SECTION IV

A REVIEW OF WORK ON SCATTERING OF LIGHT BY THE EARTH'S ATMOSPHERE
4.1 Effect of curvature of the earth on the equivalent optical path

A beam of light travelling through the earth's atmosphere from the sun (S) to a point (P) in the atmosphere, suffers a loss of intensity due to scattering by the molecules in the path SP. The intensity of the beam at P is given by

\[ I = I_0 e^{-\zeta_{SP}} \]

where

\[ I_0 = \text{intensity of the incident radiation on the top of the atmosphere} \]

and \[ \zeta_{SP} = \text{equivalent optical path from the sun to the point P.} \]

\[ \zeta_{SP} \] depends upon the zenith angle of the sun (Z) i.e. the angle which SP makes with the vertical through P. It also depends on the curvature of the earth, the distribution of air density in the earth's atmosphere and the wave-length of light. For a plane-parallel infinite atmosphere of finite optical thickness, \[ \zeta_{SP} \] is sec Z times the vertical optical path above P. For \[ Z = 90^\circ \], \[ \zeta_{SP} = \infty \].

Chapman in 1931 derived an integral expression for the equivalent optical path for different scale heights taking into account the curvature of the earth. The function which was originally denoted by the symbol \( f(x, \chi) \) is now generally referred to as Chapman Function \( Ch(x, \chi) \) in the literature. \( x \) stands for \( (a + h)/H \) where \( a \) is the radius of the earth and \( h \), height of the point under consideration. \( \chi \) is the zenith distance of the sun. For \( x = 800 \) which corresponds to \( H = 8 \) Km
and \( z = 90^\circ \), \( \text{Ch}(x, \chi) \) has a value of 35.46. In a later paper, Chapman (1953) expressed \( \text{Ch}(x, \chi) \) in the form \( \sec(\chi - \Delta \chi) \) and tabulated the values of \( \Delta \chi \) for different values of \( x \) and \( \chi \). For \( x = 800 \) and \( \chi = 90^\circ \), \( \Delta \chi = 1^\circ 37' \). Recently, Wilkes (1954) has tabulated the Chapman function for values of \( x \) between 50 and 1000 Km in steps of 50 Km up to 500 Km and steps of 100 Km between 500 and 1000 Km and for \( \chi \) varying from 20° to 100° in steps of 1° when \( \text{Ch}(x, \chi) \) does not exceed 100. The values given are correct to three decimal places. These tables are of great help in computing air paths for different temperature distributions and angles of incidence. A simple but less accurate formula for the optical path was used by the present author (1956) in his calculations on the intensity and polarization of the sky during twilight.

When Bemperlid made his calculations in 1907, he assumed a temperature distribution with height in the atmosphere as it was then known. The values adopted by him cannot be accepted today. However, as it is the temperature of the air in the first few kilometers above the point \( P \) that exerts a dominant influence on the optical path \( \gamma_{sp} \), the Bemperlid functions may still be taken as reasonably accurate when the point is situated at heights up to 10 Km above ground.

Link and Sekera (1940) have considered the effect of atmospheric refraction and calculated the equivalent optical paths for two different temperature distributions (1) corresponding to winter and (2) summer. The density data used were based on the results of sounding balloon ascents and other atmospheric
data available at that time. The values of equivalent optical paths as given by different workers are summarised in Table 1.

**Table 1**

Equivalent optical path from the sun to the point P situated at the ground as given by different workers

<table>
<thead>
<tr>
<th>Z</th>
<th>Sec 2</th>
<th>Chapman, x =</th>
<th>Calculated H = 7</th>
<th>Link &amp; Sekera B*</th>
<th>Winter</th>
<th>Summer</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>60</td>
<td>2.000</td>
<td>1.992</td>
<td>1.993</td>
<td>1.994</td>
<td>1.99</td>
<td>2.00</td>
</tr>
<tr>
<td>75</td>
<td>3.864</td>
<td>3.791</td>
<td>3.800</td>
<td>3.812</td>
<td>3.80</td>
<td>3.82</td>
</tr>
<tr>
<td>80</td>
<td>5.759</td>
<td>5.525</td>
<td>5.551</td>
<td>5.572</td>
<td>5.56</td>
<td>5.60</td>
</tr>
<tr>
<td>85</td>
<td>11.474</td>
<td>10.008</td>
<td>10.144</td>
<td>10.257</td>
<td>10.28</td>
<td>10.39</td>
</tr>
<tr>
<td>90</td>
<td>\infty</td>
<td>33.177</td>
<td>35.466</td>
<td>37.615</td>
<td>37.81</td>
<td>39.65</td>
</tr>
</tbody>
</table>

B* Bempepd.

