Chapter 7

PROOF OF THEOREM 1.0.2

Recall from §2.4.3 that \(G_0\) is a simply connected non-compact real simple Lie group and \(K_0\) is a maximal compact subgroup of \(G_0\) such that rank \((G_0) = \text{rank } (K_0)\) and \(G_0/K_0\) is not Hermitian symmetric. Also recall that \(Y = K_0/L_0\) is an irreducible Hermitian symmetric space of the compact type with the non-compact dual \(X = K_0^*/L_0\). The \(\Delta_0^+\) is a positive system of \((l, t)\) with \(\Psi_l = \Psi \setminus \{\nu\}\) the set of simple roots and \(\Delta_0^+ \cup \Delta_2\) is a positive system of \((k, t)\) with \(\Psi_k = (\Psi \setminus \{\nu\}) \cup \{\epsilon\}\) the set of simple roots. The simple root \(\epsilon\) is the unique non-compact root in \(\Psi_k\). If \(w_0^k(\epsilon) = -\epsilon\), then \(w_0^Y(\Delta_0^+) = \Delta_0^-\), \(w_0^Y(\Delta_2) = \Delta_{-2}\) and \(w_0^Y(\Delta_0^+ \cup \Delta_2) = \Delta_0^- \cup \Delta_2\). This implies \(w_0^Y = \text{Id}\). Also \(w_0^Y(\epsilon) = -\epsilon\) implies \(w_0^Y(\epsilon^*) = -\epsilon^*\). Let \(\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subset \Delta_2\) be the maximal set of strongly orthogonal roots obtained as in §2.5. If \(\gamma + \rho_\theta\) is the Harish-Chandra parameter of a Borel-de Siebenthal discrete series representation \(\pi_{\gamma + \rho_\theta}\) of \(G_0\), then the \(K_0\) finite part \((\pi_{\gamma + \rho_\theta})_{K_0}\) of \(\pi_{\gamma + \rho_\theta}\) is isomorphic to \(\bigoplus_{m \geq 0} H^m(Y; \mathcal{E}_\gamma \otimes \mathcal{S}^m(u_{-1}))\). See Theorem 2.4.1. The \(L_0\) finite part \((\pi_{\gamma + \rho_\theta})_{L_0}\) of the associated holomorphic discrete series representation \(\pi_{\gamma + \rho_\theta}\) of \(K_0^*\) is isomorphic to \(E_\gamma \otimes \mathcal{S}^*(u_{-2})\). See §3.2 in Chapter 3.

In §7.1, we establish three lemmas which will be needed in the proof of Theorem 1.0.2. We shall use Littelmann’s path model described in §2.6 to prove these lemmas. Up to the end of proof of Lemma 7.1.3 we shall use the symbols \(\pi, \pi_\lambda\), etc., paths in the sense of Littelmann and are not to be confused with discrete series. The main result of this thesis Theorem 1.0.2 is proved in §7.2.

7.1 Branching rule using Littelmann’s path model

Recall from §2.6.1 that \(\pi_{\lambda}\) denotes the path \(t \mapsto t\lambda, 0 \leq t \leq 1\), for an integral weight \(\lambda\) of \(\mathfrak{t}\). If in addition \(\lambda\) is dominant, then \(w(\pi_{\lambda}) = \pi_{w\lambda}\) is an LS-path of shape \(\lambda\) for any element \(w\) in the Weyl group of \((\mathfrak{t}, \mathfrak{t})\). We also have the action of Littelmann’s root operator \(f_\alpha\) \((\alpha \in \Psi_\lambda)\) on the concatenation of two paths. See (2.8) in the Proposition 2.6.2.

We denote by \(V_\lambda\) (respectively \(E_\lambda\)), the finite dimensional irreducible representation of \(\mathfrak{t}\) (respectively \(\mathfrak{l}\)) with highest weight \(\lambda\) (respectively \(\kappa\)). If \(V\) is a \(\mathfrak{t}\)-representation, we shall denote by \(\text{Res}_\mathfrak{t}(V)\) its restriction to \(\mathfrak{l}\). Since \(\mathfrak{l}\) is a Levi subalgebra of \(\mathfrak{t}\), we have the
branching rule (2.10) of Resₗ(V).

