Chapter 2

$H$-line graphs

This chapter deals with the graph operator $L_H(G)$ and the corresponding graph class, $H$-line graphs. We show that $H$-line graphs admit a forbidden subgraph characterization only when $H = K_{1,2}$. We also obtain a Krausz type characterization for star-line graphs. The notion of line index of a graph, $\zeta(G)$ is generalized to $\zeta_n(G)$, $n$-star-line-index of a graph $G$. We also characterize graphs in terms of $\zeta_3(G)$, $\zeta_4(G)$ and $\zeta_n(G), n \geq 5$.

Some results of this chapter are included in the following papers.
2. Seema Varghese, A. Vijayakumar, On the planarity of iterated star-line graphs (Submitted).

45
2.1 Non-existence of forbidden subgraph characterization

Let $H$ be a connected graph of order at least three. It is clear that $L_H(G)$ is a spanning subgraph of $L(G)$.

**Lemma 2.1.1.** If $H$ is a graph with the edge-independence number, $\alpha'(H) > 1$, then $K_n, n \geq 2$ is not an $H$-line graph.

*Proof.* Suppose that $\alpha'(H) > 1$ and $e_1, e_2$ are any two independent edges in $H$. Since $L_H(G)$ has an edge if and only if $G$ contains a copy of $H$, the edges $e_1, e_2$ will be independent in $G$ also. Clearly, the vertices corresponding to $e_1$ and $e_2$ are not adjacent in $L_H(G)$. \hfill \Box

**Lemma 2.1.2.** Every $K_n$ is an induced subgraph of $L_H(G)$ for some graph $G$.

*Proof.* The graph $G$ can be constructed as follows. With each pair of adjacent edges $\{vv_i, vv_j\}$ of $K_{1,n}$ construct a copy of $H$. In the newly constructed graph $G$, the edges $\{vv_i, vv_j\}$ are adjacent and there is a copy of $H$ containing both these
2.1. Non-existence of forbidden subgraph characterization

Fig 2.1: $K_4$ is an induced subgraph of $L_{C_4}(G)$

edges where $i$ and $j$ are integers such that $1 \leq i, j \leq n$. Hence $\{vv_1, vv_2, \ldots vv_n\}$ will induce a $K_n$ in $L_H(G)$.

**Note:** The case when $H = C_4$ is illustrated in Fig: 2.1.

Thus, it is clear from Lemma 2.1.2 that $H$-line graphs do not have induced hereditary property and hence, by Theorem 1.2.1 they lack forbidden subgraph characterization, if $\alpha'(H) > 1$. If $\alpha'(H) = 1$, then $H$ is either $K_{1,n}, n \geq 2$ or $K_3$. $L_{K_{1,2}}(G)$, which
is the line graph of $G$, admits a forbidden subgraph characterization (Theorem 1.2.2) but $L_{K_3}(G)$ does not admit a forbidden subgraph characterization (Theorem 1.2.4). Now, we shall show that $L_{K_{1,n}}(G), n \geq 3$ does not have the induced hereditary property.

**Lemma 2.1.3.** If $G$ is a $H$-line graph, then every edge of $G$ lies in a copy of $L(H)$.

*Proof.* Let $G = L_H(G')$. If there is an edge in $G$, then there will be a copy of $H$ in $G'$. Then the edges in $H \subseteq G'$ will induce a copy of $L(H)$ in $G$. \hfill $\Box$

**Lemma 2.1.4.** The graph $C_m, m \geq 4$ is not a $L_{K_{1,n}}(G), n \geq 3$, for any $G$.

*Proof.* If $C_m, m \geq 4$ were a $L_{K_{1,n}}(G), n \geq 3$, then by Lemma 2.1.3, every edge of $C_m$ would lie in a copy of $L(K_{1,n}) \cong K_n, n \geq 3$. \hfill $\Box$

**Lemma 2.1.5.** For $m \geq 4$, every $C_m$ is an induced subgraph of $L_{K_{1,n}}(G), n \geq 3$, for some graph $G$. 

2.1. Non-existence of forbidden subgraph characterization

Fig 2.2: $C_4$ is an induced subgraph of $L_{K_{1,4}}(G)$

Proof. Let $G = C_m \circ K_{n-2}^c$. Then $L_{K_{1,n}}(G)$ will contain $C_m$ as an induced subgraph.

