\beta+\lambda}^{\lambda} \phi = \lambda I_{u+a,\beta+\lambda}^{\lambda} a \lambda I_{x}^{\lambda} \phi \tag{1.4.8}
\]

hold for function \( \phi \in L[a,b] \).

**PROOF**

Using the definitions (1.4.1) and (0.5.1) and interchanging the order of integrations, we have for \( x > a \)

\[
a \lambda I_{x}^{\lambda} \Gamma^{a,\gamma,\delta} \left( \phi(x) \right) = \frac{1}{\Gamma(\lambda)} \int_{a}^{x} (x-u)^{\lambda-1} \left[ \int_{a}^{u} (u-t)^{\beta-1} \left( \gamma,\delta \right)_{\alpha,\beta} \left[ \omega(u-t)^{\alpha} \right] \phi(t) dt \right] du
\]
\[
\frac{1}{\Gamma(\lambda)} \int_0^x \left( \int_0^{x-u} (u-t)^{\beta-1} \gamma E_{\alpha,\beta} \left( \omega(u-t)^{\alpha} \right) du \right) \phi(t) dt
\]

\[
= \frac{1}{\Gamma(\lambda)} \int_0^x \left( \int_0^{x-t} (t-\tau)^{\beta-1} \gamma \delta E_{\alpha,\beta} \left( \omega \tau^\alpha \right) d\tau \right) \phi(t) dt
\]

\[
= \int_0^x \left[ I_{x-t}^{-\gamma,\delta} E_{\alpha,\beta} \left( \omega(x-t)^{\alpha} \right) \right] \phi(t) dt.
\]

(1.4.9)

Using the result (1.3.32), we can write (1.4.9) as

\[
\left( a I_x^{-\gamma,\delta,\alpha,\beta} \phi \right)(x) = \int_a^x (x-t)^{\beta+1-\lambda} \gamma \delta E_{\alpha,\beta+\lambda} \left( \omega(x-t)^{\alpha} \right) \phi(t) dt
\]

\[
= \left( \mathcal{H}_{\alpha,\beta+\lambda}^{-\gamma,\delta,\alpha,\beta} \phi \right)(x),
\]

which proves first part of the relation (1.4.8). The second part can be proved on similar lines.
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