Chapter 5

Gauged supergravity

5.1 Introduction

In chapter 4, we studied five dimensional homogeneous brane geometries classified by the Bianchi classification. In a recent work, Bianchi I type geometries such as the Lifshitz solution where embedded in $\mathcal{N} = 2, d = 4$ gauged supergravity using the generalised attractors procedure [14]. In this approach, one sets the bosonic fields in the theory to be constants in tangent space, which results in solvable algebraic field equations. We wish to extend the study to five dimensions and also realise the Bianchi type metrics as generalised attractor solutions. This requires some background in five dimensional gauged supergravity which we provide in this chapter.

Gauged supergravities are supersymmetry preserving deformations of ungauged supergravity. The deformations are implemented by promoting some of the global symmetries of the ungauged theory to local symmetries. Gaugings are usually done by coupling the symmetry generators to corresponding gauge fields. The first example of gauged supergravity was obtained by gauging the $SO(8)_R$ global symmetry of $\mathcal{N} = 8$ supergravity [128]. Gauged supergravities with non-compact gaugings were constructed in [129–131] and generalisation to higher dimensions were constructed in [132–134]. More recently,
gauged supergravities are understood as low energy effective theories that describe flux compactifications of string theory. For example, the low energy theory from Type IIB string theory compactified on a Calabi-Yau manifold in the presence of Ramond-Ramond and Neveu-Schwarz fluxes for the three form fields is a $\mathcal{N} = 2, d = 4$ gauged supergravity [8–10].

Ungauged supergravity contains free scalar fields called moduli that take values on a moduli space. The moduli parametrise a non-linear sigma model that defines a manifold. For example, the non-linear sigma model for the scalars in the vector multiplet of $\mathcal{N} = 2, d = 4$ supergravity defines a Kähler manifold [135]. While in $\mathcal{N} = 2, d = 5$ supergravity, the corresponding scalar manifold is real and very special [136, 137]. The scalars in the hypermultiplet parametrise a quaternionic manifold in both cases [138].

When the symmetries of the scalar manifold leave the non-linear sigma model invariant, they often extend to symmetries of the full Lagrangian. For example, in four dimensions the symmetries of the scalar manifold always extend to symmetries of the full supersymmetric Lagrangian, whereas in five dimensions symmetries of the scalar manifold can sometimes be broken by supergravity interactions [139]. The R-symmetry group, which is an automorphism of the Poincaré superalgebra is another global symmetry of the theory. Gauged supergravity is obtained by gauging some or all of the global symmetries of supergravity.

In the context of the AdS/CFT correspondence [11], gauged supergravity generically describes the supergravity regime of the bulk theory. This is due to the fact that many gauged supergravities support an AdS vaccum due to the presence of non-trivial potentials for the scalar fields in the theory. The potential terms are of the order $O(g^2)$, where $g$ is the gauge coupling constant. Supersymmetry requires the presence of the potential term to compensate for the additional terms that appear in the covariant derivatives of gauged supergravity. When the scalar fields take their extremum, the value of the potential sets the cosmological constant of the theory. For example, the $\mathcal{N} = 8, d = 5 \text{SO}(6)$ gauged super-
gravity [134] describes type IIB supergravity compactified on $AdS_5 \times S^5$. According to the AdS/CFT correspondence [11] this theory is dual to the four dimensional $SU(N), \mathcal{N} = 4$ super Yang-Mills theory as discussed in the introduction of the thesis.

In this chapter, we describe the necessary background in five dimensional gauged supergravity. We begin with a discussion on the global symmetries of the ungauged theory in §5.2. We discuss the global symmetries of the very special, quaternionic manifolds and the R symmetry. We then follow up with a discussion of gauged supergravity in §5.3. In particular, we focus on $\mathcal{N} = 2, d = 5$ gauged supergravity coupled to vector, tensor and hypermultiplets [12, 13, 140]. Towards the end of the chapter we discuss a simple gauged supergravity model with one vector multiplet in §5.4. Useful supplementary material is provided in §C.

5.2 $\mathcal{N} = 2, d = 5$ supergravity

5.2.1 Field content

The $\mathcal{N} = 2, d = 5$ ungauged supergravity, often called as Maxwell-Einstein supergravity was constructed in [132, 134, 136]. The field contents of the theory are the following.

- The gravity multiplet contains the graviton $e^a_\mu$, two gravitinos $\psi^i_\mu$ and a graviphoton.

- The vector multiplet contains a vector field $A_\mu$, $SU(2)_R$ doublet of fermions (gauginos) $\lambda^i$ and a real scalar field $\phi$.

- The hyper multiplet contains a doublet of fermions (hyperinos) $\zeta^A$ with $A = 1, 2$ and four real scalars $q^X$ with $X = 1, \ldots, 4$.