Note: The case when $m = 4$ and $n = 4$ is illustrated in Fig: 2.2.

\[\square\]

**Theorem 2.1.6.** $H$-line graphs admit a forbidden subgraph characterization if and only if $H = K_{1,2}$.

Proof. The necessary part follows from Theorem 1.2.2. The sufficiency part follows from Theorem 1.2.4 and Lemmas 2.1.1 to 2.1.5.  

\[\square\]
2.2 Krausz-type characterization for star-line graphs

Analogous to the Theorem 1.2.3, the Krausz characterization of line graphs, we have the following theorem for star-line graphs.

**Theorem 2.2.1.** A graph $G$ is a star-line graph, $L_{K_{1,n}}(G')$, $n \geq 2$ if and only if $E(G)$ has a partition into cliques of order at least $n$ using each vertex of $G$ at most twice.

*Proof.* When $n = 2$, the theorem reduces to the Krausz characterization of line graphs. Suppose, $G \cong L_{K_{1,n}}(G')$, for some $n \geq 3$. Let $v \in G'$ be such that $\deg(v) \geq n$. The edges incident to $v$ will form a clique $C_v$ of order at least $n$ in $G$. Then, $\mathcal{E} = \{C_v | v \in G' \text{ and } \deg(v) \geq n\}$ will form a clique cover of the edges of $G$ in which every vertex of $G$ is in at most two members of $\mathcal{E}$.

Conversely, suppose that $G$ has an edge clique partition $\mathcal{E}$ satisfying the condition of the theorem. Consider the intersection graph $I(\mathcal{E})$. Corresponding to every vertex of $G$, which
belong to exactly one clique $C$ of $\mathcal{E}$, draw a pendant vertex to the vertex corresponding to $C$ in $I(\mathcal{E})$ and for every isolated vertex of $G$, draw an isolated edge. Let the newly constructed graph be $G'$. Now we shall show that $L_{K_{1,n}}(G') \cong G$. Define $\phi : V(G) \longrightarrow V(L_{K_{1,n}}(G'))$ as follows: If $v \in V(G)$ is such that $v \in C_i \cap C_j$, then $C_i$ and $C_j$ are adjacent in $I(\mathcal{E})$ and define $\phi(v)$ to be the edge in $G'$ joining $C_i$ and $C_j$. If $v \in C_i$ only, then there will be a pendant vertex in $G'$ corresponding to $v$ and define $\phi(v)$ to be the pendant edge attached to $C_i$. If $v$ is an isolated vertex in $G$, define $\phi(v)$ to be the isolated edge in $G'$ corresponding to $v$. It is clear that $\phi$ is a well-defined bijection.

Let $u$ and $v$ be adjacent vertices in $G$. Then $u$ and $v$ belong to a clique $C_i$ of the partition. Since every clique of the partition is of order at least $n$, there are vertices $w_1, w_2 \ldots w_{n-2}$ in $C_i$. The construction of $G'$ is such that edges corresponding to these vertices $u, v, w_1, w_2 \ldots w_{n-2}$ will have a common vertex forming a $K_{1,n}$ in $G'$. Thus the edges corresponding to $u$ and $v$ are adjacent and lie in a common copy of $K_{1,n}$ in $G'$ and hence $u$ and $v$ are adjacent in $L_{K_{1,n}}(G')$. Therefore, $\phi$ is an isomorphism.

**Corollary 2.2.2.** $L_{K_{1,n}}(G'), n \geq 3$ is a line graph in which every edge lies in a $K_n$. 

2.3 3-star-line-index of a graph

In this section, we characterize graphs in terms of $\zeta_3(G)$.

**Lemma 2.3.1.** If $G'$ is a subgraph of $G$, then $\zeta_n(G) \leq \zeta_n(G')$.

**Proof.** Let $\zeta_n(G') = k$. Then, $L^k_{K_{1,n}}(G')$ is nonplanar and so is $L^k_{K_{1,n}}(G)$, since $G'$ is subgraph of $G$. □

**Lemma 2.3.2.** If $G$ is a graph with $\Delta(G) \geq 4$, then $\zeta_3(G) \leq 3$.

**Proof.** If $\Delta(G) \geq 4$, then $G$ contains $K_{1,4}$ as a subgraph and $L^3_{K_{1,3}}(K_{1,4})$ (Fig: 2.3) is a 6-regular graph and hence is nonplanar by Theorem 1.2.6. Therefore, $\zeta_3(K_{1,4}) \leq 3$ and by Lemma 2.3.1, $\zeta_3(G) \leq 3$. □

![Fig 2.3: $L^3_{K_{1,3}}(K_{1,4})$](image-url)
Lemma 2.3.3. For any graph $G$, $\zeta_3(G) \in \{0, 1, 2, 3, 4, \infty\}$. Also, $\zeta_3(G) = \infty$ if and only if $\Delta(G) \leq 3$ and no two vertices in $G$ of degree three are adjacent.