4.2 Scatter of light by a plane-parallel atmosphere

4.21 Work of Rayleigh, King and Tichanowsky

Lord Rayleigh worked out the effect of small dielectric spheres in scattering electromagnetic radiation. When the linear dimensions of the particle are small compared to the wave-length of the incident radiation, the secondary radiation is proportional to \( \lambda^{-4} \). It was recognised by Lord Rayleigh that the blue colour of the sky and its polarisation could be explained by the scattering of sunlight by the molecules in air.
According to Rayleigh's law of scattering, when the incident light is unpolarised natural light, the intensity of light scattered at an angle $\Theta$ to the incident radiation and confined in a small cone of solid angle $d\mathcal{W}$, is proportional to

$$(1 + \cos^2 \Theta) \, d\mathcal{W}$$

when this is integrated over all directions of $\mathcal{W}$, and since

$$(1 + \cos^2 \Theta) \, d\mathcal{W} = \frac{16 \pi}{3}$$

We have, the fraction of total incident radiation scattered in the cone $d\mathcal{W}$, given by

$$\frac{3 \sigma}{16 \pi} (1 + \cos^2 \Theta) \, d\mathcal{W}$$

where $\sigma$ is the scattering coefficient of the particle and is proportional to $\lambda^{-4}$.

If we take only primary scattering into account, the light scattered at right angles to an incident beam of unpolarised radiation would be completely polarised. Experiments show that there is a residual polarisation at right angles to the incident radiation. Lord Rayleigh explained this as being due to the anisotropy of the molecules. For pure air, laboratory experiments show a value of 0.042 for the depolarisation factor. The values of maximum polarisation observed on clear days are however significantly lower than those calculated on the basis of primary scattering by anisotropic molecules. Hence higher orders of scattering have to be taken into account as was realised by Lord Rayleigh himself.
L.V. King (1913) formulated the integral equations for the scattered radiation on the following assumptions:—

1. Owing to the random motions of the molecules in a gas, contributions to the total intensity of the scattered radiation by different molecules in an element of volume may be obtained by adding the intensities of light scattered by individual molecules.

2. Instead of considering the height of a point in the plane-parallel infinite atmosphere, its position is defined in terms of air density at that point. If the atmosphere is assumed to be isothermal with density decreasing exponentially with height, the formulae developed can be applied to any height distribution of mass.

3. Instead of treating the distribution of scattered radiation according to Rayleigh's phase law, an average value over a spherical surface was taken.

4. Anisotropy of the molecules, reflection by ground and refraction in the atmosphere were neglected.

The integral equation given by King consists of two parts (1) contribution to the intensity of sky light due to the sun's radiation which had been scattered once by the atmosphere and (2) contribution to "self-illumination" by higher orders of scattering. King found it difficult to evaluate the second part.
The complete solution of the problem of polarisation of sky radiation requires the solution of two simultaneous integral equations in three variables and as this was considered to be an almost impossible task at that time, King tried to calculate the polarisation on the assumption that primary scattered light was polarised according to Rayleigh's law and that "self-illumination" was unpolarised.

Tichanowsky (1927, 1928) formulated the equations for primary, secondary and tertiary scattered radiations in a plane-parallel stratified atmosphere taking into account the anisotropy of the molecules. The effect of loss of intensity due to scattering was neglected and hence the treatment can be assumed to apply only to weakly attenuated radiations in the red and infra-red region of the spectrum.

4.22 Theory of Chapman and Hammad

The equations for calculating the primary and secondary scattered radiations from the sky for any zenith angle of the sun above the horizon, were formulated by Chapman and Hammad (1939). The method is true for a plane-parallel atmosphere of finite optical thickness, scattering according to Rayleigh's law. The necessary functions for computing the primary and secondary scattered radiations were discussed fully in the original paper and their values were tabulated for some special cases by Hammad (1945, 1947 and 1948). The method is described in brief in this sub-section.

The height of a point is specified in terms of the
fraction of total air mass \( \frac{m}{m} \) of the atmosphere below it (level), or \( \bar{m} \) the fraction of the total mass above it (depth). \( m + \bar{m} = 1 \). A beam of incident radiation of intensity \( I_\infty \), falling on the top of the atmosphere in a direction \( \mathbf{a} \), suffers attenuation as it comes down the atmosphere. Its intensity at \( P \) (level \( m' \) and depth \( \bar{m}' \)) is given by

\[
I_\infty e^{-\bar{m}' \sec Z}
\]

where

\[
c = \sigma M = \text{scattering coefficient of the atmosphere}
\]

\[
\sigma = \text{scattering coefficient per unit mass of air}
\]

\[
M = \text{total air mass of the atmosphere}
\]

and \( Z = \text{angle which } \mathbf{a} \text{ makes with the vertical} \).