Lemma 7.1.1 (i) The restriction Resₗ(Vₘₑ) to  of the irreducible \( \tau \)-representation \( Vₘₑ \) contains \( Resₗ(V(ₘ₋p)ₑ) \otimes \mathbb{C}₋pₑ \) for \( 0 \leq p \leq m \).

(ii) Suppose that \( w₀^{(1)}(Δ₀) = Δ₀ \). Then \( Resₗ(Vₘₑ) \) contains \( Resₗ(V(ₘ₋p)ₑ) \otimes \mathbb{C}₋pₑ \).

Proof: (i) Note that \( πₘₑ \) equals the concatenation \( πₘₑ \otimes \mathbb{C}₋pₑ \).

Let \( τ \) be an LS-path of shape \( (m-p)ₑ \) which is \( τ \)-dominant. Then \( τ = f₁ₘₑ \cdots fₐₘₑ \) for some sequence \( a₁, \ldots, aₗ \) of simple roots in \( Ψₗ \). Then \( f₁ₘₑ \cdots fₐₘₑ(πₘₑ) \neq 0 \) for \( 1 \leq i \leq q \). It follows that \( f₁ₘₑ \cdots fₐₘₑ(πₘₑ) = f₁ₘₑ \cdots fₐₘₑ(πₘₑ \otimes πₑ) = f₁ₘₑ \cdots fₐₘₑ(πₘₑ \otimes πₑ) * πₑ \) since \( e_a(πₑ) = 0 \). Thus we see that if \( τ \) is any \( τ \)-dominant LS-path of shape \( (m-p)ₑ \), then \( πₑ \) is an LS-path of shape \( mₑ \). It is clear that \( πₑ \) is \( \tau \)-dominant.

Since \( Eₑ(πₑ) = Eₑ(πₑ) \otimes \mathbb{C}₋pₑ \) and since for any path \( σ \), \( σ * πₑ \) implies \( σ = τ \), it follows that \( Resₗ(Vₘₑ) \) contains \( Resₗ(V(ₘ₋p)ₑ) \otimes \mathbb{C}₋pₑ \) in view of (2.10).

(ii) Suppose that \( w₀^{(1)}(Δ₀) = Δ₀ \). This is equivalent to the condition that \( w₀^{(1)}(ε) = ε \), in turn is equivalent to the requirement that \( Vₖₑ \) is self-dual as a \( τ \)-representation for all \( q \geq 1 \). Since \( Resₗ(V(ₘ₋p)ₑ) \otimes \mathbb{C}₋pₑ \) is contained in \( Vₘₑ \), so is its dual. That is, \( Resₗ(Vₘₑ) \otimes \mathbb{C}₋pₑ \) is contained in \( Resₗ(Vₘₑ) \).

Lemma 7.1.2 Let \( 0 \leq p₁ ≤ \cdots ≤ p₁ ≤ p₀ ≤ m \) be a sequence of integers. Then \( Resₗ(Vₘₑ) \) contains \( Eₗ \) where \( κ = mₑ + p₁γ₁ + \cdots + pₗγₗ \). Moreover, if \( w₀^{(1)}(Δ₀) = Δ₀ \), then \( Eₗ \) occurs in \( Resₗ(Vₘₑ) \) where \( λ = (m - 2p₀)ₑ + (\sum_{1 ≤ j ≤ l} p_jγ_{r+1-j}) \).