Proof. If $\Delta(G) \geq 4$, by Lemma 2.3.2, we have $\zeta_3(G) \leq 3$. If $\Delta(G) \leq 2$, then $G$ does not contain $K_{1,3}$ as a subgraph and hence $L_{K_{1,3}}(G)$ is totally disconnected. Therefore $\zeta_3(G) = \infty$. If $\Delta(G) = 3$ and $G$ does not have two adjacent vertices of degree three, then $L^2_{K_{1,3}}(G)$ will be totally disconnected and hence $\zeta_3(G) = \infty$. If $G$ has two adjacent vertices of degree three, then $L^2_{K_{1,3}}(G)$ will have $K_4$ as a subgraph and $L^2_{K_{1,3}}(K_4)$ is a 6-regular graph (Fig: 1.9) which is non-planar. Hence $\zeta_3(G) = 4$. \qed

Lemma 2.3.4. For any graph $G$, $L_{K_{1,3}}(G)$ is planar if and only if $G$ satisfies the following:

(i) $\Delta(G) \leq 4$.

(ii) $G$ does not contain any of the graphs $H_1$ or $H_2$ in Fig 2.4 as a subgraph.

(iii) $G$ does not contain any subgraph homeomorphic to $K_{3,3}$ in which degree of every vertex in $G$ is at least three.

Note: An edge with a single end vertex shows the degree of
that vertex. In Fig 2.4, the degree is three.

\textit{Proof.} If $\Delta(G) \geq 5$, then $L_{K_{1,3}}(G)$ contains $K_5$ as a subgraph and hence it is nonplanar by Theorem 1.2.5. Also, if $G$ has any one of the graphs $H_1$ or $H_2$ as a subgraph, then $L_{K_{1,3}}(G)$ will contain any one of the graphs $H'_1$ or $H'_2$ in Fig 2.5 as a subgraph. Both graphs $H'_1$ or $H'_2$ are non planar by Theorem 1.2.5 and hence $L_{K_{1,3}}(G)$ is nonplanar.
For the necessity of condition (iii) we prove the following,

**Claim 2.3.1.** If $G$ has a subgraph homeomorphic to $G'$ in which degree of every vertex in $G$ is at least three, then $L_{K_{1,3}}(G)$ has a subgraph homeomorphic to $L_{K_{1,3}}(G')$.

Let $u_1u_2$ be an edge of $G'$ and $u$ be the vertex in $L_{K_{1,3}}(G')$ corresponding to the edge $u_1u_2$. Suppose that the edge $u_1u_2$ is subdivided by the vertex $u_3$ whose degree in $G$ is at least three as in Fig 2.6. Then the edges $u_3u_1, u_3u_2, u_3v_1, u_3v_2 \ldots u_3v_{n-2}$ will form a clique $C_u$ in $L_{K_{1,3}}(G)$. Now, the vertices which were adjacent to $u$ in $L_{K_{1,3}}(G')$ will be adjacent to the vertices corresponding to $u_3u_1$ and $u_3u_2$ in $L_{K_{1,3}}(G)$. Thus, corresponding to every edge of $L_{K_{1,3}}(G')$, we get a path in $L_{K_{1,3}}(G)$ and hence it
contains a subgraph homeomorphic to $L_{K_{1,3}}(G')$.

Hence, if $G$ has a subgraph homeomorphic to $K_{3,3}$ in which degree of every vertex in $G$ is at least three, then $L_{K_{1,3}}(G)$ has a subgraph homeomorphic to $L_{K_{1,3}}(K_{3,3})$ (Fig 2.7) which is non-planar.

\begin{figure}[h]
\centering
\includegraphics[width=8cm]{fig2.7.png}
\caption{$K_{3,3}$ and $L_{K_{1,3}}(K_{3,3})$}
\end{figure}

Conversely, suppose that $L_{K_{1,3}}(G)$ is nonplanar. Then, it contains a subgraph homeomorphic to $K_{5}$ or $K_{3,3}$.

