Chapter:-1
EXTENDED TOTAL DOMINATION
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It is clear from the definition of domination that a dominating set exists in any graph. However it is not true for total domination. A totally dominating set does not exist in a graph having isolated vertices. Moreover when a vertex is removed from the graph the resulting graph may have the isolated vertices. Considering this fact it is desirable to have the concept which is approximately same as total domination and it can be define for any graph. To accomplish this we introduce the concept of so called extended total domination.

In this chapter we introduce the concept of extended total domination and relevant concepts. In particular we define minimum extended totally dominating set and extended total domination number. This chapter is devoted to characterize those vertices whose removal increases, decreases or does not change the extended total domination number of a graph. We prove that if the extended total domination number changes whenever any vertex is removed then it decreases when any vertex is removed.

We may mention that a totally dominating set is assumed to have at least two vertices and all our graphs are simple.

In this chapter I will denote the set of all isolated vertices of $G$.

**Definition-1.1: Totally Dominating Set.**[2]

Let $G$ be graph and $S$ be a set of vertices. Then $S$ is said to be a totally dominating set if every $v$ in $V(G)$ is adjacent to some vertex of $S$.

We introduced the following definition.

**Definition-1.2: Extended Totally Dominating Set.**

A set $S$ is said to be an extended totally dominating set if $S = S_1 \cup I$, where $S_1$ is totally dominating set in $G - I$. 

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**Definition-1.3: Minimal Extended Totally Dominating Set.**

An extended totally dominating set $S$ is said to be minimal extended totally dominating set if every $v \in S$ then $S - \{v\}$ is not an extended totally dominating set.

**Definition-1.4: Total Private Neighborhood.**

Let $S \subseteq V(G)$. Then the total private neighborhood of $v$ with respect to $S$ is $T_{pn}[v,S] = \{w \in V(G) : N(w) \cap S = \{v\}\}$.

**Definition-1.5: Minimum Extended Totally Dominating Set.**

An extended totally dominating set with smallest cardinality is said to be minimum extended totally dominating set. It is denoted by $\gamma_{Te}$ set of graph the $G$.

**Definition-1.6: Extended Total Domination Number.**

The number of vertices in a minimum extended totally dominating set is called extended total domination number of the graph $G$. It is denoted by $\gamma_{Te}(G)$.

**Example-1.7** We give an example of graph whose domination number and extended total domination number are different.

\[
\begin{array}{c}
1 & 2 \\
\bigcirc & \bigcirc \\
\bigcirc & \bigcirc
\end{array}
\]

**Figure-1.1**

$\gamma_{set} = \{2,3\}$ and $\gamma(G) = 2$

$\gamma_{Te} = \{1,2,3\}$ and $\gamma_{Te}(G) = 3$. 
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Note: It may be noted that a set $S$ is an extended totally dominating set if and only if every vertex $v$ of $G$ is either isolated or adjacent to some vertex of $S$.

We characterizes those vertices whose removal increases, decreases or does not change the extended total domination number of the graph. For this purpose we will define three types of sets as follows. $V^+_{Te}$, $V^-_{Te}$, and $V^0_{Te}$.

$$V^+_{Te} = \{v \in V(G): \gamma_{Te}(G-v) > \gamma_{Te}(G)\}.$$ $$V^-_{Te} = \{v \in V(G): \gamma_{Te}(G-v) < \gamma_{Te}(G)\}.$$ $$V^0_{Te} = \{v \in V(G): \gamma_{Te}(G-v) = \gamma_{Te}(G)\}.$$ Obviously all the three sets are mutually disjoint and their union is $V(G)$. 

\[
\begin{align*}
\gamma_{Te} \text{ set } &= \{a, c, d\} , \quad \gamma_{Te}(G) = 3 \\
\gamma \text{ set } &= \{a, d\} \quad \gamma(G) = 2.
\end{align*}
\]
First we characterize minimal extended totally dominating sets in the following theorem.

**Theorem -1.8:** An extended totally dominating set $S$ of graph $G$ is minimal extended totally dominating set if and only if every vertex $v$ in $S$ satisfies only one of the following two conditions.

1. $T_{pd}[v,S] \neq \emptyset$.
2. $v$ is an isolated vertex of $G$.

**Proof:**

We are given $S$ is minimal extended totally dominating set of $G$. Suppose $v \in S$, if $v$ is an isolated vertex then second condition will be satisfied.