54
Now suppose \( w_\epsilon(\Delta_0) = \Delta_0 \). By Lemma 7.1.1, we have \( \text{Res}_i V_{\epsilon^*} \) contains \( \text{Res}_i V_{\epsilon^*} \otimes E_{(m-p_0)\epsilon^*} \). By what has been proved already \( \text{Res}_i V_{\epsilon^*} \) contains \( E_{p_1 \epsilon^* + p_1 \gamma_1 + p_2 \gamma_2 + \cdots + p_r \gamma_r} =: E \). Since \( V_{\epsilon^*} \) is self-dual, \( \text{Hom}(E, \mathbb{C}) \) is contained in \( \text{Res}_i V_{\epsilon^*} \). The highest weight of \( \text{Hom}(E, \mathbb{C}) = -p_0 \epsilon - \sum_{k \neq \gamma} \lambda_k w_k(\gamma_k) = -p_0 \epsilon - p_1 \gamma_1 - p_2 \gamma_2 - \cdots - p_r \gamma_r \), using Remark 2.5.4(i). Tensoring with \( E_{(m-p_0)\epsilon^*} \), we conclude that \( E \) occurs in \( \text{Res}_1 V_{\epsilon^*} \) with \( \lambda = (m - 2p_0)\epsilon - p_1 \gamma_1 - p_2 \gamma_2 - \cdots - p_r \gamma_r \).

Write \( \gamma = \gamma_0 + t \epsilon \) with \( \langle \gamma_0, \mu \rangle = 0 \). Then \( \gamma_0 \) is \( t \)-integral weight and \( t \) is an integer (\( \gamma \) being a \( t \)-integral weight). Also \( \gamma \) is \( t \)-dominant implies that \( \gamma_0 \) is \( t \)-dominant. Since \( \langle \gamma + \rho_t, \mu \rangle < 0 \), we have \( t < -2\langle \rho_t, \mu \rangle / ||\epsilon||^2 \). Assuming \( w_\epsilon(\epsilon) = -\epsilon \), we get \( \langle w_\gamma(\gamma_0), \alpha \rangle \geq 0 \) when \( \alpha \) is in \( \Delta_0^+ \) and \( \langle w_\gamma(\gamma_0), \epsilon \rangle = 0 \). So \( w_\gamma(\gamma_0) \) is \( t \)-dominant integral weight.

**Lemma 7.1.3** With the above notation, suppose that \( w_\epsilon(\epsilon) = -\epsilon \) and that \( E \) is a subrepresentation of \( \text{Res}_i (V_{\epsilon^*}) \). Then \( E_{\gamma_0 + w_\gamma(t)} \) is a subrepresentation of \( \text{Res}_i (V_{w_\gamma(\gamma_0) + \epsilon^*}) \).

**Proof:** Let \( \pi \) denote the path \( \pi_{\epsilon^*} * \pi_{w_\gamma(\gamma_0)} \). Then \( \text{Im}(\pi) \) is contained in the dominant Weyl chamber (of \( t \) and \( \pi(1) = w_\gamma(\gamma_0) + \epsilon^* \)). Since \( E \) is contained in \( \text{Res}_i (V_{\epsilon^*}) \), there exist a sequence \( \alpha_1, \ldots, \alpha_k \) of simple roots of \( t \) such that \( f_{a_1} \cdots f_{a_k}(\pi_{\epsilon^*}) =: \eta \) is \( t \)-dominant path with \( \eta(1) = \tau \). Since \( \pi_{w_\gamma(\gamma_0)} \) is \( t \)-dominant path, \( \theta := f_{a_1} \cdots f_{a_k}(\pi) = \eta * \pi_{w_\gamma(\gamma_0)} \), in view of (2.8). Clearly \( \theta \) is \( t \)-dominant and \( \theta(1) = \tau + w_\gamma(\gamma_0) \). Hence by the branching rule (2.10), \( E_{w_\gamma(\gamma_0) + \tau} \) occurs in \( \text{Res}_i (V_{w_\gamma(\gamma_0) + \epsilon^*}) \).

