Chapter 7

Stability of Bianchi attractors in gauged supergravity

7.1 Introduction

In chapter 4, we discussed the classification of homogeneous but anisotropic extremal black brane horizons known as Bianchi attractors. In chapter 6, we studied generalised attractors in gauged supergravity and constructed some explicit examples of such Bianchi attractors. One of the important issues being investigated currently is the stability of such Lorentz violating geometries [159–164]. Instabilities due to scalar field fluctuations were found to exist in a class of charged black brane geometries [165, 166]. Presence of such instabilities in these solutions plays a crucial role because they indicate that the geometry might get corrected in the deep infrared [161]. Though the stability analysis has been carried out in a number of examples, a common recipe to figure out whether certain geometry has any instability is still lacking.

In chapter 3, we discussed the attractor mechanism which has been studied quite extensively in the context of extremal black holes in Minkowski space with near horizon
geometry $AdS_2 \times S^2$. The study of a similar mechanism for generalised attractors has not yet been explored thoroughly for the new class of Lorentz violating geometries arising as gravity duals of condensed matter systems. Especially, it is not at all obvious which among these entire class of new attractor geometries are stable and can survive in the deep infrared. Since a number of such geometries can be embedded in gauged supergravity, where the scalar couplings and potential term are determined by symmetry, it is natural to ask whether these gauged supergravity attractors are stable.

In this chapter, we analyse the stability of electrically charged Bianchi attractors in gauged supergravity. For attractors which asymptote to Minkowski space the conditions for stability is well understood [95]. In such cases the attractor values of the scalar fields must correspond to an absolute minimum of the black hole potential. We discussed this in chapter 3. In this chapter we derive the analogous condition for the generalised attractors in gauged supergravity. The main reference for this chapter is [21].

We consider the fluctuations of the scalar fields about their attractor value. We take the fluctuation to be of the form,

$$\phi_c + \epsilon \delta \phi(r, t), \quad (7.1.1)$$

where $t$ denotes the time, $r$ is the radial direction, $\phi_c$ are the attractor values of the scalars and $\delta \phi$ is the perturbation with $\epsilon < 1$. We have taken the fluctuation to not depend on the $(x, y, z)$ directions to respect the Bianchi type symmetries along these directions. Besides, we are primarily interested in the radial behavior of the fluctuation as one approaches the horizon. We also assume that the black brane metric can be expanded about the near horizon geometries as follows,

$$\tilde{g}_{\mu \nu} \sim g_{\mu \nu}(r - r_h) + \epsilon \ g^1_{\mu \nu}(r - r_h) + O(\epsilon^2) + \ldots, \quad (7.1.2)$$

where $g_{\mu \nu}$ is the near horizon metric given by the Bianchi type geometries. The higher order terms like $g^1_{\mu \nu}$ are due to the back reaction of the scalar field fluctuations on the
We study the stress energy tensor in gauged supergravity and expand it in first order of scalar fluctuations. We find that the stress energy tensor in gauged supergravity depends on the scalar fluctuations even at first order perturbation due to non-trivial interaction terms in the theory. If there is a large backreaction due to scalar fluctuations, the geometry would significantly differ from the attractor geometry indicating an instability. Therefore, stable attractor geometries are those where the scalar fluctuations die out as one approaches the horizon.

We then study the scalar field equations with the fluctuations at first order, determine the general solution and the conditions under which these fluctuations can exist. These conditions are such that the generalised attractor geometries must exist at critical points which are maxima of the attractor potential. We then derive conditions for stability of the Bianchi attractors in gauged supergravity by studying the near horizon behaviour of the scalar fluctuations and demanding regularity. In particular, we find that this severely restricts the general form of these metrics. We find that metrics which factorise as

\[ ds^2 = L^2 \left[ -\hat{r}^2 u_0 dt^2 + \frac{dr^2}{\hat{r}^2} + \eta_{ij} \omega^i \otimes \omega^j \right], \]

are stable under scalar fluctuations about the attractor value. The parameter \( u_0 \) must be positive in order to have a regular horizon. In particular, when \( u_0 = 1 \) we get an \( AdS_2 \) factor and the symmetry is enhanced to \( SO(2, 1) \times M \). This factorisation is reminiscent of extremal black holes in four dimensions where the near horizon geometries factorise as \( AdS_2 \times S^2 \). We briefly mentioned such a class of metrics in chapter 4 (eq (4.5.35)) with scale invariance only along the \( \hat{r}, \hat{t} \) directions. In the previous chapter, we constructed explicit examples of such metrics from \( U(1)_R \) gauged supergravity in §6.5.