The $n_V$ vector multiplets together with the graviphoton constitute $n_V + 1$ vectors $A^I_\mu, I = 0, \ldots, n_V$. The vector multiplet contains $n_V$ scalars $\phi^x, x = 1, 2, \ldots, n_V$ and the hyper mul-
triplet contains $4n_H$ scalars $q^X$, with $X = 1, 2, \ldots, 4n_H$. The bosonic part of the Lagrangian is given by,

$$
\hat{e}^{-1} \mathcal{L}_{Bosonic}^{N=2} = -\frac{1}{2} R - \frac{1}{4} a_{IJ} F_{\mu \nu}^{I} F^{J \mu \nu} - \frac{1}{2} g_{XY} \partial_\mu q^X \partial^\mu q^Y - \frac{1}{2} g_{xY} \partial_\mu \phi^x \partial^\mu \phi^y + \hat{e}^{-1} C_{IJK} \epsilon^{\mu \nu \rho \sigma \tau} F_{\mu \nu}^I F_{\rho \sigma}^J A^K_\tau ,
$$

(5.2.1)

where $\hat{e} = \sqrt{-\text{det} g_{\mu \nu}}$, $a_{IJ}$ is the ambient metric used to raise and lower the vector indices, $g_{xy}$ is the metric on the scalar manifold and $g_{xy}$ is the metric on the quaternionic manifold. The coefficients $C_{IJK}$ that appear with the Chern-Simons term are constant symmetric tensors.

### 5.2.2 Global Symmetries

The scalars in the theory parametrise a manifold that factorises into direct product of a very special and a quaternionic manifold,

$$
\mathcal{M}_{scalar} = S(n_v) \otimes Q(n_H) .
$$

(5.2.2)

Some important references for this section are [12, 140–143].

**Very special Manifold**

The scalars in the vector multiplet are real and parametrise a very special manifold in five dimensions. A very special manifold $S$ is a real $n$ dimensional manifold defined by the hypersurface,

$$
N \equiv C_{IJK} h^I h^J h^K = 1 ,
$$

(5.2.3)
where the $h' \equiv h'(\phi)$ are co-ordinates in $\mathbb{R}^{n+1}$. The metric on the very special manifold is given by the pullback of the metric on $\mathbb{R}^{n+1}$,

$$ds_{\mathbb{R}^{n+1}}^2 = a_{IJ} dh^I \otimes dh^J ,$$

$$a_{IJ} = -\frac{1}{2} \frac{\partial}{\partial h^I} \frac{\partial}{\partial h^J} \ln N \Big|_{N=1} .$$

In a given co-ordinate frame, the metric on the scalar manifold $g_{xy}$ is then defined as,

$$g_{xy} = h'_x h'_y a_{IJ} ,$$

$$a_{IJ} = h_I h_J + h'_I h'_J g_{xy} ,$$

where $h'_I$ are defined by,

$$\frac{\partial h_I}{\partial \phi^x} \equiv h_{I,x} = \sqrt{\frac{2}{3}} h_{I,x} , \quad \frac{\partial h'_I}{\partial \phi^x} \equiv h'_I , x = -\sqrt{\frac{2}{3}} h'_I .$$

In supergravity one often works in the frame language and the following relations are useful,

$$f^a_x f^b_y \eta_{ab} = g_{xy} ,$$

$$f^a_{[x,y]} + \Omega^{a}_{[x,y]} f^b_{x} = 0 .$$

Here $f^a_x$ and $\Omega^{a}_{b}$ are the $n_V$-bein and the spin connection on $S$ respectively. The indices $a, b$ are flat indices and $\eta_{ab}$ is the flat metric with signature $\{-,+,\ldots\}$.

The symmetries of the scalar manifold are the transformations that leave (5.2.3) invariant. These symmetries can be made manifest when the kinetic term of the scalars in the vector multiplet,

$$- \frac{1}{2} g_{xy} \partial_\mu \phi^x \partial^\mu \phi^y ,$$

is written in terms of $h'$. The completeness relations (5.2.5) can be used to rewrite the
kinetic term as,

\[-\frac{1}{2} a_{IJ} h_i h_j \partial_\mu \phi^i \partial^\mu \phi^j = \frac{3}{4} a_{IJ} h_i h_j \partial_\nu \phi^i \partial^\nu \phi^j = -\frac{3}{4} a_{IJ} \partial_\mu h^j \partial^\mu h^l. \tag{5.2.9}\]

The definition of the ambient metric (5.2.4) can be simplified to obtain

\[a_{IJ} = \frac{3}{2} C_{IJK} h^K,\]

using the relations \( C_{IJK} h^J h^K = \frac{1}{\alpha^2} h_I \) and \( C_{IJK} h^K = \frac{1}{\alpha} h_I h_J \), where \( \alpha = h^I h_I \). Using the above relations the scalar kinetic term takes the form,

\[-\frac{1}{2} g_{\nu \xi} \partial_\mu \phi^i \partial^\mu \phi^j = -\frac{9}{8} C_{IJK} h^J \partial_\mu h^l \partial^\mu h^K. \tag{5.2.10}\]

In this form, the scalar kinetic term manifestly exhibits the symmetries of the scalar manifold. Consider a group of linear transformations,

\[ h^I \rightarrow B^I h^I. \tag{5.2.11}\]