Suppose $v$ is not an isolated vertex. Now $S_1 = S - I$, (because $S = S_1 U I$), and $S_1$ is minimal totally dominating set in $G - I$. So, $S_1 - \{v\}$ is not a totally dominating set in $G - I$. So, there is at least one vertex $w$ which is not adjacent to any vertex of $S_1 - \{v\}$, where $w \in G$. i.e. suppose $w = v$, then $v$ is not adjacent to any vertex of $S_1 - \{v\}$. This contradicts fact that $S_1$ is a totally dominating set in $G - I$. So, $w \neq v$.

We know that $S_1$ is totally dominating set in $G - I$. So, $w$ is adjacent to some vertex of $S_1$ and we also know that $w$ is not adjacent to any vertex of $S_1 - \{v\}$. So, $w$ is adjacent to only $v$. So, $N(w) \cap S = \{v\}$.

Now we prove converse.

Let $v \in S$. If $v$ is an isolated vertex then $S - \{v\}$ is not an extended totally dominating set in $G$.

Now let $v$ is not an isolated vertex. So, there is only one vertex $w$ is adjacent to only $v$ in $S_1$ and $w$ is not adjacent to any vertex of $S_1 - \{v\}$. So, $S_1 - \{v\}$ is not a totally dominating set in $G - I$. Hence $S - \{v\}$ is not an extended totally dominating set of $G$. 


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Lemma -1.9: Suppose v is a vertex of G such that $\gamma_{Te}(G-v) < \gamma_{Te}(G)$. Then $\gamma_{Te}(G-v) = \gamma_{Te}(G) - 1$.

Proof: Suppose $v \in V_{Te}$. If v is an isolated vertex of G then v belongs to every $\gamma_{Te}$ set of G. So, suppose v is not an isolated in G.

Let $S_1$ be a minimum extended totally dominating set of $G-\{v\}$.

Suppose v is adjacent to some vertex of $S_1$. Then w must be unique because if $w_1$ is any other vertex of $S_1$ such that v is adjacent to $w_1$ then $T = S_1 - \{w\} U \{v\}$ is an extended totally dominating set of G with $|T| < \gamma_{Te}(G)$ — a contradiction. Thus, w is unique. Let $S = S_1 U \{v\}$. Then S is a minimum extended totally dominating set of G. Therefore $\gamma_{Te}(G) = |S| = |S_1 + 1| = \gamma_{Te}(G-v) + 1$. Thus, $\gamma_{Te}(G-v) = \gamma_{Te}(G) - 1$.

Now we prove necessary and sufficient conditions under which the extended total domination number increases when a vertex v is removed. Note that these conditions are similar to those for domination.(Theorem- 0.31)

Theorem- 1.10: $v \in V^*_{Te}$ if and only if the following three conditions are satisfies.

1. v is not an isolated vertex of G.
2. v is in every minimum extended totally dominating set of G.
3. There is no set S which satisfies any one of the following two conditions.
   
   (a) S is a minimum extended totally dominating set of $G-\{v\}$ with $|S| \leq \gamma_{Te}(G)$ such that $N[v] \cap S$ is an empty set.
   
   (b) S is a minimum extended totally dominating set of $G-\{v\}$ with $|S| \leq \gamma_{Te}(G)$ and there is a neighbor of v in S which is an isolated vertex in $G-\{v\}$. 

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Proof: (1)

We assume \( v \) is an isolated vertex of \( G \). Let \( S \) be an extended totally dominating set in \( G \), then \( v \in S \). Let \( w \) be any vertex of \( G - \{v\} \).

If \( w \) is an isolated vertex in \( G - \{v\} \) then \( w \) is an isolated also in \( G \). (because \( v \) is an isolated.). Hence \( w \in S - \{v\} \).

If \( w \) is not an isolated vertex in \( G - \{v\} \) then \( w \) is adjacent to some vertex \( t \) of \( S \). Since \( v \) is an isolated vertex and \( t \neq v \). Thus, \( w \) is adjacent to some vertex of \( S - \{v\} \). Thus, \( S - \{v\} \) is an extended totally dominating set in \( G - \{v\} \). Hence \( \gamma_{Te}(G-v) < \gamma_{Te}(G) \). This is a contradiction.