**Case 1.** $L_{K_{1,3}}(G)$ contains $K_{5}$ or a subgraph homeomorphic to $K_{5}$. 
If $L_{K_{1,3}}(G)$ contains $K_5$, then there are five mutually incident edges in $G$ and $\Delta(G) \geq 5$, which is a contradiction. If $L_{K_{1,3}}(G)$ has a copy of $K_5$ with one edge subdivided once or twice, then it contains either a copy of $G_a$ or a copy of $G_b$ in Fig 2.8 as an induced subgraph. If $L_{K_{1,3}}(G)$ has a copy of $K_5$ with one edge subdivided more than twice then it contains a copy of $G_c$ as an induced subgraph. If $L_{K_{1,3}}(G)$ has a copy of $K_5$ with more than one edge subdivided, then it has a copy of $K_{1,3}$ as an induced subgraph. All the graphs $G_a, G_b, G_c, K_{1,3}$ are forbidden subgraphs for line graphs by Theorem 1.2.2 and hence are forbidden for star-line graphs also by Corollary 2.2.2. Hence, $L_{K_{1,3}}(G)$ cannot have any subgraph homeomorphic to $K_5$ other than $K_5$.

**Case 2.** $L_{K_{1,3}}(G)$ contains $K_{3,3}$ or a homeomorphic copy of $K_{3,3}$ as a subgraph.

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{fig2.8.png}
\caption{Fig 2.8: $G_a, G_b$ and $G_c$}
\end{figure}
In this case, $L_{K_{1,3}}(G)$ contains $K_{1,3}$ as an induced subgraph which is forbidden for star-line graphs. Also, any edge in $L_{K_{1,3}}(G)$ will lie in a triangle and any two cliques in the edge-clique partition of $L_{K_{1,3}}(G)$ can have at most one common vertex. These conditions will force $L_{K_{1,3}}(G)$ to have a copy of $K_5$ or a homeomorphic copy of $K_{3,3}$ in which degree of every vertex in $G$ is at least three. But, then $\Delta(G)$ will be greater than four.

**Lemma 2.3.5.** For any graph $G$, $\zeta_3(G) = 1$ if and only if $G$ is planar and contains any one of the graphs $K_{1,5}$, $H_1$ or $H_2$ in Fig 2.4 as a subgraph.

*Proof.* Follows from Lemma 2.3.4.

**Lemma 2.3.6.** For any graph $G$, $\zeta_3(G) = 2$ if and only if $L_{K_{1,3}}(G)$ is planar and $G$ contains any one of the graphs in Fig 2.9 as a subgraph.

*Proof.* By Lemma 2.3.5, $\zeta_3(G) = 2$ if and only if $L_{K_{1,3}}(G)$ is planar and has any one of the graphs $K_{1,5}$, $H_1$ or $H_2$ in Fig 2.4 as a subgraph. As in the proof of Lemma 2.3.4, it follows that this is possible if and only if $G$ has any one of the graphs in Fig 2.9 as a subgraph.
Lemma 2.3.7. For any graph $G$, $\zeta_3(G) = 4$ if and only if $\Delta(G) \leq 3$, $G$ is planar and has two adjacent vertices of degree three and does not have any one of the graphs in Fig 2.10 as a subgraph.

Proof. If $\Delta(G) \geq 4$, we have by Lemma 2.3.2 that $\zeta_3(G) \leq 3$. Also, $\zeta_3(G)$ of the graphs (1) and (2) in Fig 2.10 is two and that of the graph (3) in Fig 2.10 is three. Hence, if $G$ contains any of these graphs as subgraphs, then by Lemma 2.3.1, $\zeta_3(G) \leq 3$. Now, if $\Delta(G) \leq 3$ and $G$ does not have two adjacent vertices
of degree three, then $L_{K_{1,3}}^2(G)$ will be totally disconnected and $\zeta_3(G) = \infty$.

We thus have,

**Theorem 2.3.8.** Let $G$ be any graph. Then,

1. $\zeta_3(G) = \infty$, if and only if $\Delta(G) \leq 3$ and $G$ does not contain two adjacent vertices of degree three.

