Appendix A

Tangent space and constant anholonomy

In this chapter, we summarise our convention for tangent space and some definitions of anholonomy coefficients used throughout the thesis. Greek indices $\mu, \nu, \ldots$ denote space time indices with $\mu = 0, 1, \ldots, 4$ and $g_{\mu \nu}$ is the space time metric. Latin indices $a, b, \ldots$ denote tangent space indices with $a = 0, 1, \ldots, 4$. The tangent space metric has the signature $\eta_{ab} = \{-, +, +, +, +\}$. The vielbeins $e^a_\mu(x)$ are related to the space time metric by,

$$g_{\mu \nu} = e^a_\mu e^b_\nu \eta^{ab}. \quad (A.0.1)$$

We define the one form $e^a \equiv e^a_\mu dx^\mu$ and its dual $\tilde{e}_a \equiv e^\mu_a \partial_\mu$. The anholonomy coefficients are defined as Lie brackets of the duals $\tilde{e}_a$,

$$[\tilde{e}_a, \tilde{e}_b] \equiv c_{ab}^c \tilde{e}_c; \quad c_{ab}^c = e^a_\mu e^b_\nu (\partial_\nu e^c_\mu - \partial_\mu e^c_\nu). \quad (A.0.2)$$
The tangent space curvature can be written in terms of the anholonomy coefficients and the spin connection,

\[ R^{d}_{abc} = \partial_{a} \omega^{d}_{bc} - \partial_{b} \omega^{d}_{ac} - \omega^{e}_{ac} \omega^{d}_{be} + \omega^{e}_{be} \omega^{d}_{ae} - c^{c}_{ab} \omega^{d}_{ec}. \] (A.0.3)

In the absence of torsion the spin connection and anholonomy coefficients are related by,

\[ \omega_{a,be} = \frac{1}{2} [c_{ab,c} - c_{ac,b} - c_{be,a}], \] (A.0.4)

where \( \omega_{a,be} = -\omega_{a,eb} \) and \( c_{ab,c} = -c_{ba,c} \). The comma is used to indicate the antisymmetric indices. It follows that when one takes constant \( c_{ab,c} \), the derivatives in (A.0.3) vanish and the Riemann tensor is a function of the constant anholonomy coefficients. For tangent space covariant derivatives acting on a spinor \( \chi^{\alpha} \) we use the convention,

\[ D_{a}(\omega)\chi^{\alpha} = \partial_{a}\chi^{\alpha} - \frac{1}{4} \omega^{bc}_{a} \gamma^{\alpha}_{bc}\chi^{\alpha}, \] (A.0.5)

where \( \alpha \) is the spinor index. The action of the covariant derivative on vectors is given by,

\[ D_{a}(\omega)V^{b} = \partial_{a}V^{b} + \omega^{b}_{a,c}V^{c}. \] (A.0.6)
Appendix B

Gamma matrices and Spinors in five dimension

The Clifford algebra in 5 space-time dimensions is,

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}. \quad \text{(B.0.1)}$$

The Dirac matrices in five dimensions are given by [1],

$$\begin{align*}
\gamma^0 &= -i\sigma_2 \otimes \sigma_3 \\
\gamma^1 &= -\sigma_1 \otimes \sigma_3 \\
\gamma^2 &= I_2 \otimes \sigma_1 \\
\gamma^3 &= I_2 \otimes \sigma_2 \\
\gamma^4 &= -i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma_3 \otimes \sigma_3 \quad \text{(B.0.2)}
\end{align*}$$

where $\sigma_i, i = 1, 2, 3$ are the usual Pauli matrices and $I_2$ is the two dimensional unit matrix.

The charge conjugation matrix $C$ has the property $C' = -C = C^{-1}$ and,

$$C\gamma^aC^{-1} = (\gamma^a)' \quad \text{(B.0.3)}$$
where \( C = B \gamma^0 \), with \( B = \gamma^3 \) such that \( B^\dagger B = -1 \). The spinor indices which are usually suppressed in most places are raised and lowered by \( C_{\alpha\beta} \) using the NW-SE convention. Expressions such as \( \bar{\psi} \psi \) and \( \bar{\psi} \gamma^a \psi \) are understood as,

\[
\bar{\psi} \psi = \bar{\psi}^\alpha \psi_\alpha, \quad \bar{\psi} \gamma^a \psi = \bar{\psi}^\alpha (\gamma^a)_\alpha^\beta \psi_\beta.
\] (B.0.4)

In addition, the spinors in the theory carry an \( SU(2) \) index which is raised and lowered using \( \epsilon_{ij} \),

\[
X^i = \epsilon^{ij} X_j, \quad X_j = X^i \epsilon_{ij},
\] (B.0.5)

with \( \epsilon_{12} = \epsilon^{12} = 1 \). With these conventions the mixed \( \epsilon \) tensors are antisymmetric

\[
\epsilon^{ijk} \epsilon_{kl} = \epsilon^{ijkl} = -\delta^i_j = -\epsilon^{ij}.
\] (B.0.6)