Let \( \Phi : K_0 \longrightarrow GL(V_{\delta_0}) \) be the representation, where \( \lambda_0 := w_\gamma(\gamma_0) + \epsilon^* \). Then \( \phi := d\Phi : t_\epsilon \longrightarrow \text{End}(V_{\delta_0}) \). For \( k \in K_0 \) and \( X \in t_\epsilon \), we have

\[
\Phi(k^{-1}) \circ \phi(X) \circ \Phi(k) = \phi(Ad(k^{-1})X) \tag{7.1}
\]

Let \( v \in V_{\delta_0} \) is a weight vector of weight \( \lambda := w_\gamma(\gamma_0) + \tau \) such that it is a highest weight vector of \( E_\lambda \). Now \( w_\gamma = (Ad(k)_\lambda) \) for some \( k \in N_{K_0}(T_0) \). Then \( \Phi(k)v \) is a weight vector of weight \( w_\lambda(\lambda) \) and it is killed by all root vectors \( X_\alpha \) (\( \alpha \in \Delta_0^+ \)), in view of (7.1); since \( w_\gamma(\Delta_0^+) = \Delta_0^+ \). Hence \( \Phi(k)v \) is a highest weight vector of an irreducible \( L_0 \)-submodule of \( \text{Res}_i (V_{\delta_0}) \). Therefore \( E_{w_\gamma(\lambda)} = E_{\gamma_0 + w_\gamma(t)} \) occurs in \( \text{Res}_i (V_{\delta_0}) \). 

### 7.2 Proof of Theorem 1.0.2

We are now ready to prove Theorem 1.0.2.

**Proof of Theorem 1.0.2:** Write \( \gamma = \gamma_0 + t \epsilon \) where \( \langle \gamma_0, \mu \rangle = 0 \).

We have

\[
(\pi_{\gamma + \gamma_0})_{\delta_0} = E_\gamma \otimes S^\epsilon(\mu_\gamma) = E_\gamma \otimes E_{a_1 \gamma_1 + \cdots + a_r \gamma_r} \]

where the sum is over all integers \( a_1 \geq \cdots \geq a_r \geq 0 \). (In view of Theorem 2.5.1). So \( (\pi_{\gamma + \gamma_0})_{\delta_0} \) contains \( E_{a_1 \gamma_1 + \cdots + a_r \gamma_r} \), for all integers \( a_1 \geq \cdots \geq a_r \geq 0 \).
Let \(k \geq 1\) be the least integer such that \(S^{k(u_{-1})}\) has one-dimensional \(L_0\)-subrepresentation, which is necessarily of the form \(E_{q\gamma}\) for some \(q < 0\). Now \((\pi_{\gamma+p_{\gamma}})_{k_0}\) contains \(\oplus_{j \geq 0} H^j(Y; \mathbb{E}_{\gamma+jq})\), by Theorem 2.4.1. By Borel-Weil-Bott theorem ([3], also see [6, Th. 1.6.8, Ch. 1]), \(H^j(Y; \mathbb{E}_{\gamma+jq})\) is an irreducible finite dimensional \(K_0\)-representation with highest weight \(w_1(\gamma + jq) + \rho_t - \rho_t = w_1(\gamma_0) + (-t - jq - c)e^*\) since \(w_1^0(e^*) = -e^*\), where \(\sum_{p \in \Delta_N} \beta = ce^*\) for some \(c \in \mathbb{N}\). Define \(m_j := \gamma - jq - c\) for all \(j \geq 0\). For \(0 \leq p_r \leq \cdots \leq p_1 \leq m_1, E_{m_1,e^*+p_1\gamma_1+\cdots+p_r\gamma_1}\) is a subrepresentation of \(\text{Res}_t(V_{m_1,e^*})\), in view of Lemma 7.1.2.