(2)

Suppose there a \( \gamma_{Te} \) set \( T \) of \( G \) such that \( v \notin T \). Now we prove that \( T \) is an extended totally dominating set in \( G - \{v\} \). Now, \( T = T_1 \cup I \) where \( I \) is the set of isolated vertices of \( G \). Since \( v \notin T \). Now, we prove that \( T_1 \) is totally dominating set of \( G - \{v\} \). Let \( w \) be any vertex of \( G - \{v\} \) which is not an isolated vertex. Since \( T \) is an extended totally dominating set of \( G \). So, \( w \) is adjacent to some vertex \( z \) of \( T \). Since \( v \notin T \), and \( z \neq v \). So, \( w \) is adjacent to \( z \), for some \( z \) in \( T_1 \). Hence \( T \) is an extended totally dominating set in \( G - \{v\} \). Thus, \( \gamma_{Te}(G-v) \leq |T| = \gamma_{Te}(G) \), a contradiction (Because we are given \( v \in V_{Te}^+ \)). So, \( v \) must be in every \( \gamma_{Te} \) set \( T \) of \( G \).

(3)

Suppose there is a subset \( S \) of the graph \( G - \{v\} \) with \( |S| \leq \gamma_{Te}(G) \) and suppose \( S \) satisfies either (a) or (b). Then \( \gamma_{Te}(G-v) \leq |S| \leq \gamma_{Te}(G) \). Thus, \( v \notin V_{Te}^+ \). This is a contradiction. Thus, \( S \) can not satisfies (a) or (b).

Now we prove converse.

Conversely assume conditions (1), (2) and (3) hold for the graph \( G \).
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Suppose $v \in V_{Te}^0$. Let $S_1$ be a minimum extended totally dominating set of $G-\{v\}$. Then $|S_1| \leq \gamma_{Te}(G)$. If $v$ is not adjacent to any vertex of $S_1$ then $N[v] \cap S_1 = \emptyset$. This is not possible.

Let $w$ be any vertex of $S_1$ which is adjacent to $v$. By 3-(b), $w$ must be non isolated in $G-\{v\}$. Since $S_1$ is an extended totally dominating set of $G-\{v\}$, $w$ must be adjacent to some vertex of $S_1$. Thus, $S_1$ is a minimum extended totally dominating set of $G$ not containing $v$ which contradicts condition (2).

Suppose $v \in V_{Te}$. Let $S_1$ be minimum extended totally dominating set of $G-\{v\}$. Then $|S_1| = \gamma_{Te}(G)-1$. Suppose $v$ is not adjacent to any vertex of $S_1$. Let $w$ be neighbor of $v$ (which is not in $S_1$). Let $S = S_1 \cup \{w\}$. Then $S$ is a minimum extended totally dominating set of $G$ not containing $v$ which contradicts (2).

If $v$ is adjacent to some vertex $z$ of $S_1$ then $z$ must be non isolated in $G-\{v\}$ and therefore $z$ be adjacent to some vertex of $S_1$. This is true for any such vertex $z$ of $S_1$ which is adjacent to $v$.

Thus, $S_1$ is an extended totally dominating set of $G$ with $|S_1| \leq \gamma_{Te}(G)$. This is again a contradiction.

Thus, $v$ does not belongs to $V_{Te}$ or $V_{Te}^0$. Hence $v \in V_{Te}^+$. This complete the theorem. 

**Example-1.11:** Consider the path graph $G = P_5$ with vertices $v_1, v_2, v_3, v_4, v_5$:

(See Figure-0.4)

Note that $\gamma_{Te}(G) = 3$, and $S = \{v_2, v_3, v_4\}$ is the unique $\gamma_{Te}$ set of $G$. Also note that $v_3 \in V_{Te}^+$ and $\gamma_{Te}(G- v_3) = 4$.

Consider the graph $G-\{v_4\}$. Note that $\gamma_{Te}(G- v_4) = 3$. The sets $S_1 = \{v_1, v_2, v_5\}$ and $S_2 = \{v_2, v_3, v_5\}$ are $\gamma_{Te}$ sets of $G-\{v_4\}$. Also note that there is a neighbor of $v_4$ (namely $v_5$) which is an isolated vertex of $G-\{v_4\}$. Also note that $|S_1| = |S_2| = \gamma_{Te}(G)$.

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**Theorem -1.12:** Let $G$ be a graph and $v$ be a vertex of $G$. $v \in V_{Te}$ if and only if one of the following conditions is satisfied.

1. $v$ is an isolated vertex of $G$ and $v$ is in every $\gamma_{Te}$ set.
2. There is $\gamma_{Te}$ set $S$ not containing $v$ and a vertex $w$ in $S$ such that $T_{pn}[w,S] = \{v\}$.