These are symmetries of the scalar manifold if (5.2.3) is invariant, which requires the \( C_{IJK} \) to transform as,

\[ B^I B^N B^K C_{MNP} = C_{IJK}. \tag{5.2.12}\]

From (5.2.10) and (5.2.12), it is clear that (5.2.11) are symmetries of the sigma model. The transformations (5.2.11) extend to the full Lagrangian (5.2.1) provided the gauge fields transform as,

\[ A^I \rightarrow B^I A^I. \tag{5.2.13}\]

This can be seen by using the relation \( a_{IJ} = \frac{3}{2} C_{IJK} h^K \) and (5.2.12) in the kinetic term for the gauge fields. Thus the symmetries of the scalar manifold are global symmetries of the Lagrangian. Note that so far the \( C_{IJK} \) are unspecified and arbitrary. Due to this, for a fixed number of vector multiplets several target manifolds are possible. In fact, from (5.2.12) it is evident that the classification of the \( C_{IJK} \) (with (5.2.3) satisfied) is equivalent to classification of the very special manifolds. This approach has been pursued in the
literature and the classification of symmetric very special manifolds was done in [136, 144, 145]. This was extended to include very special manifolds that are homogeneous spaces in [141].

As a simple example, consider the symmetric very special manifold which belongs to the “generic Jordan class” in the classification with a coset structure [136,137],

\[ M = \frac{SO(n-1,1) \times SO(1,1)}{SO(n-1)}, \quad n \geq 1. \]  

(5.2.14)

The symmetry group of this manifold is given by \( G = SO(n-1,1) \times SO(1,1) \). This symmetry group can be made manifest by choosing a suitable parametrisation to satisfy the constraint (5.2.3). For example, in terms of co-ordinates \( \xi^I \) on \( \mathbb{R}^{n+1} \), (5.2.3) takes the form [146],

\[
N(\xi) = \left(\frac{2}{3}\right)^{\frac{1}{2}} C_{IJK} \xi^I \xi^J \xi^K ,
\]

\[ = \sqrt{2} \xi^0 \left[ (\xi^1)^2 - (\xi^3)^2 - (\xi^5)^2 - \ldots - (\xi^n)^2 \right]. \]  

(5.2.15)

One can see that the symmetry group \( G \) is manifest in this parametrisation. Of course the parametrisation also has to satisfy \( N(\xi) = 1 \), which can be solved in terms of the scalar fields in the Lagrangian by choosing \( \xi \equiv \xi(\phi) \) as,

\[
\xi^0 = \frac{1}{\sqrt{2||\phi||^2}},
\]

\[
\xi^1 = \phi^1,
\]

\[ \vdots \]

\[
\xi^n = \phi^n , \]  

(5.2.16)

where,

\[ ||\phi||^2 = (\phi^1)^2 - (\phi^2)^2 - \ldots - (\phi^n)^2 , \]  

(5.2.17)
In any parametrisation, the scalar fields must be restricted to suitable domains such that the metric on scalar manifold \( g_{xy} \) and \( a_{IJ} \) are positive definite. This ensures that the kinetic terms in the Lagrangian (5.2.1) have proper sign. In principle, the above information is sufficient to determine the metrics \( a_{IJ} \) and \( g_{xy} \) completely using (5.2.4) and (5.2.5) respectively. We will show this in §5.4, where we consider a simple gauged supergravity model with one vector multiplet and the very special manifold is an example of the symmetric spaces discussed above.

**Quaternionic Kähler manifold**

The hypermultiplet in a \( \mathcal{N} = 2 \) supergravity theory contains four real scalars which are locally considered as components of a quaternion \( q^X \). For \( n_H \) hypermultiplets, the \( q^X \) parametrise a quaternionic Kähler manifold\(^1\) \( Q \) endowed with a metric,

\[
 ds^2 = g_{XY}(q) dq^X \otimes dq^Y , \quad X, Y = 1, 2, \ldots 4n_H ,
\]

and three complex structures \( J^x \) that satisfy the quaternionic identity,

\[
 (J^x)_{[X}^Z (J^y)_{Y]}^V = -\delta^V_Y (Id)_X^Z + \epsilon^{YZ}(J^z)_X^Y ,
\]

where \( x = 1, 2, 3 \). The metric on \( Q \) is hermitian with respect to \( J^x \),

\[
 (J^x)_Y^X (J^y)_W^V g_{XY} = g_{VW} .
\]

The existence of a hermitian metric together with a complex structure defines a natural two form on the manifold. This can be seen by multiplying the above equation with \( (J^x)_Z^V \),

\(^1\)Hypermultiplets can appear in rigid supersymmetric Yang-Mills theories as well as supergravity theories. In the former case, the scalar manifold is HyperKähler, while in the latter it is Quaternionic. The difference in the two cases lies in the curvature of the principal bundle.
using (5.2.19) and defining \( K^x_{xy} = g_{yz}(J^x)^z_y \) to get,

\[
K^x_{wz} = -K^z_{wx} \quad , \quad K^x = K^x_{xy} d\theta^y \land d\theta^y .
\]  

(5.2.21)