The chapter is organised as follows. In §7.2 we expand the stress energy tensor under

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1In deriving this result, we make certain technical assumption on the killing vectors used in gauging, as well as on the nature of the critical points giving rise to the attractor geometry which will be discussed in due course.
scalar fluctuations about the attractor value and discuss the backreaction. We then derive
the general solutions for the scalar fluctuations and describe the conditions under which
these fluctuations exist in §7.3. Following this we study the near horizon behaviour of
the fluctuations and derive stability conditions for the Bianchi attractors and discuss the
constraints on the metric in §7.4. We conclude and summarise our results in §7.5.

7.2 Backreaction at first order

In this section, we analyse the stress energy tensor in gauged supergravity under scalar
fluctuations. The stress energy tensor calculated from (5.3.9) takes the form,

\[
T_{\mu\nu} = g_{\mu\nu} \left[ \frac{1}{4} a_{IJ} F_{\mu\nu}^I F_{\mu\nu}^J + \frac{1}{2} g_{xy} D_\mu \phi^x D_\nu \phi^y + V(\phi) \right]
- \left[ a_{IJ} F_{\mu\nu}^I F_{\lambda\sigma}^J + g_{xy} D_\mu \phi^x D_\nu \phi^y \right].
\]

(7.2.1)

We now expand the stress energy tensor (7.2.1) up to first order in \( \epsilon \) under the scalar
perturbations (7.1.1) to get,

\[
T_{\mu\nu}(\phi_c + \delta \phi) = g_{\mu\nu} \left[ \frac{1}{4} \left( a_{IJ} \right|_{\phi_c} \left. + \frac{\partial a_{IJ}}{\partial \phi^z} \right|_{\phi_c} \delta \phi^z \right) F_{\mu\nu}^I F_{\lambda\sigma}^J + g \left( g_{xy} K_i^I \right|_{\phi_c} A_i^I \delta \phi^J \\
+ \frac{1}{2} g^2 A_i^I A_i^J \left( K_{IJ} \right|_{\phi_c} \left. + \frac{\partial K_{IJ}}{\partial \phi^z} \right|_{\phi_c} \delta \phi^z \right) \left( V(\phi_c) + \left. \frac{\partial^2 V}{\partial \phi^z} \right|_{\phi_c} \delta \phi^z \right) \right]
- \left[ \left( a_{IJ} \right|_{\phi_c} \left. + \frac{\partial a_{IJ}}{\partial \phi^z} \right|_{\phi_c} \delta \phi^z \right) F_{\mu\nu}^I F_{\lambda\sigma}^J + g \left( g_{xy} K_i^I \right|_{\phi_c} \left( A_i^I \delta \phi^J \right. \\
+ A_i^J \delta \phi^I \right) + \left( K_{IJ} \right|_{\phi_c} \left( \frac{\partial K_{IJ}}{\partial \phi^z} \right|_{\phi_c} \delta \phi^z \right) g^2 A_i^I A_i^J \right].
\]

(7.2.2)
where we have defined $K_{IJ} = K^*_I K^*_J g_{yy}$. The above equation can be further simplified and written as,

\[
T_{\mu\nu}(\phi_c + \delta \phi) = T^\text{attr}_{\mu\nu}|_{\phi_c} + g K_{IJ}|_{\phi_c} \left( A^{IL} \partial_L (\delta \phi^c) g_{\mu\nu} - A^I_{\mu} \partial_\nu (\delta \phi^c) - A^I_{\nu} \partial_\mu (\delta \phi^c) \right) \\
+ \left[ \frac{\partial a_{IJ}}{\partial \phi^c} \right]_{\phi_c} \left( \frac{1}{4} g_{\mu\nu} F^{\lambda LR} F_{\lambda LR} - F^I_{\mu \lambda} F^J_{\nu \lambda} \right) \\
+ g^2 \frac{\partial K_{IJ}}{\partial \phi^c} \left[ \frac{1}{2} g_{\mu\nu} A^I_{\lambda} A^J_{\lambda} - A^I_{\mu} A^J_{\nu} \right] + \frac{\partial V}{\partial \phi^c} \bigg|_{\phi_c} \delta \phi^c. \tag{7.2.3}
\]