**Proof:**

Suppose $v \in V_{Te}$. If $v$ is an isolated vertex then $v$ is in every $\gamma_{Te}$ set $S$.

Suppose $v$ is not an isolated vertex in $G$. Since $v \in V_{Te}$, then there is a $\gamma_{Te}$ set $S$ of $G\{-v\}$ with $|S| = \gamma_{Te}(G) - 1$. If $v$ is adjacent to some vertex $w$ of $S$ then $w$ must be unique. Because if $w_1$ is any other vertex of $S$ adjacent to $v$ then $(S - \{w_1\}) \cup \{v\}$ is an extended totally dominating set of $G$ with cardinality less than $\gamma_{Te}(G)$. This is a contradiction. Thus, $w$ is unique. Also $w$ is not adjacent to any other vertex of $S$ because otherwise $S$ would be an extended totally dominating set in $G$ with cardinality less than $\gamma_{Te}(G)$.

Let $S_1 = S \cup \{v\}$ then $S_1$ is a $\gamma_{Te}$ set of $G$ and $T_{pn}[w,S_1]$ contains $v$. Since $v$ is not an isolated vertex in $G$ then there is a vertex $w$ in $G\{-v\}$ which is adjacent to $v$. Note that $w$ cannot be an isolated vertex in $G\{-v\}$, because $w \notin S$ and $S$ is an extended totally dominating set of $G\{-v\}$. Let $w_1$ be vertex of $S$ which is adjacent to $w$. Let $S_1 = S \cup \{w\}$, then $S_1$ is a $\gamma_{Te}$ set in $G$. Since $v$ is not adjacent to any vertex of $S$ and $v \in T_{pn}[w,S_1]$.

Thus, in both the cases we have proved that there is a vertex $w$ in $S_1$ such that $T_{pn}[w,S_1]$ contains $v$.

Let $t$ be a vertex of $G$ where $t \neq v$. If $t$ is an isolated vertex of $G\{-v\}$ then $t$ cannot be adjacent to $w$ because $w \neq v$. If $t$ is not an isolated in $G - \{v\}$ then $t$ is adjacent to some vertex $z$ of $S$. If $t$ is adjacent to $w$ implies that $t$ is adjacent to two distinct vertices of $S_1$, i.e. $t$ does not belong to $T_{pn}[w,S_1]$. So, $T_{pn}[w,S_1] = \{v\}$.
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Now we prove converse.

Let $S$ be an extended totally dominating set of $G$ such that condition (1) or (2) holds.

If $v$ is an isolated vertex of $G$ then $S – \{v\}$ is an extended totally dominating set of $G – \{v\}$ and we have $v \in V_{Te}$.

If $v$ is not an isolated vertex of $G$ then there is a vertex $w$ in $S$ such that $T_{pn}[w,S] = \{v\}$. We prove that $S – \{w\}$ is an extended totally dominating set of $G – \{v\}$.

Let $x$ be any vertex of $G-\{v\}$. If $x$ is an isolated vertex in $G – \{v\}$ and also isolated in $G$ then $x \in S$ and obviously $x \in S-\{v\}$.

If $x$ is an isolated in $G – \{v\}$ but not isolated in $G$ then since $S$ is an extended totally dominating set in $G$, $x$ must be adjacent to some vertex $z$ of $S$. Since $x \notin T_{pn}[w,S]$, we may assume that $z \neq w$, thus $x$ is adjacent to some vertex of $S-\{w\}$. Hence $S-\{w\}$ is an extended totally dominating set of $G-\{v\}$. This implies that $v \in V_{Te}$.

If $x$ is not an isolated vertex in $G-\{v\}$ then $x$ is not an isolated in $G$ also and since $S$ is an extended totally dominating set in $G$, there is a vertex $z$ in $S$ different from $w$ which is adjacent to $x$. Thus, $x$ is adjacent to some vertex of $S-\{w\}$. This proves that $S-\{w\}$ is an extended totally dominating set of $G-\{v\}$.

Hence $v \in V_{Te}$. □
Theorem-1.13: Let $G$ be a graph and $v$ be a vertex which belongs to $V^+_Te$, then for any $\gamma_Te$ set $S$, $v \in S$ and $T_{pn}[v,S]$ contains at least two vertices.

Proof:

First we prove that if $T_{pn}[v,S] = \{w\}$ then $w \notin S$.