2. $\zeta_3(G) = 0$, if and only if $G$ is non-planar.

3. $\zeta_3(G) = 1$, if and only if $G$ is planar and contains any one of the graphs $K_{1,5}, H_1$ or $H_2$ in Fig 2.4 as a subgraph.

4. $\zeta_3(G) = 2$, if and only if $L_{K_{1,3}}^2(G)$ is planar and $G$ contains any one of the graphs in Fig 2.9 as a subgraph.
(5) \( \zeta_3(G) = 4 \), if and only if \( \Delta(G) \leq 3 \), \( G \) is planar and has two adjacent vertices of degree three and does not contain any one of the graphs in Fig 2.10 as a subgraph.

(6) \( \zeta_3(G) = 3 \), otherwise.

\[ \square \]

### 2.4 4-star-line-index of a graph

In this section, we characterize all graphs in terms of \( \zeta_4(G) \). We first state two lemmas which can be proved as in the previous section and use it to compute the value of \( \zeta_4(G) \).

**Lemma 2.4.1.** Let \( G \) be any graph. Then \( L_{K_{1,4}}(G) \) is planar if and only if \( G \) satisfies the following:

(i) \( \Delta(G) \leq 4 \).

(ii) \( G \) does not contain any one of the graphs \( H_3 \) or \( H_4 \) in Fig 2.11 as a subgraph.

(iii) \( G \) does not contain any subgraph homeomorphic to \( K_{3,3} \) in which degree of every vertex in \( G \) is at least four.

**Lemma 2.4.2.** For any graph \( G \), \( \zeta_4(G) = 2 \) if and only if \( L_{K_{1,4}}(G) \) is planar and \( G \) has any one of the graphs in Fig 2.12 as a subgraph.
Lemma 2.4.3. Let $G$ be any graph. Then, $\zeta_4(G) = \{0, 1, 2, \infty\}$.

Proof. For any graph $G$, $\zeta_4(G) = 3$ if and only if $L_{K_{1,4}}^2(G)$ contains any one of the graphs $K_{1,5}$, $H_3$ or $H_4$ as a subgraph. Also, if $L_{K_{1,4}}^2(G)$ contains any of these graphs, then $G$ has any
2.5. n-star-line-index of a graph

one of the graphs in Fig 2.12 as a subgraph, which implies that
$L^2_{K_{1,4}}(G)$ is nonplanar and $\zeta_4(G) = 2.$

We summarize these results as follows.

**Theorem 2.4.4.** Let $G$ be any graph. Then,

(1) $\zeta_4(G) = 0,$ if and only if $G$ is non-planar.

(2) $\zeta_4(G) = 1,$ if and only if $G$ is planar and contains any one
of the graphs $K_{1,5}, H_3$ or $H_4$ in Fig 2.11 as a subgraph.

(3) $\zeta_4(G) = 2,$ if and only if $L_{K_{1,4}}(G)$ is planar and $G$ contains
any one of the graphs in Fig 2.12 as a subgraph.

(4) $\zeta_4(G) = \infty,$ otherwise.

2.5 n-star-line-index of a graph

**Theorem 2.5.1.** For $n \geq 5$ and for any graph $G,$ $\zeta_n(G) \in \{0, 1, \infty\}.$ Also,

(1) $\zeta_n(G) = 0,$ if and only if $G$ is non-planar.
\( (2) \) \( \zeta_n(G) = \infty \), if and only if \( G \) is planar and \( \Delta(G) \leq 4 \).

\( (3) \) \( \zeta_n(G) = 1 \), otherwise.

**Proof.** \( L_{K_{1,n}}(G) \), \( n \geq 5 \) will have an edge if and only if \( \Delta(G) \geq 5 \) and in that case the edges incident on the vertex with maximum degree will induce a \( K_5 \) in \( L_{K_{1,n}}(G) \) which makes it nonplanar. Hence, \( \zeta_n(G) = 1 \). If \( G \) is nonplanar, then \( \zeta_n(G) = 0 \). If \( G \) is planar and \( \Delta(G) \leq 4 \), then \( L_{K_{1,n}}(G) \), \( n \geq 5 \) is an edgeless graph and hence \( \zeta_n(G) = \infty \). \( \square \)