At \( P \), direct sunlight gets scattered along the direction \( k' \) making an angle \( \gamma' \) with \( \mathbf{a} \). The amount of energy scattered in a small cone of solid angle \( dk' \) is given by

\[
\frac{3 \sigma}{16 \pi} I_\infty (1 + \cos^2 \gamma') e^{-\bar{m}' \sec Z} \quad dk'
\]

In travelling from \( P \) to another point \( Q \) (level \( m \), depth \( \bar{m} \)), the primary scattered radiation undergoes a further attenuation given by

\[
-c(\bar{m} - \bar{m}') \sec \theta'
\]

where \( \theta' \) is the angle which \( k' \) makes with the vertical. The contribution to the intensity of primary scattered radiation passing from \( P \) through a surface normal to \( k' \) at \( Q \) is proportional to
At $Q$, let the primary scattered radiation given by (6) undergo a second scattering process in the direction of observation $k'$ such that the angle between $k'$ and $k$ is $\chi$. The secondary emission at $Q$ in the direction $k$ is proportional to

$$\frac{g \sigma_2}{256 \pi} I_\infty (1 + \cos^2 \psi') (1 + \cos^2 \chi) e^{-(cm') \sec Z} e^{-c(m \sim m')} \sec \theta'$$

Integrating over all directions around $Q$ and the air mass of the entire atmosphere, the total secondary emission $E_2$ at $Q$ in the direction $k$ is given by

$$E_2 = \frac{g \sigma_2 M}{256 \pi^2} I_\infty \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{2\pi} e^{-(cm') \sec Z} e^{-c(m \sim m')} \sec \theta'$$

$$x (1 + \cos^2 \psi') (1 + \cos^2 \chi) \sec \theta' \sin \theta' \, dm' \, d\theta' \, d\phi'$$

$\phi'$ is the azimuth with respect to the vertical plane through the sun and the observer. Since the exponential terms are independent of $\phi'$, the integration over $\phi'$ can be easily carried out taking $\theta'$ and $m'$ as constant. The solution of the integral
\[
\frac{1}{\pi} \int_0^{2\pi} (1 + \cos^2 \psi') (1 + \cos^2 \chi) \, d\phi'
\]
can be written in the form

\[A_0 + A_1 \cos^2 \theta' + A_2 \cos^4 \theta' \]

where \(A_0\), \(A_1\) and \(A_2\) are the functions of direction of incident radiation \((Z, 0)\) and the direction of secondary scattered radiation \((\theta, \phi')\). Please see Table 2. Substituting this in eq. (8),

\[
E_2 = \frac{9 \sigma^2 M}{256 \pi} \int \int e^{-cm' \sec \theta} \left( -c (m \sim m') \sec \theta' \sec \theta \right) \sin \theta' \sin \theta' \, dm' \, d\theta'
\]

changing the variable from \(\theta'\) to \(\sec \theta'\) and as

\[d \sec \theta' = \sec \theta' \tan \theta' \, d\theta'\]

we have,

\[
E_2 = \frac{9 \sigma^2 M}{256 \pi} \int \int e^{-cm' \sec \theta} \left( -c (m \sim m') \sec \theta' \sec \theta \right) \sin \theta' \sin \theta' \, dm' \, d\sec \theta'
\]

\[
x \left\{ A_0 (\sec \theta')^{-1} + A_1 (\sec \theta')^{-3} + A_2 (\sec \theta')^{-5} \right\} \cdot x
\]

\[
x \, \, dm' \, d\sec \theta'
\]
Integration in eq. (11) is replaced by $L_n$ functions ($n = 1, 3$ and $5$) which depend upon the scattering coefficient of the atmosphere, solar zenith angle and the position of $Q$. Three consecutive odd values of $n$ occur, because of the consecutive negative odd powers of $\sec \theta'$.

$$E_2 = \frac{2 \sigma^2 M}{256 \pi} \int_0 (A_0 L_1 + A_1 L_3 + A_2 L_5)$$

$L_n$ can be expressed in the form

$$L_n (c, z, m) = \frac{1}{c} e^{-cm} \sec Z \times$$

$$\times \left\{ E_{kn} (c_m, -\sec Z) + E_{kn} (c_m, \sec Z) \right\}$$

where the first term in the bracket represents the contribution to $E_2$ at $Q$ from the part of the atmosphere below level $m$ and the second term corresponds to the contribution from the part of the atmosphere above level $m$.

$$E_{kn} (\tau, b) = \int_{0}^{\tau} e^{bt} E_{jn} (t) \, dt$$

where

$$E_{jn} (t) = \int_{-\infty}^{0} e^{-tx} x^{-n}$$

shows that the contribution $E_{kn}$ is obtained by two steps: (1) integrating over $\sec \theta'$ from $1$ to $\infty$ keeping $m'$ constant and (2) then summing up for different levels above or below.
m as the case may be. \( E_{jn} \) functions have been tabulated by

1. Gold (1908) \( n = 1 \) to 3 and several values of \( t \) from 0 to 6

2. Hammad (1947) \( n = 1 \) to 5 and \( t = 0.00 \) to 1.00 at an interval of 0.01

3. Placzek (1947) \( n = 1 \) to 20, and \( t = 0.00 \) to 2.00 at an interval of 0.01, and \( t = 2.0 \) to 10.0 at an interval of 0.1.