where,

\[
T^\text{attr}_{\mu\nu}|_{\phi_c} = V^\text{attr}(\phi_c) g_{\mu\nu} - \left[ a_{IJ}|_{\phi_c} F^I_{\mu \lambda} A^J_{\lambda} + g^2 K_{IJ}|_{\phi_c} A^I_{\mu} A^J_{\nu} \right]. \tag{7.2.4}
\]

The attractor equations (6.3.1) can be used for further simplification to get,

\[
T_{\mu\nu}(\phi_c + \delta \phi) = T^\text{attr}_{\mu\nu}|_{\phi_c} + g K_{IJ}|_{\phi_c} \left( A^{IL} \partial_L (\delta \phi^c) g_{\mu\nu} - A^I_{\mu} \partial_\nu (\delta \phi^c) - A^I_{\nu} \partial_\mu (\delta \phi^c) \right) \\
- \left[ \frac{\partial a_{IJ}}{\partial \phi^c} \right]_{\phi_c} \left( F^I_{\mu \lambda} F^J_{\nu \lambda} + 8 g^2 K_{IJ}|_{\phi_c} A^I_{\mu} A^J_{\nu} \right) \delta \phi^c. \tag{7.2.5}
\]

It is already clear that for general perturbations of the scalar field, there is backreaction at first order even after using the attractor equations. In particular this requires the fluctuations and their derivatives to be well behaved as one approaches the horizon. Any divergent fluctuation would cause infinite backreaction and deviation from the attractor geometry indicating an instability. Taking the trace of (7.2.5) we get,

\[
T^\mu_{\mu}(\phi_c + \delta \phi) = T^\text{attr}_{\mu\mu}|_{\phi_c} + (d - 2) g K_{IJ}|_{\phi_c} A^{IL} \partial_L (\delta \phi^c) \\
- \left[ \frac{\partial a_{IJ}}{\partial \phi^c} \right]_{\phi_c} \left( F^I_{\mu \nu} F^J_{\nu \mu} + 8 g^2 K_{IJ}|_{\phi_c} A^I_{\mu} A^J_{\nu} \right) \delta \phi^c, \tag{7.2.6}
\]

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where $d$ is the space time dimension. Once again we can use the attractor equations (6.3.1) to simplify, and the Einstein equations take the form,

$$R (2 - d) = T_{\mu}^{\text{attr}} |_{\phi_c} + (d - 2) g K_{\gamma \lambda} |_{\phi_c} A^{\lambda} \partial_\gamma (\delta \phi^\nu)$$

$$+ \left[ g^2 \frac{\partial K_{IJ}}{\partial \phi^c} |_{\phi_c} A^I_{\mu} A^J_{\mu} + 4 \frac{\partial \mathcal{V}}{\partial \phi^c} |_{\phi_c} \right] \delta \phi^z .$$  (7.2.7)

Suppose if the critical points of the attractor potential are also simultaneous critical points of the gauged supergravity scalar potential (as was the case with all the examples discussed in chapter 6), we see that the terms relevant for the backreaction are proportional to $g$:

$$R (2 - d) = T_{\mu}^{\text{attr}} |_{\phi_c} + (d - 2) g K_{\gamma \lambda} |_{\phi_c} A^{\lambda} \partial_\gamma (\delta \phi^\nu) + g^2 \frac{\partial K_{IJ}}{\partial \phi^c} |_{\phi_c} A^I_{\mu} A^J_{\mu} \delta \phi^z .$$  (7.2.8)

Thus, for gauging of $R$ symmetry, $g = 0$ and hence the backreaction is absent:

$$R (2 - d) = T_{\mu}^{\text{attr}} |_{\phi_c} .$$  (7.2.9)

