Chapter-4: Perfect Domination

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PERFECT DOMINATION
Perfect Domination is closely related to Perfect Codes and Perfect Codes have been used in Coding Theory. In this chapter we study the effect of removing a vertex from the graph on perfect domination.

**Definition-4.1: Perfect dominating set.**[42]

A subset $S$ of $V(G)$ is said to be a perfect dominating set if for each vertex $v$ not in $S$, $v$ is adjacent to exactly one vertex of $S$.

Consider the path $P_4$ with four vertices $1,2,3,4$. The set $S = \{2, 3\}$ is perfect dominating set in this graph.

It may be noted that if $G$ is a graph then $V(G)$ is always a perfect dominating set of $G$.

**Definition-4.2: Minimal perfect dominating set.**

A perfect dominating set $S$ of the graph $G$ is said to be minimal perfect dominating set if for each vertex $v$ in $S$, $S-\{v\}$ is not a perfect dominating set.

It may be noted that it is not necessary that a proper subset of minimal perfect dominating set is not a perfect dominating set.

**Example-4.3:**

Consider the cycle graph $G = C_6$ with six vertices $1, 2, 3, 4, 5, 6$. Then obviously $V(G)$ is a minimal perfect dominating set of $G$.

However the set $\{1, 4\}$ is proper subset of $V(G)$ and is a perfect dominating set in the graph $G$. 

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Definition-4.4: Minimum perfect dominating set.

A perfect dominating set with smallest cardinality is called minimum perfect dominating set. It is called $\gamma_{pf}$ set of the graph $G$.

Definition-4.5: Perfect domination number.

The cardinality of a minimum perfect dominating set is called the perfect domination number of the graph $G$. It is denoted as $\gamma_{pf}(G)$.

The perfect domination number of cycle $C_6$ is 2 and that of the path $P_3$ is also 1.

Definition-4.6: Perfect private neighborhood.

Let $S$ be a subset of $V(G)$ and $v \in S$. Then the perfect private neighborhood of $v$ with respect to $S$ is

$$P_{pf}[v,S] = \{ w \in V(G) - S: N(w) \cap S = \{v\} \} \cup \{ v, \text{ if } v \text{ is adjacent to no vertex of } S \text{ or at least } \text{vertices of } S \}.$$

Theorem-4.7: A perfect dominating set $S$ of $G$ is minimal perfect dominating set if and only if for each vertex $v$ in $S$ $P_{pf}[v,S]$ is non-empty.

Proof:

Suppose $S$ is minimal and $v \in S$. Therefore there is a vertex $w$ not in $S\{v\}$ such that either $w$ is adjacent to no vertex of $S\{v\}$ or $w$ is adjacent to at least two vertices of $S\{v\}$.

If $w = v$ then this implies that $v \in P_{pf}[v,S]$.

If $w \neq v$ then it is impossible that $w$ is adjacent to at least two vertices of $S\{v\}$ because $S$ is a perfect dominating set. Therefore $w$ is not adjacent to any vertex of $S\{v\}$. Since $S$ is a perfect dominating set $w$ is adjacent to only $v$ in $S$. That is $N(w) \cap S = \{v\}$. Thus, $w \in P_{pf}[v,S]$. 

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Conversely suppose \( v \in S \) and \( P_{pf}[v,S] \) contains some vertex \( w \) of \( G \).

If \( w = v \) then \( w \) is either adjacent to at least two vertices of \( S-\{v\} \) or \( w \) is adjacent to no vertex of \( S-\{v\} \). Thus, \( S-\{v\} \) is not a perfect dominating set.

If \( w \neq v \) then \( N(w) \cap S = \{v\} \) implies that \( w \) is not adjacent to any vertex of \( S-\{v\} \).

Thus, in all cases \( S-\{v\} \) is not a perfect dominating set if \( v \in S \). Thus, \( S \) is minimal.

**Example-4.8:**

Consider the path \( G = P_5 \) (See Figure-0.4) with five vertices \( v_1, v_2, v_3, v_4, v_5 \).