Suppose $w \in S$, then $w$ is not adjacent to any other vertex of $S$. Suppose also that $w$ is not adjacent to any vertex outside $S$, then $w$ is an isolated vertex in $G-\{v\}$. Now we prove that $S_1 = S-\{v\}$ is an extended totally dominating set in $G-\{v\}$. For this let $z$ be any vertex of $G-\{v\}$. If $z$ is an isolated in $G$ then $z \in S-\{v\}$. Suppose $z$ is an isolated in $G-\{v\}$ but is not isolated in $G$. Since $S$ is an extended totally dominating set in $G$, $z$ is adjacent to some vertex $t$ of $S$. Since $z \notin T_{pn}[v,S]$, we may assume that $t \neq v$. Thus, $z$ is adjacent to at least two vertices of $S$ which is a contradiction. Therefore this possibility does not arise.

If $z$ is not an isolated in $G-\{v\}$ then $z$ is adjacent to some vertex of $S$ different from $v$. Therefore $z$ is adjacent to some vertex of $S-\{v\}$. Therefore $S_1 = S-\{v\}$ is an extended totally dominating set in $G-\{v\}$. So, $\gamma_Te(G-v) < \gamma_Te(G)$. So, $v \in V^Te$. This is a contradiction. So, $w$ must be adjacent to some vertex $w_1$ outside $S$. Now $w_1$ is adjacent to some vertex $w_{11}$ of $S$. Now let $S_1 = S-\{v\} U \{w_1\}$. Then $S_1$ is a $\gamma_Te$ set of $G$ not containing $v$. This contradict the fact that $v \in V^+_Te$. Thus, $w$ can not be in $S$. So, $w \in V(G)-S$.

Now suppose $w$ is an isolated in $G-\{v\}$ then $S_1 = S-\{v\} U \{w\}$ is an extended totally dominating set in $G-\{v\}$. So, $\gamma_Te(G-v) \leq \gamma_Te(G)$. So, $v \notin V^+_Te$. This is a contradiction. Thus, $w$ can not be an isolated vertex in $G-\{v\}$. So, there is a vertex $z \notin S$ such that $z$ is adjacent to $w$. Now, $z$ is adjacent to $z_1$ of $S$. Let $S_1 = S-\{v\} U \{z\}$ is a minimum extended totally dominating set in $G$ not containing $v$. This contradicts the fact that $v \in V^+_Te$. Thus, if we assume that $T_{pn}[v,S] = \{w\}$ then we have a contradiction. Thus, $T_{pn}[v,S]$ must contain at least two vertices.
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Remark-1.14:

The above Theorem 1.13 implies that if $v \in V^+_Te$ then $d(v) \geq 2$. Thus, any vertex of degree one is either in $V^-Te$ or $V^0Te$. Of course we know that any vertex of degree zero is always in $V^-Te$.

Example 1.15: It may be noted that if $w_1$ and $w_2$ belongs to $T_{pn}[v,S]$ and $w_1$ and $w_2$ does not belongs to $S$ then $w_1$ and $w_2$ may or may not be adjacent.

In Figure-1.31, the graph has vertices 0,1,2,3,4,5. And $0 \in V^+_Te, T = \{0,1\}$ then $T_{pn}[0,T] = \{4,3\}$, where 4 and 3 adjacent

In Figure-1.4, $0 \in V^+_Te, T = \{0,2\} T_{pn}[0,T] = \{2,5\}$, where 2 and 5 are non adjacent.

Theorem -1.16: Let $G$ be a graph and $v$ and $w$ are distinct vertices of $G$ such that $v \in V^+_Te$ and $w \in V^-Te$ then $v$ and $w$ are non adjacent vertices.

Proof:
If $w$ is an isolated vertex of $G$ then $v$ and $w$ are non adjacent.

In Figure-1.31, the graph has vertices 0,1,2,3,4,5. And $0 \in V^+_Te, T = \{0,1\}$ then $T_{pn}[0,T] = \{4,3\}$, where 4 and 3 adjacent

In Figure-1.4, $0 \in V^+_Te, T = \{0,2\} T_{pn}[0,T] = \{2,5\}$, where 2 and 5 are non adjacent.
If \( v \) and \( w_1 \) are distinct vertices then \( w \) is adjacent to two vertices of \( S \) and one of them is \( v \) which implies that \( w \notin T_{pn}(w_1,S) \). This is a contradiction. Therefore \( v \) and \( w \) must be non adjacent.