(See §6.5 for some examples of generalised attractor in gauged supergravity with just $R$ symmetry gauging). However, in gauged supergravity with a generic gauging of symmetries of the scalar manifold, the equation depends on the first order fluctuations in the scalar fields. Thus, the generalised attractor geometries in gauged supergravity with a generic gauging can get backreacted by fluctuations of scalar fields. It then follows that the relevant boundary conditions to have stable attractors should be such that the fluctuations and derivatives of fluctuations vanish as one approaches the horizon.
7.3 Scalar fluctuations

In this section, we will analyse the scalar fluctuations in detail using the equation of motion for the scalar fields. The field equation (6.2.12) can be rewritten as:

\[
\hat{e}^{-1} \partial_{\mu} \left[ \hat{e} g_{\gamma\gamma} \nabla_{\mu} \phi \right] - \frac{1}{2} g_{\gamma\gamma} \partial_{\phi^x} \partial_{\phi^y} - g \frac{\partial K_{I\gamma}}{\partial \phi^z} A_{\mu} \nabla_{\mu} \phi - \partial V_{\text{attr}} = 0 .
\] (7.3.1)

We will now expand the scalar fields about their attractor values and keep terms of \(O(\epsilon)\) to get:

\[
g_{\gamma\gamma} |_{\phi_c} \nabla_{\mu} \delta \phi^y \bigg|_{\phi_c} \delta \phi^y + g \left[ \frac{\partial K_{I\gamma}}{\partial \phi^z} - \frac{\partial K_{I\gamma}}{\partial \phi^z} \right]_{\phi_c} A_{\mu} \nabla_{\mu} \delta \phi^y + g \left[ K_{I\gamma} |_{\phi_c} - \frac{\partial K_{I\gamma}}{\partial \phi^z} \delta \phi^y \right]_{\phi_c} A_{\mu} = 0 .
\] (7.3.2)

Here the covariant derivative \(\nabla_{\mu}\) is taken with respect to the zeroth order metrics which represent the near horizon Bianchi geometries. Note that the higher order metric terms which are undetermined are not required at \(O(\epsilon)\). We choose the gauge condition \(\nabla_{\mu} A_{\mu} = 0\) to eliminate the last term. Finally we get,

\[
\nabla_{\mu} \nabla^\mu \delta \phi^y - g^{\gamma\gamma} \nabla_{\mu} \delta \phi^y |_{\phi_c} \delta \phi^y + 2 g \left( g^{\gamma\gamma} \nabla_{\mu} K_{I\gamma} |_{\phi_c} A_{\mu} \nabla_{\mu} \delta \phi^y = 0 ,
\] (7.3.3)

where \(\tilde{\nabla}\) is the covariant derivative with respect to the metric on the scalar manifold \(g_{\gamma\gamma}\).

The Laplacian operator can be written as,

\[
\nabla_{\mu} \nabla^\mu = g^{\mu\nu} \partial_\mu \partial_\nu + \left( g^{\mu\nu} \frac{\partial^2}{\partial \hat{e} \partial \hat{e}} + \partial_\mu \right) \partial_\mu ,
\] (7.3.4)

since the scalar fluctuations depend only on the radial and time co-ordinates.

Before substituting the details, we would like to make some comments on the co-ordinate system used for writing the Bianchi attractor geometries. In [7], the horizon for the Bianchi metrics was located at \(r = -\infty\), where as in chapters 4 and 6 we have chosen
the co-ordinate $\hat{r} = e^r$ such that the horizon lies at $\hat{r} = 0$ instead. As can be seen from the
general form of the Bianchi metrics (4.5.3), the constants $u_0, u_i$ must be positive in order
to have a regular horizon. Thus one can see that the general form of the determinant is,

$$\hat{e} = \sqrt{-\text{det}g_{\mu\nu}} \sim L^5 \hat{r}^m f(x, y, z),$$  \hspace{1cm} (7.3.5)