The natural two form \( K^x \) on the quaternionic manifold is called a HyperKähler form. It also implies that the manifold \( Q \) has a symplectic structure. Supersymmetry requires the existence of an \( SU(2) \) principal bundle \( SU \to Q \) and \( \omega^x \) is the connection on such a bundle. The HyperKähler form is covariantly closed with respect to the connection \( \omega^x \),

\[
\nabla K \equiv dK^x + \epsilon^{xyz} \omega^y \land K^z = 0 .
\]  

(5.2.22)

The curvature on the \( SU(2) \) bundle can then be defined as,

\[
\Omega^x \equiv d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \land \omega^z .
\]  

(5.2.23)

Hence, for quaternionic manifolds the curvature on the principal bundle is proportional to the HyperKähler form,

\[
\Omega^x = \lambda K^x ,
\]  

(5.2.24)

where \( \lambda \) is real number related to the scale of the Quaternionic manifold. Since the tangent space is not flat, the holonomy group of \( Q \) is \( SU(2) \otimes \mathcal{H} \) with \( \mathcal{H} \subset Sp(2n_H) \). One can introduce Quaternionic vielbeins \( f^X_iA \) (with \( i \in SU(2) \) and \( A \in Sp(2n_H) \)) as follows,

\[
\begin{align*}
    f^X_iC_j + f^Y_iC_j &= g^{XY} \epsilon_{ij} , \\
    g_{XY} f^X_iA f^Y_jB &= \epsilon_{ij} C_{AB} , \\
    f^X_iB f^Y_iB + f^Y_iB f^X_iB &= \frac{1}{n_H} g^{XY} C_{AB} ,
\end{align*}
\]  

(5.2.25)

where \( \epsilon_{ij} \) and \( C_{AB} \) are \( SU(2) \) and \( Sp(2n_H) \) invariant tensors respectively [13].

As discussed above, the quaternionic manifold is Riemannian, has a complex structure
and a compatible symplectic structure. Thus the quaternionic manifold is also a Kähler manifold [105, 106]. The classification of homogeneous Quaternionic manifolds first appeared in the mathematics literature in [147], and is further discussed in [141–143].

We now describe a simple example of a Quaternionic Kähler manifold [54, 105, 106],

$$\frac{SU(2, 1)}{SU(2) \times U(1)},$$

(5.2.26)

and illustrate its symmetries. As argued before, the quaternionic manifold is also Kähler and hence the metric can be derived from a suitable Kähler potential. Following [54], let us denote the quaternion \(q^X = \{V, \sigma, \theta, \tau\}\) and define the variables,

\[
S = V + \theta^2 + \tau^2 + i\sigma, \quad C = \theta - i\tau.
\]

(5.2.27)

The Kähler potential has the form,

\[
K = -\frac{1}{2} \log(S + \bar{S} - 2C\bar{C}).
\]

(5.2.28)

The metric on \(Q\) is defined by,

\[
g_{a\bar{b}} = \frac{\partial^2 K}{\partial z_a \partial \bar{z}_b}, \quad z_a = S, C.
\]

(5.2.29)

and simplifies to,

\[
ds^2 = e^{4K} \left[ \frac{1}{2} dS d\bar{S} + 2C\bar{C} dC d\bar{C} - \bar{C} dC d\bar{S} - CdC d\bar{S} \right] + e^{2K} dCd\bar{C}.
\]

(5.2.30)

This can be rewritten in terms of the original co-ordinates \(q^X = \{V, \sigma, \theta, \tau\}\) as,

\[
ds^2 = \frac{1}{2V^2} (dV^2 + (d\sigma + 2\theta d\tau - 2\tau d\theta)^2) + \frac{2}{V} (d\theta^2 + d\tau^2).
\]

(5.2.31)

The symmetries of this metric are the symmetries of the sigma model. The full set of
Killing vectors $k^X_a$ given below generate an $SU(2, 1)$ algebra [54].

$$
k_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 \\ 2\theta \\ 0 \\ 1 \end{pmatrix}, \quad k_3 = \begin{pmatrix} 0 \\ -2\tau \\ 0 \\ 0 \end{pmatrix}, \quad k_4 = \begin{pmatrix} 0 \\ 0 \\ -\tau \\ \theta \end{pmatrix}, \quad k_5 = \begin{pmatrix} V \\ \sigma \\ \theta/2 \\ \tau/2 \end{pmatrix},
$$

$$
k_6 = \begin{pmatrix} 2V\sigma \\ \sigma^2 - (V + \theta^2 + \tau^2)^2 \\ \sigma\theta - \tau(V + \theta^2 + \tau^2) \\ \sigma\tau + \theta(V + \theta^2 + \tau^2) \end{pmatrix}, \quad k_7 = \begin{pmatrix} -2V\sigma \\ -\sigma\theta + V\tau + \tau(\theta^2 + \tau^2) \\ (V - \theta^2 + 3\tau^2)/2 \\ -2\theta\tau - \sigma/2 \end{pmatrix},
$$

$$
k_8 = \begin{pmatrix} -2V\tau \\ -\sigma\tau - V\theta - \theta(\theta^2 + \tau^2) \\ -2\theta\tau + \sigma/2 \\ (V + 3\theta^2 - \tau^2)/2 \end{pmatrix}. \quad (5.2.32)
$$