Total secondary scattered radiation \( (R_2) \) received by the observer at ground level, can be obtained by integrating \( E_2 \) over \( dm \). We have,

\[
R_2 = \frac{9e^2}{256\pi} \lambda_0 \sec \theta \left\{ A_0 \, Ld_1 + A_1 \, Ld_3 + A_2 \, Ld_5 \right\}
\]

where

\[
Ld_n = \frac{-c \sec \theta}{e} \left\{ \int_0^1 \frac{-cm \sec \theta}{e} \, L_n (c, Z, m) \, dm \right\}
\]

In practice, \( Ld_n \) functions are calculated starting from the values of \( E_{jn} \) and dividing the atmosphere into a number of layers containing equal air masses.

The functions \( A_0, A_1 \) and \( A_2 \) occurring in the above formulae as originally given by Chapman and Hammad (1939).
IV.12

assume that the primary scattered radiation is unpolarised. Eq. (7). This is not correct. This was corrected by Hammad (1948) while considering the polarisation of secondary scattered radiation and also independently by Sakihara (1951). Their values are given in Table 2 in terms of $Z$ for the direction of observation along the zenith ($\theta = 0^\circ, \phi = 0^\circ$).

**Table 2**

Functions $A_0$, $A_1$ and $A_2$ for $\theta = 0^\circ, \phi = 0^\circ$

<table>
<thead>
<tr>
<th>$A_r$</th>
<th>Chapman and Hammad (1939)</th>
<th>Hammad (1948)</th>
<th>Sakihara (1951)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Normal</td>
<td>Transverse</td>
<td></td>
</tr>
<tr>
<td>$A_0$</td>
<td>$2 + \sin^2 Z$</td>
<td>$2 - \frac{1}{2} \sin^2 Z$</td>
<td>$2 - \frac{3}{2} \sin^2 Z$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$4 - 2 \sin^2 Z$</td>
<td>$3 \sin^2 Z$</td>
<td>$\sin^2 Z$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$2 - 3 \sin^2 Z$</td>
<td>$2 - \frac{5}{2} \sin^2 Z$</td>
<td>$2 - \frac{7}{2} \sin^2 Z$</td>
</tr>
</tbody>
</table>

Hammad (1948) modified the original theory so as to include the polarisation of the scattered light and the anisotropy of the molecules. This modified theory is given briefly in the paper presented in article 1.2 of Section I. Hammad (1953) showed that the values of polarisation and the positions of neutral points calculated on the basis of his theory, were in close agreement with the values observed on clear skies.
4.23 Chandrasekhar's theory of radiative equilibrium

The problem of sky illumination taking into consideration all orders of scattering and reflection by ground according to a definite law, has been solved by Chandrasekhar. The necessary corrections for molecular anisotropy have also been dealt with. The theory is described fully by Chandrasekhar in his book on "Radiative Transfer" (1950). The theoretical values of polarisation, and positions of neutral points and neutral lines are presented by Chandrasekhar and Elbert (1951) for $\gamma_1 = 0.15$. A brief summary with the necessary tables to compute the intensity and polarisation of the sunlit sky at any point for visible and near ultra-violet parts of the spectrum, have been given by Chandrasekhar and Elbert (1954). Coulson (1954) has computed the positions of neutral points for different values of $\gamma_1$ such as 0.02, 0.15, 0.25 and 1.00. He has shown the existence of double Arago point for large optical thicknesses. The effect of ground reflection on the positions of neutral points, has also been considered. In this sub-section, the summary of the theory is given keeping in view the problem of sky illumination.

4.23a Stokes parameters

A beam of unpolarised natural light on scattering, gets polarised and for complete representation of the radiation scattered in any direction, it is essential to know its intensity, direction of polarisation and degree of
polarisation. These quantities are different in nature and two or more independently polarised beams cannot be combined together by simple addition. This necessitates a search for a new representation and it has been observed by Chandrasekhar that the Stokes parameters satisfy all the required conditions. An elliptically polarised beam of light can be fully represented by four parameters and a partially polarised beam of light by three. Let \( I_1 \) and \( r \) be two directions at right angles to each other and \( I_1 \) and \( I_r \) represent the intensity along these two directions. \( I(\Psi) \), the intensity in a direction at an angle \( \Psi \) to \( l \) (measured clockwise) is given by

\[
I(\Psi) = I_1 \cos^2 \Psi + I_r \sin^2 \Psi + \frac{1}{2} U \sin 2\Psi \quad \ldots 18
\]