Spinors in \( d = 5 \) satisfy a symplectic majorana condition. To apply this condition one needs \( B^\dagger B = -1 \), even number of Dirac spinors \( \psi_i, i = 1, \ldots, 2n \) and an antisymmetric real matrix \( \Omega_{ij} \) with \( \Omega^2 = -1_{2n} \). The symplectic majorana condition on a generic spinor reads as,

\[
\psi_i^* = \Omega_{ij} B \psi_j,
\] (B.0.7)

or equivalently as,

\[
\bar{\psi}^i \equiv (\psi_i^*)^0 = (\psi^i)^C.
\] (B.0.8)

For \( N = 2 \) supersymmetry \( i = 1, 2 \), and using \( \Omega_{ij} = \epsilon_{ij} \) (B.0.7) reads as,

\[
\psi_1^* = \gamma^3 \psi_2.
\] (B.0.9)

Note that this condition does not reduce the degrees of freedom as compared to a single unconstrained Dirac spinor. This is because one needs at least a pair of Dirac spinors to apply the symplectic majorana condition (B.0.7). However, the action of the R-symmetry is manifest with this condition. Let us start with a pair of generic Dirac spinors in five
dimensions,
\[
\begin{pmatrix}
\epsilon_{11R} + i\epsilon_{11I} \\
\epsilon_{12R} + i\epsilon_{12I} \\
\epsilon_{13R} + i\epsilon_{13I} \\
\epsilon_{14R} + i\epsilon_{14I}
\end{pmatrix}
\quad \text{and}
\begin{pmatrix}
\epsilon_{21R} + i\epsilon_{21I} \\
\epsilon_{22R} + i\epsilon_{22I} \\
\epsilon_{23R} + i\epsilon_{23I} \\
\epsilon_{24R} + i\epsilon_{24I}
\end{pmatrix},
\]  
\tag{B.0.10}

where all the components are real valued constants. Using (B.0.7) one finds that,
\[
\begin{align*}
\epsilon_{21R} &= -\epsilon_{13I} \\
\epsilon_{21I} &= -\epsilon_{13R} \\
\epsilon_{22R} &= -\epsilon_{14I} \\
\epsilon_{22I} &= -\epsilon_{14R} \\
\epsilon_{23R} &= \epsilon_{11I} \\
\epsilon_{23I} &= \epsilon_{11R} \\
\epsilon_{24R} &= \epsilon_{12I} \\
\epsilon_{24I} &= \epsilon_{12R}.
\end{align*}
\tag{B.0.11}
\]

Therefore,
\[
\begin{pmatrix}
\epsilon_{11R} + i\epsilon_{11I} \\
\epsilon_{12R} + i\epsilon_{12I} \\
\epsilon_{13R} + i\epsilon_{13I} \\
\epsilon_{14R} + i\epsilon_{14I}
\end{pmatrix}
\quad \text{and}
\begin{pmatrix}
-\epsilon_{13I} - i\epsilon_{13R} \\
-\epsilon_{14I} - i\epsilon_{14R} \\
\epsilon_{11I} + i\epsilon_{11R} \\
\epsilon_{12I} + i\epsilon_{12R}
\end{pmatrix}.
\tag{B.0.12}
\]

As one can see, there are 8 independent real components, which is same as the number of degrees of freedom of a single unconstrained Dirac spinor. The minimum supersymmetry that one can have in five dimensions is then \( \mathcal{N} = 2 \) and thus the R symmetry group of the Poincaré superalgebra is \( USp(2)_R \). The advantage of using the symplectic majorana condition is that the action of the \( SU(2)_R \) symmetry is manifest. For example, rewriting the symplectic majorana spinors in two component notation one sees that,
\[
\epsilon_i = \begin{pmatrix} i\epsilon_{ij}\lambda_j \\ \lambda_i \end{pmatrix},
\tag{B.0.13}
\]

where,
\[
\lambda_1 = \begin{pmatrix} \epsilon_{13R} - i\epsilon_{13I} \\ \epsilon_{14R} - i\epsilon_{14I} \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \epsilon_{11I} - i\epsilon_{11R} \\ \epsilon_{12I} - i\epsilon_{12R} \end{pmatrix}.
\tag{B.0.14}
We do not require the two component notation for our purposes, we will use \((B.0.12)\). Antisymmetrisation is done with the following convention,

\[
\gamma_{a_1a_2...a_n} = \gamma_{(a_1a_2...a_n)} = \frac{1}{n!} \sum_{\sigma \in P_n} \text{Sign}(\sigma) \gamma_{a_{\sigma(1)}} \gamma_{a_{\sigma(2)}} \cdots \gamma_{a_{\sigma(n)}}, \tag{B.0.15}
\]