where $m = -1 + \sum_i c_i u_i$, $u_i$ are the various exponents and $c_i$ is a positive number with $c_0 = 1$ for all Bianchi attractors. For example, in the Bianchi II case (see (6.4.33)) $m = -1 + u_0 + 2(u_1 + u_3)$. We can also see that,

$$g^{\hat{r}\hat{r}} = \hat{r}^2, \quad g^{\hat{t}\hat{t}} = -\frac{1}{\hat{r}^{2m_0}} L^2, \hspace{1cm} (7.3.6)$$

for all Bianchi attractors. Using the above data, the Laplacian (7.3.4) can be expressed as,

$$\nabla_\mu \nabla^\mu = \frac{1}{L^2} \left[ \hat{r}^2 \partial_\hat{r}^2 + (m + 2)\hat{r} \partial_\hat{r} - \frac{1}{\hat{r}^{2m_0}} \partial_\hat{t}^2 \right].$$  \hspace{1cm} (7.3.7)

Substituting (7.3.7) in (7.3.3) and using the ansatz (6.4.5) for $A^I_\mu$ we get,

$$\left[ \hat{r}^2 \partial_\hat{r}^2 + (m + 2)\hat{r} \partial_\hat{t} - \frac{1}{\hat{r}^{2m_0}} \partial_\hat{t}^2 \right] \delta \phi^\mu - M_\phi^\mu |_{\phi_c} \delta \phi^\mu + N_\phi^\mu |_{\phi_c} \frac{1}{\hat{r}^{2m_0}} \partial_\hat{t} \delta \phi^\mu = 0$$  \hspace{1cm} (7.3.8)

where,

$$M_\phi^\mu |_{\phi_c} = L^2 g^{\hat{x}\hat{x}} \frac{\partial^2 V_{\text{attr}}}{\partial \phi^\mu \partial \phi^\nu} \bigg|_{\phi_c}, \quad N_\phi^\mu |_{\phi_c} = 2 g L \hat{r}^0 (g^{\hat{x}\hat{y}} \nabla_y K_{\text{ct}})|_{\phi_c}. \hspace{1cm} (7.3.9)$$

The metric on the moduli space $g_{\phi\phi}$ is chosen to be positive definite and the nature of the
critical point is given by the sign of the double derivative of the attractor potential. We
further assume that $M_\phi^\mu |_{\phi_c}$ is diagonal so that,

$$M_\phi^\mu |_{\phi_c} \delta \phi^\nu = \lambda \delta \phi^\mu.$$  \hspace{1cm} (7.3.10)

The term $N_\phi^\mu |_{\phi_c}$ can be non-zero in general, but vanishes trivially for the gauged supergravity
model where we found some examples Bianchi attractors (see §(5.4)). There is only one Killing vector (5.4.14) that generates the $SO(2)$ isometry on the scalar manifold, and the critical point is such that $\phi^2_c = \phi^3_c = 0$. Therefore one is left with just the $\bar{\nabla}_i K_{ix}$ component which vanishes due to the Killing vector equation on the manifold.\footnote{Here, the single surviving component of the Killing vector is along the direction of $\phi^1$ on the scalar manifold.}

Thus, the scalar fluctuation equation (7.3.3) has the final form,

\[
\begin{split}
\hat{r}^2 \partial_{\hat{r}}^2 + (m + 2)\hat{r} \partial_{\hat{r}} - \frac{1}{\hat{r}^2 \kappa_0} \partial_{\hat{r}}^2 - \lambda \delta \phi^x = 0.
\end{split}
\] (7.3.11)

The above equation admits a simple solution when the fluctuations $\delta \phi^x$ are time independent. In this case, we have,

\[
\delta \phi^x = C_1 \hat{r} \left( \frac{\sqrt{4 \lambda + (1 + m)^2 - (1 + m)}}{2} \right) + C_2 \hat{r} \left( - \frac{\sqrt{4 \lambda + (1 + m)^2 - (1 + m)}}{2} \right).
\] (7.3.12)

Thus, one of the modes vanishes as $r \to 0$ provided $\lambda$ is positive and it is possible to get stable attractors upon setting $C_2 = 0$. However, all the explicit examples we discussed in chapter 6, do not admit a critical point with $\lambda > 0$. Thus, such fluctuations are unstable.

