Chapter 3

HOLOMORPHIC DISCRETE SERIES ASSOCIATED TO A BOREL-DE SIEBENTHAL DISCRETE SERIES

Unless explicitly stated, from here onwards we keep the notations of §2.4.3. In §3.1, we discuss the irreducible bounded symmetric domain dual to $Y = K_0/L_0 \cong K/K \cap Q$. In §3.2, we will see that for every Borel-de Siebenthal discrete series representation of $G_0$, there is a naturally associated holomorphic discrete series representation of $K^*_0$ which is the dual of $K_0$ in $K$.

3.1 Hermitian symmetric space dual to $Y$

Recall that $Y = K_0/L_0 = K_1/L_1$ is an irreducible Hermitian symmetric space of the compact type. Also recall that $\theta = \text{Ad}_K(\exp \frac{i\pi}{2} h_{\nu^*})$ and $(t_0, \theta_{|t_0})$ is an orthogonal symmetric Lie algebra of the compact type with $t_0$ the set of fixed points of $\theta_{|t_0}$. Notice that $\theta(t_1) \subset t_1$ and $t_1$ is the set of fixed points of $\theta_{|t_1}$. Hence $(t_1, \theta_{|t_1})$ is an irreducible orthogonal symmetric Lie algebra of the compact type and is associated with $Y$. Let $t^*_0 \subset t$ (respectively, $t^*_1 \subset t^*_1$) denote the non-compact real form of $t$ (respectively, $t^*_1$) dual to $(t_0, \theta_{|t_0})$ (respectively, $(t_1, \theta_{|t_1})$). We have $t^*_0 = t^*_1 \oplus t_2$. Let $K^*_0$ denote the connected Lie subgroup of $K$ with Lie algebra $t^*_0$ and $K^*_1$ the connected Lie subgroup of $K^*_0$ corresponding to the Lie subalgebra $t^*_1$. We have $K^*_0 = K^*_1 K_2$ and $X := K^*_0/L_0 = K^*_1/L_1$ (denoting $L_0$, $L_1$ by the same notation $L_0$ and similarly for $L_1$) is an irreducible Hermitian symmetric space of the non-compact type dual to $Y$.

A well-known result of Harish-Chandra (see [10, Ch. VIII] or §2.3.1) is that $X$ is naturally imbedded as a bounded symmetric domain in $u_2 = T_o(Y)$, the holomorphic tangent space at $o = eK_0$ of $Y$. Denote by $U_{u_2} \subset K$ the (unipotent) Lie subgroup of $K$ with Lie algebra $u_{u_2} \subset t$. Then the exponential map is a diffeomorphism from $u_{u_2}$ onto $U_{u_2}$. The image $U_2$ in $K/(L.U_{-2})$ is an open neighbourhood of $o$ in $K/(L.U_{-2}) \cong Y$. Thus $X$ is imbedded in $Y$ as an open complex analytic submanifold. See §2.3.1.
3.2 Holomorphic discrete series associated to a Borel-de Siebenthal discrete series

Recall that \( t = t_0^* \otimes_{\mathbb{R}} \mathbb{C} \) and that \( t \subset t \) is a Cartan subalgebra of \( t \). The sets of compact and non-compact roots of \( (t_0^*, t_0) \) are \( \Delta_0 \) and \( \Delta_2 \cup \Delta_3 \) respectively. The unique non-compact simple root of \( \Psi_t \) is \( \epsilon \in \Delta_2 \).

Note that the group \( K_0^* \) admits holomorphic discrete series. See §2.4.2 or [13, Theorem 6.6, Chapter VI]. The positive system \( \Delta_0^+ \) is a special positive system of \( (t, t) \) as in §2.4.2.

Let \( \gamma + \rho_\theta \) be the Harish-Chandra parameter for a Borel-de Siebenthal discrete series representation of \( G_0 \). Thus \( \gamma \) is the highest weight of an irreducible \( L_0 \)-representation and \( \langle \gamma + \rho_\theta, \beta \rangle < 0 \) for all \( \beta \in \Delta_1 \cup \Delta_2 \). Clearly \( \langle \gamma + \rho_\theta, \alpha \rangle > 0 \) for all positive compact roots \( \alpha \in \Delta_0^+ \). We claim that \( \langle \gamma + \rho_\theta, \beta \rangle < 0 \) for all positive non-compact roots \( \beta \in \Delta_2 \). To see this, let \( \beta_i \in \Delta_i, i = 1, 2 \). Observe that \( \beta_1 + \beta_2 \) is not a root and so \( \langle \beta_1, \beta_2 \rangle \geq 0 \). It follows that \( \langle \rho_\theta, \beta_2 \rangle = \langle \rho_\theta - 1/2 \sum_{\beta_i} \beta_i, \beta_2 \rangle = \langle \rho_\theta, \beta_2 \rangle - 1/2 \sum_{\beta_i} \langle \beta_1, \beta_2 \rangle \leq \langle \rho_\theta, \beta_2 \rangle \). So \( \langle \gamma + \rho_\theta, \beta \rangle \leq \langle \gamma + \rho_\theta, \beta \rangle < 0 \) for all \( \beta \in \Delta_2 \). Thus \( \gamma + \rho_\theta \) is the Harish-Chandra parameter for a holomorphic discrete series representation \( \pi_{\gamma + \rho_\theta} \) of \( K_0^* \), which is naturally associated to the Borel-de Siebenthal discrete series representation \( \pi_{\gamma + \rho_\theta} \) of \( G_0 \).

The \( L_0 \)-finite part of \( \pi_{\gamma + \rho_\theta} \) equals \( E_\gamma \otimes S^*(u_{-2}) \), where \( E_\gamma \) is the irreducible \( L_0 \)-representation with highest weight \( \gamma \) (see §2.4.2). Write \( \gamma = \lambda + \kappa \) where \( \lambda \) and \( \kappa \) are dominant weights of \( t_1^0 \) and \( t_0^0 \) respectively. We have \( E_\gamma = E_\lambda \otimes E_\kappa \). Hence \( (\pi_{\gamma + \rho_\theta})_{L_0} = E_\lambda \otimes (E_\kappa \otimes S^*(u_{-2})) = E_\lambda \otimes (\pi_{\lambda + \rho_\kappa})_{L_0} \), where \( \pi_{\lambda + \rho_\kappa} \) is the holomorphic discrete series representation of \( K_1^* \) with Harish-Chandra parameter \( \lambda + \rho_\kappa \).

We have \( (\pi_{\gamma + \rho_\theta})_{L_0} = E_\lambda \otimes (\pi_{\lambda + \rho_\kappa})_{L_0} \). Therefore \( \pi_{\gamma + \rho_\theta} \) is \( L_0' \)-admissible if and only if \( \pi_{\lambda + \rho_\kappa} \) is \( L_1' \)-admissible, where \( L_0' \) (respectively, \( L_1' \)) denote the connected Lie subgroup of \( L_0 \) (respectively, \( L_1 \)) corresponding to the semisimple ideal \([l_0, l_0] \) (respectively, \([l_1, l_1] \)) of \( l_0 \) (respectively \( l_1 \)). Since \( K_1 \) is simple, and since \( w_0^0(\epsilon) = w_{l_0}^0(\epsilon) \), it follows from the Proposition 2.5.3 of Chapter 2 that \( \pi_{\gamma + \rho_\theta} \) is \( L_0' \) admissible if and only if \( w_0^0(\epsilon) \neq -\epsilon \).