Note that \( S = \{v_2, v_5\} \) is minimum and therefore minimal perfect dominating set.

\[ P_{pf}[v_2,.S] = \{v_1, v_2, v_3\}. \]

Now we define the following symbols.

\[ V^+_{pf} = \{v \in V(G): \gamma_{pf}(G) < \gamma_{pf}(G-v)\}. \]

\[ V^-_{pf} = \{v \in V(G): \gamma_{pf}(G) > \gamma_{pf}(G-v)\}. \]

\[ V^0_{pf} = \{v \in V(G): \gamma_{pf}(G) = \gamma_{pf}(G-v)\}. \]

Note that the above sets are mutually disjoint and their union is \( V(G) \).

Now we prove the following lemma.

**Lemma-4.9:** Let \( v \in V(G) \) and suppose \( v \) is a pendent vertex and \( w \) has a neighbor \( w \) of degree at least two. If \( v \in V^-_{pf} \) then \( \gamma_{pf}(G-v) = \gamma_{pf}(G) - 1. \)

**Proof:**

Let \( S_1 \) be a minimum perfect dominating set of \( G-\{v\} \). If \( w \in S_1 \) then \( S_1 \) is a perfect dominating set of \( G \) with \( |S_1| < \gamma_{pf}(G) \). That is \( \gamma_{pf}(G) \leq |S_1| < \gamma_{pf}(G) \), this is a contradiction. Therefore \( w \notin S_1 \). Let \( S = S_1 \cup \{w\} \). Then \( S \) is a minimum perfect dominating set of \( G \). Therefore \( \gamma_{pf}(G) = |S| = |S_1| + 1 = \gamma_{pf}(G-v) + 1. \)

This proves the lemma.
Next we prove the necessary and sufficient conditions for a pendent vertex (with a neighbor of degree at least two) to be in $V_{pf}^+$. 

**Theorem-4.10:** Let $v$ be a vertex of $G$ Then $v \in V_{pf}^+$ if and only if the following conditions are satisfies.

1. $v$ belongs to every $\gamma_{pf}$ set of $G$.
2. No subset $S$ of $G-\{v\}$ which is either disjoint from $N[v]$ or intersects $N[v]$ in at least two vertices and $|S| \leq \gamma_{pf}(G)$ can be a perfectly dominating set of $G-\{v\}$.

**Proof:**

1. Suppose $v \in V_{pf}^+$. Suppose $S$ is a $\gamma_{pf}$ set of $G$ which does not contain $v$ then $S$ is a perfect dominating set of $G-\{v\}$. Therefore $\gamma_{pf}(G-v) \leq |S| = \gamma_{pf}(G)$. Thus, $v \notin V_{pf}^+$. This is a contradiction. Thus, $v$ must belong to every $\gamma_{pf}$ set of $G$.

2. If there is set $S$ which satisfies the condition stated in (2). Then $S$ is a perfect dominating set of $G-\{v\}$ and therefore $\gamma_{pf}(G-v) \leq \gamma_{pf}(G)$. This is a contradiction.

Conversely assume that (1) and (2) hold.

Suppose $v \in V_{pf}^0$. Let $S$ be a minimum perfect dominating set of $G-\{v\}$. Then $|S| = \gamma_{pf}(G)$.

Suppose $v$ is not adjacent to any vertex of $S$. Then $S$ is disjoint from $N[v]$, $|S| = \gamma_{pf}(G)$ and $S$ is a perfectly dominating set of $G-\{v\}$. This violates (2).

Suppose $v$ is adjacent to exactly one vertex of $S$ then $S$ is a minimum perfect dominating set of $G$ not containing $v$ which violates (1).
Suppose \( v \) is adjacent to at least two vertices of \( S \). Then \( S \cap N[v] \) in at least two vertices and \( S \) is a perfectly dominating set of \( G-\{v\} \) with \( |S| = \gamma_{pf}(G) \), which again violate (2).

Thus, \( v \in V_{pf}^0 \) implies (1) or (2) violated.

Suppose \( v \in V_{pf} \). Let \( S_1 \) be a minimum perfect dominating set of \( G-\{v\} \). Then \( |S_1| < \gamma_{pf}(G) \). If \( v \) is not adjacent to any vertex of \( S_1 \) then as above (2) is violated.