In \(d = 5\) only \(I, \gamma_a, \gamma_{ab}\) form an independent set, other matrices are related by the general identity for \(d = 2k + 3\),

\[
\gamma^{\mu_1\mu_2...\mu_s} = \frac{-i^{k+s(s-1)}}{(d - s)!} \epsilon^{\mu_1\mu_2...\mu_s} \gamma_{\mu_{s+1}...\mu_d}. \tag{B.0.16}
\]

We also list some useful identities involving various Dirac matrices \([29]\),

\[
\begin{align*}
[\gamma_a, \gamma_b] &= 2\gamma_{ab}, \\
[\gamma_h, \gamma_{abc}] &= 2\gamma_{habc}, \\
[\gamma_{abc}, \gamma_{efh}] &= \eta_{ef} \eta_{gp} \eta_{hk} (2\gamma_{abc} f^{pk} \gamma_{i} - 36 \delta_{[abc} \gamma_{i]}). \tag{B.0.17}
\end{align*}
\]
Appendix C

Origin of tensor multiplets in gauged supergravity

A novel feature of the gauged supergravity in five dimensions is the entry of tensor multiplets upon gauging. Once the gauge group $K \subset G$ is identified, if one chooses to gauge the $n_V + 1$ vector fields $A^I$, one is left with $n_T = \dim(G) - n_V$ vector fields $A^M$ charged under $K$. These $n_T$ gauge fields are dualised into antisymmetric tensor fields $B^M_{\mu\nu}$ and give rise to the tensor multiplets in the theory. It is important to note that there is no need for a tensor multiplet in the ungauged supergravity (5.2.1) since the vectors and tensors are equivalent by the duality relation [151],

$$\partial_{[\mu} A_{\nu]} = \epsilon_{\mu\nu}^{\lambda\rho\sigma} \partial_\lambda B_{\rho\sigma}. \quad (C.0.1)$$

This is however not true when the tensors carry massive degrees of freedom. The “self-duality” condition for a massive tensor field in five dimensions is given by,

$$B_{\mu\nu} = \frac{i}{3!m} \epsilon_{\mu\nu\lambda\rho\sigma} H^{\lambda\rho\sigma}. \quad (C.0.2)$$
where $H$ is the three form field strength of $B$ and $m$ is a mass parameter. In fact, the condition (C.0.2) follows from the generalisation of the Proca Lagrangian for tensor fields,

$$L_{\text{proca}} = B^{\mu \nu} B_{\mu \nu} - \frac{i}{3! m} \epsilon_{\mu \nu \lambda \rho \sigma} B^{\mu \nu} H^{\lambda \rho \sigma}. \quad (C.0.3)$$

One can compare the above Lagrangian with (5.3.9) and see that in the gauged supergravity the tensor fields appear exactly as in the Proca Lagrangian, except that there are covariant derivatives that appear due to the gauging. The presence of the $i$ also implies that these tensor fields are complex. In this discussion we will consider the tensor fields to be decomposed of real and imaginary parts and hence the index $M$ is always even.

We now explain briefly how the vectors $A^{M}_{\mu}$ lose the degrees of freedom to tensors $B^{M}_{\mu \nu}$ via a Higgs type mechanism. Remember that the $A^{M}_{\mu}$ are those vector fields which are neither adjoint nor singlets under $K$, hence they cannot describe Yang-Mills gauge fields. In particular, they describe massive degrees of freedom. The field strength $F^{M}_{\mu \nu}$ is replaced by the combination,

$$B^{M}_{\mu \nu} = F^{M}_{\mu \nu} + b^{M}_{\mu \nu}, \quad (C.0.4)$$

where $b^{M}_{\mu \nu}$ is an antisymmetric tensor invariant under the gauge transformation for a tensor field,

$$\delta b^{M}_{\mu \nu} = \partial_{[\mu} A^{M}_{\nu]}, \quad (C.0.5)$$

This immediately implies that the $A^{M}_{\mu}$ transform as,

$$\delta A^{M}_{\mu} = -A^{M}_{\mu}, \quad (C.0.6)$$

so that (C.0.4) is gauge invariant. The above form of the gauge transformation also follows from the closure of the supersymmetry algebra [13]. Using this gauge transformation one can always choose $A^{M} = A^{M}$ and get $B^{M}_{\mu \nu} = b^{M}_{\mu \nu}$. Thus the massive degrees of freedom of $A^{M}_{\mu}$ are absorbed by $b^{M}_{\mu \nu}$ by a Higgs mechanism. It follows that the tensor fields $B^{M}_{\mu \nu}$ describe massive degrees of freedom.