Now we turn to the case of time dependent fluctuations. Since the equation for $\delta \phi^x$ is separable, we try the ansatz $\delta \phi(\hat{r}, \hat{t}) = f(\hat{r}) e^{ik\hat{t}}$ (with $k$ real) to get the Bessel equation:

\[
\begin{split}
\hat{r}^2 \partial_{\hat{r}}^2 + (m + 2)\hat{r} \partial_{\hat{r}} + \left( \frac{k^2}{\hat{r}^2 \kappa_0} - \lambda \right) f(\hat{r}) = 0.
\end{split}
\] (7.3.13)

The general solutions for this equation are given by the standard Bessel functions (see, for example, [167], page 932):

\[
f(X) = \left( \frac{X}{2} \right)^m \left[ C_1 \Gamma(1 - \nu_i) J_{-\nu_i}(X) + C_2 \Gamma(1 + \nu_i) J_{\nu_i}(X) \right],
\] (7.3.14)
where,

\[ X = \frac{k}{u_0^2}, \quad \nu = \frac{\sqrt{(1 + m)^2 + 4\lambda}}{2u_0}, \quad \nu_0 = \frac{(1 + m)}{2u_0}, \quad (7.3.15) \]

\[ C_1 \text{ and } C_2 \text{ are arbitrary constants, and the Bessel functions are,} \]

\[ J_{\nu_0}(X) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j + \nu_0 + 1)} \left( \frac{X}{2} \right)^{2j}, \]

\[ J_{-\nu_0}(X) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j - \nu_0 + 1)} \left( \frac{X}{2} \right)^{2j}. \quad (7.3.16) \]

The power series representation is valid in the small \( X \) or equivalently, in the large \( r \) regime. We can rewrite the solution in terms of the Hankel functions,

\[ J_{\nu}(X) = \frac{1}{2}(H^1_{\nu}(X) + H^2_{\nu}(X)), \]

\[ J_{-\nu}(X) = \frac{1}{2}(H^1_{\nu}(X)e^{i\nu\pi} + H^2_{\nu}(X)e^{-i\nu\pi}), \quad (7.3.17) \]

to get,

\[ f(X) = \left( \frac{X}{2} \right)^{\nu} \left[ C_1 H^1_{\nu_0}(X) \left( \Gamma(1 - \nu_0) e^{i\nu\pi} + \Gamma(1 + \nu_0) \right) \right. \]

\[ + C_2 H^2_{\nu_0}(X) \left[ \Gamma(1 - \nu_0) e^{-i\nu\pi} + \Gamma(1 + \nu_0) \right]. \quad (7.3.18) \]

As one can see from above equation, there is already a restriction on \( \nu_0 \) from the Gamma function that appears in the general solution. First let us consider the case \( \nu_0 \) real, then we have the condition,

\[ \nu_0 = \sqrt{(1 + m)^2 + 4\lambda} = \sqrt{(\sum_l c_l u_l)^2 + 4\lambda} \leq 1, \quad (7.3.19) \]

for,

\[ -\frac{(\sum_l c_l u_l)^2}{4} \leq \lambda < 0. \quad (7.3.20) \]
Note that only negative $\lambda$ can satisfy (7.3.19). Since $c_i > 0$ and all the $u_l$ have to be positive for the existence of a regular horizon, we conclude that $\lambda$ has to be negative. Remember that the sign of $\lambda$ is provided by the double derivative of the attractor potential eqs. (7.3.9,7.3.10). This implies that the critical points correspond to maxima of the attractor potential. For the case of imaginary $\nu_{\lambda}$ we have,

$$\lambda < \frac{-(\sum l c_l u_l)^2}{4}, \quad (7.3.21)$$

and hence, even in this case the critical points correspond to a maxima of the attractor potential. Thus we have determined the general solution for the scalar fluctuation (7.3.18) and we find that they are well behaved at large distance provided they satisfy the conditions (7.3.20,7.3.21). This may be useful for the study of attractor flow equations for black holes in $AdS$.