**Theorem-1.17:** Let \( G \) be a graph then \( |V^{0}_{Te}| \geq 2|V^{+}_{Te}| \).

**Proof:**

We will prove that every \( v \in V^{+}_{Te} \) give rise at least two vertices \( v_1 \) and \( v_2 \) in \( V^{0}_{Te} \). Let \( S \) be a \( \gamma_{Te} \) set containing \( v \) then (by above Theorem-1.13), \( T_{pn}(v,S) \) contains at least two vertices \( w_1 \) and \( w_2 \).

**Case 1:** Suppose \( w_1 \) and \( w_2 \in S \). If \( w_1, w_2 \in V^{0}_{Te} \). Let \( v_1 = w_1 \) and \( v_2 = w_2 \).

Suppose \( w_1 \notin V^{0}_{Te} \) then \( w_1 \in V^{+}_{Te} \) or \( w_1 \in V^{+}_{Te} \). Since \( w_1 \) and \( v \) are adjacent, \( w_1 \notin V^{+}_{Te} \) then \( w_1 \in V^{+}_{Te} \). Since \( w_1 \in V^{+}_{Te} \), \( T_{pn}(w_1,S) \) contains a vertex \( z \) different from \( v \) (by above theorem 1.13) so, \( z \notin S \), again by similar above argument \( z \notin V^{+}_{Te} \). So, \( z \in V^{0}_{Te} \).

Let \( v_1 = z \). If \( w_1 \in V^{0}_{Te} \) then \( v_1 = w_1 \).

If \( w_2 \in V^{+}_{Te} \) then by similar above arguments there is a vertex \( z_1 \) not is \( S \) such that \( z_1 \in T_{pn}(w_2,S) \). Let \( v_2 = z_1 \).

If \( w_2 \in V^{0}_{Te} \), then \( v_2 = w_2 \).

**Case 2:** If \( w_1, w_2 \notin S \) then \( v_1 = w_1 \) and \( v_2 = w_2 \).

**Case 3:** Suppose \( w_1 \in S \) and \( w_2 \notin S \) if \( w_1 \in V^{0}_{Te} \) then \( v_1 = w_1 \) and \( v_2 = w_2 \).

If \( w_1 \in V^{+}_{Te} \) then as in case (1) there is a vertex \( z \) not in \( S \) such that \( z \) is adjacent to \( w_1 \) and \( z \in V^{0}_{Te} \). Let in this case let \( v_1 = z \) and \( v_2 = w_2 \).
Case 4:

If $w_1 \not\in S$ and $w_2 \in S$. The proof of this part is as in above case. So, in all cases we get two vertices $v_1$ and $v_{11}$ in $V^0_{Te}$ corresponding to vertex $v \in V^+_{Te}$. It can be proved if $v_1$ and $v_2$ are distinct vertices of $V^+_{Te}$. Then the sets $\{v_{11}, v_{12}\}$ and $\{v_{21}, v_{22}\}$ are disjoint. So, $|V^0_{Te}| \geq 2|V^+_{Te}|$. □

Corollary-1.18: If $G$ is a graph such that $\gamma_{Te}(G-v) \neq \gamma_{Te}(G)$ then $\gamma_{Te}(G-v) < \gamma_{Te}(G)$ for every $v \in V(G)$.

Proof:

Suppose for every vertex $v$ of $G$, $\gamma_{Te}(G-v) \neq \gamma_{Te}(G)$ then $v \in V^+_{Te}$ or $v \in V^-_{Te}$. If for some vertex $v \in V^+_{Te}$, then there are two vertices $v_1$ and $v_2$ such that $v_1$ and $v_2$ belongs to $V^0_{Te}$. This contradicts the hypothesis of corollary. Hence $V(G) = V^-_{Te}$. □

Theorem-1.19: Let $G$ be a graph and $v$ be a non isolated vertex of $G$. If for every vertex $w \in N(v)$ and $N(w)$ is complete then $v \not\in V^-_{Te}$.

Proof:

Suppose $v \in V^-_{Te}$ then there is a $\gamma_{Te}$ set $S$ not containing $v$ and a vertex $w$ in $S$ such that $T_{pn}[w, S] = \{v\}$. Now, $w$ is adjacent to some vertex $w^1$ in $S$. Since $N(w)$ is complete it implies that $v$ is adjacent to $w^1$. This contradicts the fact that $v \in T_{pn}[w, S]$.

Hence $v \not\in V^-_{Te}$. □