The Killing vectors $(k_1, k_2, k_3)$ generate translations in the $(\sigma, \theta, \tau)$ respectively, $k_4$ generates rotations in $(\sigma, \tau)$, while $k_5$ corresponds to dilatations and $(k_6, k_7, k_8)$ generate other isometries of (5.2.31). The manifold (5.2.26) has the symmetry group $SU(2) \times U(1)$, this can be seen by rewriting the Killing vectors as,

$$
T_1 = \frac{1}{4}(k_2 - 2k_8), \quad T_5 = \frac{-i}{2}(k_1 - k_6), \\
T_2 = \frac{1}{4}(k_3 - 2k_7), \quad T_6 = \frac{-i}{4}(k_3 + 2k_7), \\
T_3 = \frac{1}{4}(k_1 + k_6 - 3k_4), \quad T_7 = \frac{-i}{4}(k_2 + 2k_8), \\
T_4 = ik_5, \quad T_8 = \frac{\sqrt{3}}{4}(k_1 + k_4 + k_6). \quad (5.2.33)
$$
Here, \( T_1, T_2, T_3 \) generate the \( SU(2) \) algebra, \( T_8 \) is the \( U(1) \) generator and \( T_4, T_5, T_6, T_7 \) generate \( \frac{SU(2,1)}{U(2)} \).

**R symmetry**

The R-symmetry group, which is an automorphism of the Poincaré superalgebra is a global symmetry of the supergravity theory. In this section, we motivate the \( SU(2) \) action of the R symmetry group by describing the \( N = 2 \) superconformal algebra in \( d = 5 \) \([148, 149]\). The spinors in 5d are symplectic majorana and the conventions are described in the appendix (B). According to Nahm’s classification \([150]\), the bosonic subgroup of the superconformal group is a direct product of the conformal group and the R symmetry group. For \( N = 2, d = 5 \), the bosonic subgroup is \( SO(5,2) \times SU(2)_R \). The \( SO(5,2) \) conformal group generated by translations \( P_\mu \), Lorentz transformations \( M_{\mu\nu} \), dilatations \( D \) and special conformal transformations \( K_\mu \) is given by,

\[
[M_{\mu\nu}, M_{\lambda\sigma}] = \eta_{\mu[\lambda} M_{\sigma]\nu} - \eta_{\nu[\lambda} M_{\sigma]\mu} , \\
[P_\mu, M_{\nu\lambda}] = \eta_{\nu[\lambda} P_{\sigma]\mu} , \quad [K_\mu, M_{\nu\lambda}] = \eta_{\nu[\lambda} K_{\sigma]\mu} , \\
[D, P_\mu] = P_\mu , \quad [D, K_\mu] = -K_\mu , \\
[P_\mu, K_\nu] = 2(\eta_{\mu\nu} D + 2M_{\mu\nu}) .
\]  

(5.2.34)

The R symmetry group \( SU(2)_R \) is generated by,

\[
U_i^j = i(U_1 \sigma_1 + U_2 \sigma_2 + U_3 \sigma_3)^j_i, \quad U_i^i = 0, \quad U_i^j = -(U_i^j)^* ,
\]  

(5.2.35)

where \( U_i \) are real and \( \sigma_i \) are the usual Pauli matrices. The \( SU(2) \) algebra is given by,

\[
[U_i^j, U_k^l] = \delta_i^k U_j^l - \delta_j^k U_i^l .
\]  

(5.2.36)
The fermionic part is generated by the usual supersymmetry $Q^i$ and special supersymmetry $S^i$,

$$\{Q_{\alpha}, Q^{\beta}\} = -\frac{1}{2} \delta^i_j \gamma^\mu \beta \alpha P_\mu, \quad \{S_{\alpha}, S^{\beta}\} = -\frac{1}{2} \delta^i_j \gamma^\mu \beta \alpha K_\mu,$$

$$\{Q_{\alpha}, S^{\beta}\} = -\frac{i}{2} (\delta^i_j \delta^\beta_\alpha D + \delta^i_j (\gamma^\mu)^\beta_{\alpha \mu} + 3 \delta^\beta_\alpha U_{ij}).$$  \hspace{1cm} (5.2.37)

The action of the conformal group on the supersymmetries is given by,

$$[M_{\mu \nu}, Q_{\alpha}^i] = -\frac{1}{4} (\gamma_{\mu \nu} Q^i)_{\alpha}, \quad [M_{\alpha \beta}, S^i_{\alpha}] = -\frac{1}{4} (\gamma_{\mu \nu} S^i)_{\alpha},$$

$$[D_\mu, Q_{\alpha}^i] = \frac{1}{2} Q_{\alpha}^i, \quad [D_\mu, S^i_{\alpha}] = -\frac{1}{2} S_{\alpha}^i,$$

$$[K_\mu, Q_{\alpha}^i] = i (\gamma^\mu S^i)_{\alpha}, \quad [P_\mu, S_{\alpha}^i] = -i (\gamma^\mu Q^i)_{\alpha}. \hspace{1cm} (5.2.38)$$

