Part VI

Conclusion
In this thesis, the main topic of study was generalizations of the problem of finding the minimum set of vertices disconnecting two vertices in graph, also called graph separation problems. These problems not only have independent applications, motivating their study, but a variety of classical problems which are not graph separation problems or even graph problems in some cases, have been found to have a graph separation problem at their core with examples including, but not restricted to **VERTEX COVER**, **ALMOST 2-SAT**, **DELETION q-Horn BACKDOOR SET DETECTION**, and **SATISFIABILITY**. Since almost all natural graph separation problems turn out to be **NP-complete**, we studied these problems in the framework of Parameterized Complexity and we designed new techniques and frameworks to obtain new as well as improved **FPT algorithms** for certain kinds of parameterized graph separation problems. We also give applications of these techniques by showing certain problems which not graph separation problems themselves to have some variant of graph separation at their core, to give new as well as improved **FPT algorithms**. In particular,

- We introduced a generalization of important separators and show how to use it as
a generic technique to design algorithms for problems which are not amenable to
the existing machinery.

- We gave a generic subroutine which can be used to impose certain structure on
a given graph which in turn leads to faster algorithms. In this thesis, we present
a version of this subroutine which is very useful for graph separation problems
which are parity based.

- We introduced a framework in which we can obtain greedy factor-OPT approxi-
mation algorithms for graph separation problems.

11.1 New FPT algorithms

- VERTEX COVER parameterized above the optimum value of the LP can be solved
in time $O(2.3146^k n^{O(1)})$, where $n$ is the number of vertices in the input graph
(Chapter 7).

- PARITY MULTIWAY CUT can be solved in time $O(2^{O(k^3)} n^{O(1)})$, where $n$ is the
number of vertices in the input graph (Chapter 8).

- $d$-SKEW-SYMMETRIC MULTICUT can be solved in time $O((4d)^k k^4(m + n))$
where $m$ and $n$ are the number of edges and vertices in the input graph respec-
tively (Chapter 11).

- DELETION q-Horn BACKDOOR SET DETECTION can be solved in time $O(12^k k^5 \ell)$
where $\ell$ is the length of the input formula (Chapter 10).

- SATISFIABILITY can be solved in time $O(12^k k^5 \ell)$ where $k$ is the size of the small-
est q-Horn deletion backdoor set of the given formula and $\ell$ is the length of the
input formula (Chapter 10).
11.2 FPT algorithms with improved dependence on the parameter

- We obtained $O(2.3146^k n^{O(1)})$ algorithms for \textsc{Vertex Cover} parameterized above the size of the maximum matching and \textsc{Odd Cycle Transversal} where $n$ is the number of vertices in the input graph. This is the first improvement (with respect to the parameter) for \textsc{Odd Cycle Transversal} after the first algorithm of Reed et al. (2004) (Chapter 7).

- We obtained $O(2.3146^k \ell^{O(1)})$ algorithms for \textsc{Almost 2-SAT} where $\ell$ is the length of the input formula (Chapter 7).

11.3 FPT algorithms with improved dependence on the input size

- We obtained an algorithm for \textsc{Odd Cycle Transversal} which runs in time $O(4^k k^4 (m + n))$, which is the first linear time FPT algorithm for this problem and answers a question asked by Reed et al. (2004) (Chapter 11).

- We obtained an algorithm for \textsc{Almost 2-SAT} which runs in time $O(4^k k^4 \ell)$ and an algorithm for \textsc{Vertex Cover} parameterized above the size of the maximum matching running in time $O(4^kk^4(m+n))$ given a graph with a matching (Chapter 11).
11.4 Future Directions

The following are potential future directions which our work in this thesis points towards.

- Consider the following parity based generalization of MULTICUT, namely PARITY MULTICUT where given a graph, integer $k$, vertex pairs $(s_1, t_1), \ldots, (s_r, t_r)$ and a set of parities $(p_1, \ldots, p_r)$ one for each pair, the objective is to test if there are $k$ vertices which intersect all paths from $s_i$ to $t_i$ of parity $p_i$ for every $1 \leq i \leq r$. This problem is also a clear generalization of PARITY MULTIWAY CUT. We remark that the parameterized complexity of this problem parameterized by the solution size $k$ is not known and it could be an interesting direction to work towards and might lead to newer techniques or a much better understanding of the existing graph separation machinery.

- Recall that our algorithm for $d$-SKEW-SYMMETRIC MULTICUT does not have a polynomial dependence on $d$. It would be interesting to study the parameterized complexity of this problem for unbounded $d$. Since even for $d = 1$ and $d = 3$ this problem generalizes a lot of well studied problems, an FPT result for unbounded $d$ could have significant consequences for a much larger number of problems. On the other hand, if this problem turns out to be $W[1]$-hard, then it could be interesting to classify the “boundary of hardness” with respect to $d$.

- Kratsch and Wahlström [64] studied the kernelization complexity of ABOVE GUARANTEE VERTEX COVER and obtained a randomized polynomial sized kernel for this problem through matroid based techniques. This implied a randomized polynomial kernel for all problems which have a parameter preserving polynomial
time reduction to **ABOVE GUARANTEE VERTEX COVER**. However, for two main problems studied in this thesis, namely PMWC and *d*-**SKew-SYMMETRIC MULTICUT**, the kernelization complexity is open. We believe that any result on the kernelization complexity of these two very general problems will help to understand and classify the structures in graph separation problems which may act as obstacles to polynomial kernels.