So by Lemma 7.1.3, \(E_{\gamma_0-m_r,e^*+\cdots+p_r\gamma_1}\) is a subrepresentation of \(\text{Res}_t(V_{w_1(\gamma_0)+m_1,e^*})\) since \(w_1(\gamma_0) = -\gamma_{r+1-j}\), for all \(1 \leq j \leq r\) by Remark 2.5.4(i). Now \(H^j(Y; \mathbb{E}_{\gamma+jq,e^*})\) is isomorphic to \(V_{w_1(\gamma_0)+m_1,e^*}\). So, for \(0 \leq p_r \leq \cdots \leq p_1 \leq m_1, E_{\gamma_0-m_r,e^*+\cdots+p_r\gamma_1}\) is an \(L_0\)-submodule of \(H^j(Y; \mathbb{E}_{\gamma+jq,e^*})\).

Fix \(a_1 \geq \cdots \geq a_r \geq 0\), where \(a_1, \ldots, a_r \in \mathbb{Z}\). In view of §4.1 and Lemma 4.3.2, \(q\) is odd when \(c\) is odd. Let \(N' = \{ j \in \mathbb{N} | (jq + c)\text{ is even}\}\). There exists \(j_0 \in \mathbb{N}\) such that for all \(j \in N'\) with \(j \geq j_0\), \(-(jq + c)/2 \geq a_i\). Define \(p_{r+1-i} := -(jq + c)/2 - a_i, 1 \leq i \leq r\). Then \(0 \leq p_r \leq \cdots \leq p_1 < m_j\).

Now \(\sum_{1 \leq i \leq r} p_i \gamma_{r+1-i} = \sum_{1 \leq i \leq r} p_i \gamma_i = \sum_{1 \leq i \leq r} (-a_i -(jq + c)/2) \gamma_i = (jq + c)e^* - \sum_{1 \leq i \leq r} a_i \gamma_i\), in view of Proposition 2.5.2(i), since \(w_1^0(e^*) = -e^*\) by hypothesis. It follows that \(\gamma_{0-m_r,e^*} = \sum_{1 \leq i \leq r} a_i \gamma_i\). So for all \(j \in N'\) with \(j \geq j_0\), \(E_{\gamma+\sum_{1 \leq i \leq r} a_i \gamma_i}\) is an \(L_0\)-submodule of \(H^j(Y; \mathbb{E}_{\gamma+jq,e^*})\). That is, for all integers \(a_1 \geq \cdots \geq a_r \geq 0\), the \(L_0\)-type \(E_{\gamma+\sum_{1 \leq i \leq r} a_i \gamma_i}\) occurs in \(\pi_{\gamma+p_{\gamma}}\) with infinite multiplicity.

In particular, if \(\gamma = t\nu^\gamma\), each \(L_0\)-type in \(\pi_{\gamma+p_{\gamma}}\) occurs in \(\pi_{\gamma+p_{\gamma}}\) with infinite multiplicity. This completes the proof.

There are three major obstacles in obtaining complete result in the non-quaternionic case. The first is the decomposition of \(S^m(u_{-1})\) into \(L_0\)-types \(E_{\lambda}\). Secondly, one has the problem of decomposing the tensor product \(E_{\gamma} \otimes E_\lambda\) into irreducible \(L_0\)-representations \(E_\kappa\). Finally, one has the restriction problem of decomposing the irreducible \(K_0\)-representation \(H^j(K_0/L_0; \mathbb{E}_c)\) into \(L_0\)-subrepresentations. The latter two problems can, in principle, be solved using the work of Littelmann [17]. The problem of detecting occurrence of an infinite family of common \(L_0\)-types in the general case appears to be intractable.

We conclude this thesis with the following questions:

**Questions:** Suppose that there exist infinitely many common \(L_0\)-types between a Borel-de Siebenthal discrete series representation \(\pi_{\gamma+p_{\gamma}}\) of \(K_0\) and the holomorphic discrete series representation \(\pi_{\gamma+p_{\gamma}}\) of \(K_0\). Then (i) Does there exist a one dimensional \(L_0\)-subrepresentation in \(S^m(u_{-1})\)? (ii) Is it true that \(w_1^0(\Delta_0) = \Delta_0\)?

56