Finally, the action of the R symmetry group on the fermionic generators is given by,

$$[U_{ij}^k, Q_{\alpha}^i] = \delta_{ij}^k Q_{\alpha}^j - \frac{1}{2} \delta_{ij}^k Q_{\alpha}^k,$$

$$[U_{ij}^k, S_{\alpha}^i] = \delta_{ij}^k S_{\alpha}^j - \frac{1}{2} \delta_{ij}^k S_{\alpha}^k.$$

which is an $SU(2)$ rotation. Thus in the $\mathcal{N} = 2$ theories the R symmetry acts as an $SU(2)$ rotation on the fermions of the theory (the gravitino $\psi_{\mu i}$, gaugino $\lambda_i$ and the hyperino $\xi_i$).

So far we have discussed the global symmetries of five dimensional $\mathcal{N} = 2$ supergravity. In the next section, we gauge the symmetries and describe gauged supergravity.

## 5.3 $\mathcal{N} = 2, d = 5$ gauged supergravity

In the previous section, we saw the global symmetries of five dimensional supergravity. This group of global symmetries is a direct product of the symmetry group of the very special manifold and the quaternionic manifold. Let us call the isometry group of the
scalar manifold as $G$. In addition, we also saw that there is an $SU(2)_R$ symmetry group. The global symmetry group of $\mathcal{N} = 2, d = 5$ supergravity is of the form $G \times SU(2)_R$. In general one has various possibilities for constructing a gauged supergravity from an ungauged supergravity.

Firstly, one can just gauge a subgroup of the $R$ symmetry group. Note that the gauge fields are inert under the $R$ symmetry group as the group acts only on the fermions. Theories of this type are called Maxwell-Einstein supergravity theories and possess a scalar potential [137]. One can also gauge a subgroup of the symmetries of the scalar manifold $K \subset G$. A subset of the gauge fields from the ungauged theory has to transform in the adjoint representation of $K$ so that they can act as Yang-Mills gauge fields. If such a group $K$ exists, the gauge fields can in general transform under $K$ as,

$$Gauge\ fields \rightarrow \text{Adj}(K) + \text{Singlets}(K) + \text{Nonsinglets}(K) . \quad (5.3.1)$$

For the singlets the structure constants of $K$ are assumed to be zero and if $K$ is abelian the presence of singlets do not change anything. If some of the gauge fields are charged under $K$, they lead to mass terms and break supersymmetry. This issue is resolved by dualising the charged vectors to tensor fields satisfying self dual field equations [151]. We have given some background on the origin of tensor multiplets in Appendix C. Finally, One can do the most general gauging of the subgroups of $SU(2)_R$ and $K \subset G$ simultaneously. This leads to the most general gauged supergravity in five dimensions [13].

In this section, we review the most general gauged supergravity in five dimensions with $n_V$ vector multiplets, $n_T$ tensor multiplets and $n_H$ hypermultiplets with a generic gauging of the symmetries of the scalar manifold and gauging of $U(1)_R \subset SU(2)_R$ $R$ symmetry group. Before proceeding further, we highlight some important differences of the five dimensional theory as compared with the four dimensional theory. Usually, the gauging can be described in terms of what is called as the momentum map associated with the scalar manifold. For the $d = 4, \mathcal{N} = 2$ theories the scalar manifold in the vector multiplet
is special Kähler and there exists a momentum map for the isometries [140]. Whereas in
the case of $d = 5$ the scalar manifold in the vector multiplet is very special, real and non-
symplectic. Hence momentum maps do not exist for the isometries. However, the possible
symmetry groups for the very special manifold have been classified for the homogeneous
cases [141–143] and one need to identify the subgroup of these group of symmetries for
gauging. Another significant difference in the $N = 2, d = 5$ theory is the presence of
tensor multiplets which originate due to the gauging. \(^{2}\) It is interesting to observe that the
quaternion structure is the same in both $d = 4$ and $d = 5$ theories. Consequently there
exist Killing prepotentials (i.e. there exist Killing vectors which are given in terms of
the derivatives of prepotentials) as in the case of four dimensional gauged supergravity.
We now discuss the gauging of global symmetries of the five dimensional supergravity
(5.2.1).