The coefficients \( I_1 \), \( I_r \) and \( U \) are called the Stokes parameters. In terms of these, the angle \( \chi \) which the direction of polarisation makes with \( l \) is given by

\[
\tan 2\chi = \frac{U}{I_1 - I_r} \quad \ldots 19
\]

and the degree of polarisation by

\[
\delta = \frac{I_1 - I_r}{I_1 + I_r} \sec 2\chi \quad \ldots 20
\]

A law of transformation of Stokes parameters for a rotation of the axis through an angle \( \phi \) can be easily derived and is given below in its final form.
\( \mathbf{I} = (I_1, I_r, U) \) denote the components of a partially polarised beam of light, then the effect of rotation of the axis through an angle \( \phi \) in a clockwise direction is to subject \( \mathbf{I} \) to the linear transformation

\[
\mathbf{I}(\phi) = \begin{pmatrix}
\cos^2 \phi & \sin^2 \phi & \frac{1}{2} \sin 2\phi \\
\sin^2 \phi & \cos^2 \phi & -\frac{1}{2} \sin 2\phi \\
-\sin 2\phi & \sin 2\phi & \cos 2\phi
\end{pmatrix}
\]

E.g. \( U \phi = -I_1 \sin 2\phi + I_r \sin 2\phi + U \cos 2\phi \)

4.23b Rayleigh's law in terms of phase matrix

In order to relate the Stokes parameters of the scattered light to those of the incident radiation, Rayleigh's law of scattering can be enunciated as follows:

If \( \mathbf{I} (I_1, I_r, U) \) represent the incident light, the scattered intensity in the direction \( \Theta \) is given by

\[
\sigma \frac{d\omega}{4\pi} R \mathbf{I} d\omega
\]

where

\[
R = \frac{3}{2} \begin{pmatrix}
\cos^2 \Theta & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \cos \Theta
\end{pmatrix}
\]

\( R \) is called the phase matrix for Rayleigh's scattering and \( \sigma \) the scattering coefficient per particle.
Absorption coefficient

A pencil of radiation of intensity $I$ after traversing a distance $ds$ in a medium will be weakened by

$$dI = - K \rho I ds$$

where $\rho$ is the density of the medium and $K$ the mass absorption coefficient for radiation of frequency $\nu$. The loss $dI$ may reappear partly or fully in other directions as scattered light.

Considering the case of scattering, a material is said to have a mass scattering coefficient $K_s$, if from a pencil of radiation incident on an element of mass of cross-section $d\sigma$ and height $ds$, energy scattered from it at the rate of

$$K_s \rho \sigma \nu \sigma \nu \sigma \omega$$

in all the directions. But $dm = \rho \sigma \nu \sigma \nu \omega ds$, therefore, energy scattered at the rate of

$$K_s \nu \nu \sigma \sigma dm \nu \sigma \sigma \omega$$

On considering the angular distribution of the scattered radiation according to a phase function $P(\cos \theta)$,

$$K_s \nu \nu P(\cos \theta) \frac{d\omega}{4\pi} dm \nu \sigma \omega$$

gives the rate at which energy is being scattered into an element of solid angle $d\omega$ and in a direction at an angle $\theta$. 
to the direction of radiation incident on an element of mass $dm$.

Total loss of energy in all the directions, from the incident beam is given by

$$K_{\nu} \cdot I_{\nu} \cdot dm \cdot d\nu \cdot d\Omega \int P(\cos \theta) \frac{d\omega'}{4\pi}$$

...25

For scattering according to Rayleigh's law,

$$P(\cos \theta) = \frac{3}{4} (1 + \cos^2 \theta)$$

...26

4.23d Emission coefficient

The emission coefficient $j_{\nu}$ is defined in such a way that an element of mass $dm$ emits in the directions confined to solid angle $d\Omega$, in the frequency interval $(\nu, \nu + d\nu)$ and in time $dt$, an amount of radiant energy given by

$$j_{\nu} \cdot dm \cdot d\Omega \cdot d\nu \cdot dt.$$ 

In the case of a medium which scatters radiation, there will be a contribution to the emission coefficient from the scattering of radiation from all the other directions into the pencil of direction considered. Thus it follows from (24) that the scattering of a pencil of radiation from the direction $(\theta', \phi')$ contributes to a pencil in the direction $(\theta, \phi)$ energy at the rate of
\[ K_{\gamma} \cdot \Delta m \cdot dv \cdot d\Omega \cdot P(\theta, \phi; \theta', \phi') \cdot I_\gamma(\theta', \phi') \times \]
\[ \frac{\sin \theta' \cdot d\theta' \cdot d\phi'}{4\pi} \]

where \( P(\theta, \phi; \theta', \phi') \) denotes the phase function for the angle between the directions \((\theta, \phi)\) and \((\theta', \phi')\). Hence the contribution \( j(s) \) to the emission coefficient by scattering alone is
\[ j(s) = K_{\gamma} \cdot \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} P(\theta, \phi; \theta', \phi') \times \]
\[ \frac{I_\gamma(\theta', \phi') \cdot \sin \theta' \cdot d\theta' \cdot d\phi'}{4\pi} \]