If \( v \) is adjacent to exactly one vertex of \( S_1 \) then \( S_1 \) is a perfect dominating set of \( G \) with \( |S_1| < \gamma_{pf}(G) \) – which is a contradiction.

If \( v \) is adjacent to at least two vertices of \( S_1 \) then \( S_1 \cap N[v] \) in at least two vertices, \( |S_1| \leq \gamma_{pf}(G) \) and \( S_1 \) is a perfect dominating set of \( G-\{v\} \) – which again violates (2).

Thus, \( v \in V_{pf} \) implies that (2) is violated.

Thus, \( v \) does not belongs to \( V_{pf}^0 \) or \( V_{pf}^- \). Hence \( v \in V_{pf}^+ \). 

**Theorem 4.11:** Let \( v \) be a pendent vertex which has the neighbor \( w \) of degree at least two then \( v \in V_{pf}^- \) if and only if there is \( \gamma_{pf} \) set \( S \) containing \( w \) and not containing \( v \) such that \( P_{pf}[w, S] = \{v\} \).

**Proof:**

Suppose \( v \in V_{pf}^- \). Let \( S_1 \) be a minimum perfect dominating set of \( G-\{v\} \). Then as proved Lemma 4.9, \( w \notin S_1 \). Let \( S = S_1 \cup \{w\} \). Then \( S \) is \( \gamma_{pf} \) containing \( w \).

Since \( S_1 \) is a perfect dominating set of \( G-\{v\} \), \( w \) is adjacent to some vertex of \( S_1 \). Therefore \( w \notin P_{pf}[w, S] \). If \( x \) is any vertex different from \( v \) such that \( x \) is adjacent to \( w \) then \( x \) is also adjacent to some vertex of \( S_1 \) because \( S_1 \) is a perfect dominating set of \( G-\{v\} \). Thus, \( x \notin P_{pf}[w, S] \). Further \( v \) is adjacent to only \( w \) of \( S \) therefore \( P_{pf}[w, S] = \{v\} \).
Conversely suppose there is a $\gamma_{pf}$ set $S$ containing $w$ such that $P_{pf}[w,S] = \{v\}$. Let $S_1 = S - \{w\}$. Let $x$ be any vertex of $G-\{v\}$ which is not in $S-\{v\}$. Since $x \notin P_{pf}[w,S]$, $x$ must be adjacent to some unique vertex $S_1$. Thus, $S_1$ is a minimum perfect dominating set of $G-\{v\}$ with $|S_1| < \gamma_{pf}(G)$. Thus, $v \in V_{pf}$. □

**Example-4.12:**
Consider the path $G= P_4$ with vertices 1,2,3,4. Then $\gamma_{pf}(G) = 2$. Let $v = 1$ and $w = 2$.

Now $\gamma_{pf}(G-1) = 1$. Thus, $1 \in V_{pf}$ also $S = (2, 3)$ is $\gamma_{pf}$ set of $G$, containing $w = 2$ and $P_{pf}[2,S] = \{1\}$.

**Theorem-4.13:** Let $S_1$ and $S_2$ be two disjoint perfect dominating sets of $G$. Then $|S_1| = |S_2|

**Proof:**
For every vertex $x$ in $S_1$ there is a unique vertex $v(x)$ in $S_2$ which is adjacent to $x$. Also for every vertex $y$ in $S_2$ there is a unique vertex $u(y)$ in $S_1$ which is adjacent to $y$. It may be noted that these functions are inverses of each other. Therefore $|S_1| = |S_2|$. □

**Corollary-4.14:** If in a graph $G$ there are perfect dominating sets $S_1$ and $S_2$ such that $|S_1| \neq |S_2|$ then $S_1 \cap S_2 \neq \phi$. □

**Corollary-4.15:** Let $G$ be a graph with $n$ vertices. If there is a perfect dominating set $S$ with $|S| < n/2$ or $\geq n/2$ then $V(G) - S$ is not a perfect dominating set. □