### 5.3.1 Gauging the symmetries

In the previous section, we studied the global symmetries if five dimensional $N = 2$
supergravity. The global symmetry group $G$ is a direct product of the group of symmetries
on the very special manifold $S$, the quaternionic Kähler manifold $Q$ and the $SU(2)_R$
symmetry group. One then identifies a subgroup of symmetries $K$ for gauging. The gauging
of symmetries on scalar manifolds is done by introducing Killing vectors $K^I_\xi(\phi)$ and $K^I_X(q)$
that act on $S$ and $Q$,

$$
\phi^\xi \rightarrow \phi^\xi + \epsilon^I K^I_\xi(\phi),
$$

$$
q^X \rightarrow q^X + \epsilon^I K^X_I(q),
$$

(5.3.2)

where $\epsilon^I$ are infinitesimal parameters. Then one replaces the ordinary derivatives on scalar
and fermions by the $K$-covariant derivatives. The bosonic part of the theory then gets the

\(^{2}\)See Appendix C for more details.
following replacements [13, 18]:

\[ \partial_\mu \phi^i \to D_\mu \phi^i \equiv \partial_\mu \phi^i + g A_\mu^I K_f^i(\phi), \]

\[ \partial_\mu q^X \to D_\mu q^X \equiv \partial_\mu q^X + g A_\mu^I K_f^X(q), \]

\[ \nabla_\mu B^M_{\nu \rho} \to D_\mu B^M_{\nu \rho} \equiv \nabla_\mu B^M_{\nu \rho} + g A_\mu^I \Lambda_{IN} B^N_{\nu \rho}, \tag{5.3.3} \]

where \( g \) is the gauge coupling and \( \nabla_\mu \) is the Lorentz covariant derivative. The \( \Lambda_{IN}^M \) are constant matrices which are valued in certain representations of \( K \). The derivatives acting on the fermions are also modified, but they have additional terms due to the gauging of \( R \) symmetry. In this case, the \( SU(2)_R \) connection is replaced by,

\[ \omega^i_j \to \omega^i_j + g_R A^I P^j_i(q), \tag{5.3.4} \]

where \( g_R \) is the \( SU(2)_R \) gauge coupling and \( P^j_i(q) \) are Killing prepotentials that exist due to the quaternionic structure on the hypermultiplet sector. The fermions get the replacements,

\[ \nabla_\mu \psi_{\mu i} \to \nabla_\mu \psi_{\mu i} + g_R A^I P^j_i(q) \psi_{\nu j}, \]

\[ \nabla_\mu \lambda^\tilde{a}_i \to \nabla_\mu \lambda^\tilde{a}_i + g_R A^I P^j_i(q) \lambda^\tilde{a}_j + g A^I_\mu \Lambda^\tilde{a}_\mu \lambda^\tilde{a}_i, \]

\[ \nabla_\mu \zeta^A \to \nabla_\mu \zeta^A + g A^I_\mu \omega_{1B} \zeta^B, \tag{5.3.5} \]

where \( g_R \) is the gauge coupling constant associated with gauging of \( R \) symmetry and,

\[ L^\tilde{a}_I^\tilde{b} \equiv \partial^\tilde{b} K^\tilde{a}_I, \quad \omega_{1B}^A(q) \equiv K_{IX} f^X_i f^Y_j. \tag{5.3.6} \]

We have defined \( K_f^i = K^X f^X_i \) using the vielbein on scalar manifold (5.2.7), and the covariant derivative on \( K_{IX} \) is with respect to the metric \( g_{XY} \) on the quaternionic Kähler manifold.
5.3.2 Field content

The five dimensional supergravity with a generic gauging of the symmetries of the scalar manifold and the $SU(2)$ R symmetry was constructed by Ceresole and Dall’Agata [13]. The theory contains gravity coupled to vector, tensor and hyper multiplets. The gravity multiplet contains the graviton $e_\mu^a$, two gravitinos $\psi_\mu^i$ and a graviphoton $A_\mu$. The hypermultiplet contains a doublet of spin $1/2$ fermions (hyperinos) $\zeta^A$ with $A = 1, 2$ and four real scalars $q^X$ with $X = 1, \ldots, 4$. The vector multiplet contains a vector field $A_\mu$, $SU(2)_R$ doublet of fermions (gauginos) $\lambda^i$ and a real scalar field $\phi$. The tensor multiplet contains a massive antisymmetric self-dual tensor field $B_{\mu\nu}$, $SU(2)_R$ doublet of fermions $\lambda^i$ and a real scalar field $\phi$.

To summarise, for $n_V$ vector, $n_T$ tensor and $n_H$ hypermultiplets the field content is given by,

$$\{e_\mu^a, \psi_\mu^i, A_\mu^I, B^{M}_{\mu\nu}, \lambda^{\tilde{a}}, \xi^A, \phi^{\tilde{x}}, q^X \} .$$ (5.3.7)

The scalars in the vector and tensor multiplets are collectively denoted by $\phi^{\tilde{x}}$, where $\tilde{x} = 1, 2, \ldots, n_v + n_T$. The constraint equation (5.2.3) on the scalar fields is now written as,

$$C^{IJK} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1 , \quad h^{\tilde{I}} \equiv h^{\tilde{I}}(\phi^{\tilde{x}}) ,$$ (5.3.8)

where only $C^{IJK}$ and $C^{IMN}$ are non zero as required by supersymmetry. The vector field index is $I = 0, 1, \ldots, n_V$ and $I = 0$ refers to the graviphoton. The index $M = 1, 2, \ldots, n_T$ counts the number tensor multiplets. The vector and tensor field strengths are collectively written as $H^{\tilde{I}}_{\mu\nu} = (F^{\tilde{I}}_{\mu\nu}, B^{M}_{\mu\nu})$ where $\tilde{I} = (I, M)$.