For a scattering atmosphere,
\[ j = j(s) \]

4.23e Source function

The ratio of emission to the absorption coefficient which plays an important part in the theory of radiative transfer is called the source function \( J_\gamma \). For a scattering atmosphere,
\[ J_\gamma(\theta, \phi) = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} P(\theta, \phi; \theta', \phi') \times \]
\[ \frac{I_\gamma(\theta', \phi') \cdot \sin \theta' \cdot d\theta' \cdot d\phi'}{4\pi} \]

4.23f Equation of transfer

With these definitions, the equation of transfer can be written down as follows: In an element of cross-section
dσ and height ds, the difference in radiant energy in the frequency interval \((\nu, \nu + d\nu)\) crossing the two faces normally, in a given time dt, and confined to an element of solid angle, is given by

\[
\frac{dI_\nu}{ds} \cdot ds \cdot d\nu \cdot d\sigma \cdot d\Omega \cdot dt.
\]

This difference in energy must arise from the excess of emission over absorption. The amount of energy absorbed is

\[
\int K_\nu \cdot \rho \cdot ds \cdot I_\nu \cdot d\nu \cdot d\sigma \cdot d\Omega \cdot dt
\]

while the amount of energy emitted is

\[
j_\nu \cdot \rho \cdot d\sigma \cdot ds \cdot d\nu \cdot d\Omega \cdot dt.
\]

Therefore,

\[
\frac{dI_\nu}{ds} = - K_\nu \cdot \rho \cdot I_\nu + j_\nu \cdot \rho
\]

or in the terms of source function \(J_\nu\) (article 4.23e)

\[
- \frac{dI_\nu}{K_\nu \cdot \rho \cdot ds} = I_\nu - J_\nu
\]

This is the equation of transfer.

For a plane-parallel atmosphere, it is convenient to measure linear distances normal to the plane of stratification. If \(Z\) is this vertical distance, the equation of transfer becomes, on suppressing the suffix \(\nu\),

\[
- \frac{dI}{ds} = I - J
\]
\[ \frac{-\cos \theta \cdot dI (Z, \theta, \phi)}{K \cdot \rho \cdot dZ} = I (Z, \theta, \phi) - J (Z, \theta, \phi) \]

where \( \theta \) denotes the inclination to the outward normal and \( \phi \) the azimuth referred to suitably chosen axes. Introducing

\[ \int_{Z}^{\infty} K \cdot \rho \cdot dZ = \gamma \]

and \( \mu = \cos \theta \)

the equation of transfer for a plane-parallel atmosphere can be written as

\[ \mu \frac{dI (\tau, \mu, \phi)}{d\tau} = I (\tau, \mu, \phi) - J (\tau, \mu, \phi) \quad \ldots 30 \]

The evaluation of the source function \( J (\theta, \phi) \) can be carried out with the help of Eqs. 21 and 22. The contribution \( dI (\theta, \phi ; \theta', \phi') \) to the source function arising from the scattering of a pencil of radiation of solid angle \( d\omega' \) in the direction \( (\theta', \phi') \) will be given by

\[ R \cdot \frac{1}{4\pi} \frac{d\omega'}{d\tau} \]

if \( I (\theta', \phi') \) is referred to the direction parallel and perpendicular to the plane of scattering, but for a plane-parallel atmosphere, \( I (\theta', \phi') \) refers to a vertical axis and a plane, therefore it becomes necessary to multiply by appropriate phase matrices for the rotation of axes at two places. The resultant source function \( P (\mu, \phi ; \mu', \phi') \)
can be expressed in the form

\[ P(\mu, \phi; \mu', \phi') = Q \left[ P^{(0)}(\mu, \mu') + (1 - \mu^2) \frac{1}{2} \left(1 - \mu'^2\right) P^{(1)}(\mu, \phi; \mu', \phi') + \right. \]

\[ \left. + P^{(2)}(\mu, \phi; \mu', \phi') \right] \]

where

\[ Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \]

\[ P^{(0)}(\mu, \mu') = \frac{3}{4} \begin{bmatrix} 2(1 - \mu^2)(1 - \mu'^2) + \mu^2 \mu'^2 & \mu^2 & 0 \\ \mu'^2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ P^{(1)}(\mu, \phi; \mu', \phi') = \frac{3}{4} \begin{bmatrix} 4 \mu \mu' \cos(\phi' - \phi) & 0 & 2 \mu \sin(\phi' - \phi) \\ 0 & 0 & 0 \\ -2 \mu' \sin(\phi' - \phi) & 0 & \cos(\phi' - \phi) \end{bmatrix} \]

and

\[ P^{(2)}(\mu, \phi; \mu', \phi') = \]
and the equation of transfer Eq. (30) for a plane-parallel atmosphere can be written in the form

\[ \mu \frac{dI(\tau; \mu, \phi)}{d\tau} = I(\tau; \mu, \phi) - \]

\[ - \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P(\mu, \phi; \mu', \phi') I(\tau; \mu', \phi') d\mu' d\phi' \]