### 7.4 Stable Bianchi attractors

In this section, we will analyse the stability of the Bianchi attractors by studying the behaviour of the solution in the $r \to 0$ limit. We are interested in the question which class of the Bianchi attractors can be stable attractor geometries in gauged supergravity. This can be answered by looking at the near horizon behaviour of the scalar fluctuations (7.3.18). From our analysis of the stress energy tensor in gauged supergravity (7.2.8), we find that there is dependence on the fluctuations and their derivatives at first order perturbation. Hence, we only require that the fluctuations do not blow up near the horizon as that would backreact strongly and deviate from the geometry. This requirement places some constraints on the form of the metric itself as we explain in the rest of the section.

Both the solutions in (7.3.18) are given in terms of the Hankel functions, the behaviour near the horizon can be determined by considering the asymptotic expansions of the Han-
Hankel functions. Remember that the horizon for the Bianchi metrics (4.5.3) is located at $\hat{r} = 0$. The form of the solution (7.3.18) makes it convenient to use the asymptotic expansions of the Hankel functions, since from (7.3.15) $X \to \infty$ as $\hat{r} \to 0$. The asymptotic expansions are given by,

$$H^1_{\nu}(X) \sim \sqrt{\frac{2}{\pi X}} e^{i(X - \frac{\pi}{2}(\nu + \frac{1}{2}))},$$

$$H^2_{\nu}(X) \sim \sqrt{\frac{2}{\pi X}} e^{-i(X - \frac{\pi}{2}(\nu + \frac{1}{2}))}.$$

(7.4.1)

Substituting (7.4.1) in (7.3.18) we determine the behaviour of the fluctuation near the horizon as,

$$f(X) \sim \left(\frac{X}{2}\right)^{\nu_0 - \frac{1}{2}} \sqrt{\frac{\Gamma(1 - \nu_A)}{\Gamma(1 + \nu_A)}} [C_1 e^{i(X - \frac{\pi}{2}(\nu_A + \frac{1}{2}))[\Gamma(1 - \nu_A) e^{i\nu_A \pi} + \Gamma(1 + \nu_A)]}
+ C_2 e^{-i(X - \frac{\pi}{2}(\nu_A + \frac{1}{2})[\Gamma(1 - \nu_A) e^{-i\nu_A \pi} + \Gamma(1 + \nu_A)]}].$$

(7.4.2)

Since $X \sim \frac{1}{\hat{r}^0}$ and $u_0 > 0$, there is a leading divergent term as $\hat{r} \to 0$ unless,

$$\frac{1 - 2\nu_0}{2} \geq 0,$$

(7.4.3)

which can be rewritten as,

$$\nu_0 = \frac{(1 + m)}{2u_0} = \sum_{l,l\neq 0} C_l u_l \leq 1,$$

(7.4.4)

Since $c_0 = 1$, this implies,

$$\sum_{l,l\neq 0} C_l u_l \leq 0,$$

(7.4.5)

which can never be satisfied without some of the exponents $u_l$ being negative. Since we require a regular horizon, all the exponents have to be positive. Thus the only possibility for which eq. (7.4.5) can be satisfied is,

$$u_0 \neq 0, \quad u_l = 0 \quad \forall \ l \neq 0.$$

(7.4.6)
The conditions on \( \lambda \) (7.3.20),(7.3.21) for the general solution (7.3.18) to exist can now be written as,

\[
- \frac{u_0^2}{4} \leq \lambda < 0 ,
\]

(7.4.7)

for real \( \nu_\lambda \) and,

\[
\lambda < - \frac{u_0^2}{4} ,
\]

(7.4.8)

for imaginary \( \nu_\lambda \). To summarise, Bianchi attractors are stable against scalar fluctuations about the attractor value for the class of metrics which satisfy the condition (7.4.6).

The condition (7.4.6), is highly restrictive on the form of the Bianchi metrics. In particular it follows from (7.4.6) that \( v_0 = \frac{1}{2} \) for any \( u_0 > 0 \) and the scalar fluctuations (7.4.2) do not diverge near the horizon.\(^4\) In particular this restricts the metrics (4.5.3) to be of the form,

\[
ds^2 = L^2 \left[ - \hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \eta_{ij} \omega^i \otimes \omega^j \right] .
\]

(7.4.9)

It is very interesting to note that the symmetry group of this metric form factorises into a direct product of the \((1+1)\) dimensional Lifshitz group and a group in the Bianchi classification. This is similar to what happens for example in four dimensional extremal black holes where the near horizon geometry factorises as \( AdS_2 \times S^2 \).