It should be noted that \( I(\tau; \mu', \phi') \) characterises the diffused radiation but in addition there is a reduced incident flux \( \eta F e^{-\tau/\mu_0} \) at the level \( \tau \), in the direction \((-\mu_0, \phi_0)\) and hence the equation of transfer appropriate for the problem of diffused reflection and transmission has the form

\[ \mu \frac{dI(\tau; \mu, \phi)}{d\tau} = I(\tau; \mu, \phi) - \]

\[ - \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P(\mu, \phi; \mu', \phi') I(\tau; \mu', \phi') d\mu' d\phi' - \]

\[ - \frac{1}{4} F e^{-\tau/\mu_0} P(\mu, \phi; -\mu_0, \phi_0) \]
The boundary conditions for solving this equation are

\[
\begin{align*}
I(0; -\mu, \phi) &= 0 \\
I(\tau; \mu, \phi) &= 0
\end{align*}
\]

\(0 < \mu < 1, \; 0 \leq \phi \leq 2\pi\)

\[\ldots 38\]

i.e. there is no diffused radiation falling on the top or from below the bottom of the atmosphere. The case of reflection by ground will be taken up in article 4.23i.

4.23g **Modification for the molecular anisotropy**

The scattering of a partially plane polarised beam of light by an anisotropic particle may be regarded as equivalent to a superposition of Rayleigh's scattering with a weight \((1 - \gamma)/(1 + 2\gamma)\) and isotropic scattering of each of the components \(I_1\) and \(I_r\) with a weight \(3\gamma/2 (1 + 2\gamma)\) where \(\gamma\) is given by \(d = 2\gamma/(1 + \gamma)\). This means that the following phase matrix should be used in the Eq. (37) for \(I=(I_1, I_r, U)\).

\[
\frac{1 - \gamma}{1 + 2\gamma} P(\mu, \phi; \mu', \phi') + \frac{3\gamma}{2(1 + 2\gamma)} \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[\ldots 39\]

4.23h **Solution of the equation of transfer**

It has been found convenient to express the intensity reflected at the top of the atmosphere in the direction \((\mu, \phi)\) by
and the intensity transmitted below the layer $\tau_1$ by

$$ I (\tau_1; -\mu, \phi) = \frac{1}{4\mu} T (\mu, \phi; \mu_0, \phi_0) $$

where $S$ and $T$ are called scattering and transmitting matrices respectively. They are functions of $\mu$, $\phi$, and $\mu_0$, $\phi_0$. $\tau_1$ is taken as a parameter. In terms of these matrices, and with the boundary conditions given in Eq. (38), the solution of equation of transfer (37) becomes easy.

The nature of $F (\mu, \phi; \mu', \phi')$ suggests that

$$ I (\tau; \mu, \phi) = I^{(0)} (\tau; \mu) + (1 - \mu^2)^{1/2} (1 - \mu_0^2)^{1/2} I^{(1)} (\tau; \mu, \phi) + I^{(2)} (\tau; \mu, \phi) $$

The solutions of $I^{(1)}$ and $I^{(2)}$ suggest that if transmitting function is expressed in the form
\[
T(\mu, \phi; \mu_o, \phi_o) = a \left[ \frac{3}{4} T^{(0)}(\mu; \mu_o) + (1 - \mu^2)^{\frac{3}{2}} (1 - \mu_o^2)^{\frac{3}{2}} T^{(1)}(\mu,\phi; \mu_o, \phi_o) + T^{(2)}(\mu, \phi; \mu_o, \phi_o) \right] \]

\[T^{(1)} \text{ is expressible in the form}\]

\[\left( \frac{1}{\mu_o} - \frac{1}{\mu} \right) T^{(1)} = P^{(1)}(-\mu, \phi; -\mu_o, \phi_o) \times \]

\[\left[ Y^{(1)}(\mu) X^{(1)}(\mu) - X^{(1)}(\mu) Y^{(1)}(\mu) \right] \]

\[(i = 1, 2) \]

\[\Psi^{(1)}(\mu) = \frac{3}{8} (1 - \mu^2) (1 + 2 \mu^2) \] and \[\Psi^{(2)}(\mu) = \frac{3}{16} (1 + \mu^2)^2 \]

where \(X^{(1)}, Y^{(1)} \) and \(X^{(2)}, Y^{(2)}\) are defined in terms of characteristic functions.

\[X \text{ and } Y \text{ functions are solved in Chapter VIII of "Radiative Transfer".} \]