The simplest non-trivial example of this class is the \( L_i f_{u_0}(2) \times M_I \) solution,

\[
ds^2 = L^2 \left[ - \hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + (d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2) \right] ,
\]

(7.4.10)

one obtains the \( AdS_2 \times \mathbb{R}^3 \) solution when \( u_0 = 1 \). Another less trivial example is the \( L_i f_{u_0}(2) \times M_{II} \) solution,

\[
ds^2 = L^2 \left[ - \hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + (d\hat{x}^2 + d\hat{y}^2 - 2 \hat{x} d\hat{y} d\hat{z} + (\hat{x}^2 + 1) d\hat{z}^2) \right] .
\]

(7.4.11)

\(^4\)Note that there are still oscillatory terms in the fluctuation.
Table 7.1: Bianchi attractor geometries in gauged supergravity, nature of critical points and stability. The first three entries are for the solutions found in [17] and discussed in §6.4. The next three entries are generalised attractors in $U(1)_R$ gauged supergravity discussed in §6.5. The last entry with the * is the most general possible Bianchi attractor geometry (7.4.9) that satisfies our stability criteria.

We have constructed the $Lif_{u_0}(2) \times M_I$ for any $u_0 > 0$ and a $Lif_{u_0}(2) \times M_{II}$ in a simple $U(1)_R$ gauged supergravity theory with one vector multiplet. These solutions were discussed in §6.5. It can be seen from Table (7.1), that these solutions satisfy our stability criteria (7.4.6) and hence are examples of stable Bianchi attractors in gauged supergravity.

The examples we constructed earlier in [17] (discussed in §6.4) all have $\lambda < 0$ and exist at maxima of the attractor potential. Therefore the condition (7.3.19) allows scalar fluctuations about the attractor values. However as one can see from table (7.1) all the metrics have some $u_l \neq 0$ for $l \neq 0$ and do not satisfy our stability condition(7.4.6). Hence the radial fluctuation of the scalar field diverges near the horizon for all these metrics. To complicate matters further, as one can see from (7.2.8) the fluctuations and their derivatives backreact on the geometry strongly. Thus there would be significant deviation of the geometry even at the first order and we conclude that these geometries are unstable attractors in the theory. These results are summarised in Table (7.1).
7.5 Summary

In this chapter, we have studied the stability of Bianchi attractors in gauged supergravity by considering scalar fluctuations about the attractor value. In general, the stress energy tensor in a generic gauged supergravity depends on the scalar fluctuations and their derivatives even at first order perturbation. Therefore, it is important that the scalar fluctuations are well behaved near the horizon. In particular, if there is a large backreaction then the geometry would deviate from the attractor geometry. Hence the fluctuations must vanish as one approaches the horizon for the attractor geometry to be stable.

We analysed the scalar fluctuation equations and found that the fluctuations can exist in general when the attractor geometries in consideration exist at critical points which, in the present case, correspond to maxima of the attractor potential. By demanding that the fluctuations vanish as one approaches the horizon we determined the conditions of stability for the metric. We found that the Bianchi attractors are stable if the metric factorises as,

\[ ds^2 = L^2 \left( -\hat{r}^{2u_0} d\hat{r}^2 + \frac{d\hat{r}^2}{\hat{r}^2} \right) + L^2 (\eta_{ij} \omega^i \otimes \omega^j), \]  

which is a subclass of the Bianchi attractors discussed in chapter 4. We have referred to this class of metrics as \( Li f_{\text{ref}}(2) \times M \), where \( M \) refers to three dimensional manifolds invariant under the nine groups given by the Bianchi classification. As stated before, these solutions exist for critical points which are maxima of the attractor potential and they satisfy all the conditions of stability. It would be interesting to explore whether this is a generic feature of attractors in gauged supergravity or an artifact of the gauged supergravity models considered.