On the other hand, the solution of azimuth independent term \(T^{(0)}\) is much more complicated and is given by

\[\left( \frac{1}{\mu_o} - \frac{1}{\mu} \right) T^{(0)}(\mu, \mu_o) = \]
where the eight functions \( \psi, \xi, \phi, \gamma, \lambda, \sigma, \eta, \theta \) are defined in terms of \( x_1, y_1 \) and \( x_r, y_r \) and the constants \( \gamma_1, \gamma_2, \gamma_3, \gamma_4, u_3, u_4 \) and \( Q \) whose values can be obtained in terms of the moments of \( x_1, y_1 \) and \( x_r, y_r \) functions. These are defined in terms of the characteristic functions

\[
\psi_1(\mu) = \frac{3}{4} (1 - \mu^2) \quad \text{and} \quad \psi_r(\mu) = \frac{3}{8} (1 - \mu^2)
\]

Substituting \( F = (F_1, F_r, U) = (\frac{1}{2}F, \frac{1}{2}F, 0) \) in Eq. (41) because the incident light is natural, the equations of \( I_1 \) etc. corresponding to \( (\zeta_1; -\mu, \phi; \lambda, \theta) \) can be written down from the Eqs. (43, 44 and 46). Using notations of Coulson (1954),
\[ I_1 = C \left\{ K \xi (\mu) + L \eta (\mu) - M \Psi (\mu) - N \phi (\mu) \\
+ \mu (1 - \mu^2)^{1/2} G \cos (\phi - \phi) \\
- \mu^2 H \cos 2 (\phi - \phi) \right\} \]

\[ I_r = C \left\{ K \sigma (\mu) + L \theta (\mu) - M \chi (\mu) - N \gamma (\mu) \\
+ H \cos 2 (\phi - \phi) \right\} \]

and

\[ U = C \left\{ (1 - \mu^2)^{1/2} G \sin (\phi - \phi) - 2 \mu H \sin 2 (\phi - \phi) \right\} \]

where

\[ C = \frac{3}{32} \frac{\rho_{\infty} F}{\mu - \mu^*} \]

\[ G = 4 \mu_0 (1 - \mu_0^2)^{1/2} \left\{ X(1) (\mu) Y(1) (\mu) - Y(1) (\mu) X(1) (\mu) \right\} \]

\[ H = (1 - \mu^2) \left\{ X(2) (\mu) Y(2) (\mu) - Y(2) (\mu) X(2) (\mu) \right\} \]

\[ K = \Psi (\mu) + \chi (\mu) \]

\[ M = \xi (\mu) + \sigma (\mu) \]

\[ L = 2 \left\{ \phi (\mu) + \eta (\mu) \right\} \]

\[ N = 2 \left\{ \theta (\mu) + \gamma (\mu) \right\} \]

4.23i **The effect of reflection by ground**

The equations of \( I_1, I_r \) and \( U \) given by Eq. (48) are derived on the assumption that the ground is a perfect absorber Eq. (38). In practice, however, the ground always
reflects a part of the total radiation falling on it. It can be assumed that ground reflects according to Lambert's law, i.e., the light reflected by ground is unpolarised and uniform in the outward direction independently of the state of incident flux. Further, the outward flux of the reflected light is always a certain fixed fraction \( \lambda_o \) (albedo) of the total inward flux of radiation falling on the surface of the earth.

The total inward flux of radiation falling on the ground consists of three parts:

1. Directly transmitted light \( \Pi F e \)
2. Diffusely transmitted light \( I \left( \tau_1; -\mu, \phi \right) \)
3. The flux of radiation re-reflected by the atmosphere towards the ground.

It has been shown that the effect of ground reflection is to increase \( I \left( \tau_1; -\mu, \phi \right) \) by an amount

\[
I^* \left( \tau_1; -\mu, \phi \right) = I \left( \tau_1; -\mu, \phi \right) + \lambda_o \left( \gamma_1 \left( \mu \right) + \gamma_x \left( \mu \right) \right)
\]

where

\[
c^* = \frac{\lambda_o F}{\left( 1 - \lambda_o \right) \beta} \lambda_o \left( \gamma_1 \left( \mu_o \right) + \gamma_x \left( \mu_o \right) \right)
\]

... 50

... 51
### References

<table>
<thead>
<tr>
<th>Reference</th>
<th>Year</th>
<th>Journal/Title</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Author(s)</td>
<td>Year</td>
</tr>
<tr>
<td>---</td>
<td>----------------------</td>
<td>------</td>
</tr>
<tr>
<td>13</td>
<td>Hammad, A.</td>
<td>1947</td>
</tr>
<tr>
<td>14</td>
<td>Hammad, A.</td>
<td>1948</td>
</tr>
<tr>
<td>17</td>
<td>King, L.V.</td>
<td>1913</td>
</tr>
<tr>
<td>20</td>
<td>Sakihara, K.</td>
<td>1951</td>
</tr>
<tr>
<td>22</td>
<td>Tichanowsky, J.J.</td>
<td>1928</td>
</tr>
</tbody>
</table>