The gauginos $\lambda^{\tilde{a}}$ in the vector and tensor multiplets transform as vectors under $SO(n_V + n_T)$ and $\tilde{a} = 1, 2, \ldots, n_v + n_T$ is a flat index. The quaternions $q^X, X = 1, 2, \ldots, 4n_H$ are the scalars in the $n_H$ hypermultiplets. The hyperinos $\xi^A, A = 1, 2, \ldots, 2n_H$ form fundamental representations of $USp(2n_H)$ and $USp(2) \approx SU(2)$. The conventions on the $SU(2)$ tensor $e^{ij}$ are summarised in Appendix B.
5.3.3 Lagrangian

The bosonic part of the five dimensional $\mathcal{N} = 2$ gauged supergravity is given by,

$$
\hat{e}^{-1} L_{\text{Bosonic}}^{N=2} = -\frac{1}{2} R - \frac{1}{4} a_{IJ} H_{\mu}^I H_{\nu}^J - \frac{1}{2} g_{XY} D_{\rho} q^X D^\rho q^Y - \frac{1}{2} g_{\tilde{X} \tilde{Y}} D_{\phi} \phi^X D^\phi \phi^\tilde{Y} \\
+ \frac{\hat{e}^{-1}}{6 \sqrt{6}} C_{IJK} e^{I_{\mu\nu\rho\sigma\tau}} F_{\mu\nu} F_{\rho\sigma} A_{\tau}^K \\
+ \frac{\hat{e}^{-1}}{4 g} e^{I_{\mu\nu\rho\sigma\tau}} \Omega_{MN} B_{\mu
u} B_{\rho\sigma} B_{\tau}^N \\
- V(\phi, q),
$$

(5.3.9)

where $\hat{e} = \sqrt{-\det g_{\mu\nu}}$ and $\Omega_{MN}$ is a constant real symplectic matrix that satisfies the following conditions,

$$
\Omega_{MN} = -\Omega_{NM}, \quad \Omega_{MN} \Omega^{NP} = \delta^P_M.
$$

(5.3.10)

Gauging the supergravity introduces a non-trivial scalar potential which is given by,

$$
V(\phi, q) = 2 g^2 W^a W^\bar{a} - g_R^2 [2 P_{ij} P_{ij} - P_{ij}^a P_{ij}^\bar{a}] + 2 g^2 N_{iA} N^{iA},
$$

(5.3.11)

where,

$$
P_{ij} \equiv h^I P_{ij}, \\
P_{ij}^\bar{a} \equiv h^I P_{ij}, \\
W^a \equiv \frac{\sqrt{6}}{4} h^I K_{ij}^a f^a_{ij} = -\frac{\sqrt{6}}{8} \Omega^{MN} h_{ij}^a h_N, \\
N^{iA} \equiv \frac{\sqrt{6}}{4} h^I K_{ij}^X f^A_{ij}. 
$$

(5.3.12)

The bosonic part of the supersymmetry transformation rules are:

$$
\delta \epsilon_{\psi_{\mu i}} = \sqrt{6} \nabla_{\mu} \epsilon_i + \frac{i}{4} h_j (\gamma_{\rho\nu} \epsilon_i - 4 g_{\mu\nu} \gamma_{\rho} \epsilon_i) H^{\rho\nu} + i g_R \gamma_{\mu} \epsilon_j P_{ij}, \\
\delta \epsilon_{A_{i}^a} = -\frac{i}{2} f^a_{ij} \gamma^\mu \epsilon_i D_{\mu} \phi^X + \frac{1}{4} h_j^a \gamma^\mu \epsilon_i H_{\mu\nu}^{I} + g_R e^I_{ij} P_{ij}^a + g W^a \epsilon_i, \\
\delta \epsilon^A = -\frac{i}{2} f^A_{ij} \gamma^\mu \epsilon_i D_{\mu} q^X + g e^I_{i} N^{iA}.
$$

(5.3.13)
The terms that are proportional to the gauge coupling constants in the supersymmetry transformation are called fermionic shifts. These appear due to supersymmetric completion of the additional terms that appear due to gauging. A supersymmetric ward identity relates the potential $\mathcal{V}(\phi, q)$, the gravitino mass matrix $P_{ij}$ and the fermionic shifts [12, 54, 152–154]. As one can see from (5.3.13) the scalar potential (5.3.11) can be written in terms of the squares of the gravitino mass matrix and the fermion shifts in the supersymmetry transformations that appear due to the gauging.

Each term in (5.3.11) has its origin from different sectors in the theory. The terms proportional to $g^2$ arise due to gauging of the symmetries of the scalar manifold. In particular the terms $W^\alpha W^\beta$ arise due to tensor multiplets and $N_{iA} N^{iA}$ appear due to hypermultiplets. The terms proportional to $g_s^2$ occur due to gauging the R symmetry.

We now discuss the possibilities of $AdS$ vacuum in this theory which occurs whenever,

$$\mathcal{V}(\phi, q)' = 0, \quad \mathcal{V}(\phi_c, q_c) < 0,$$

(5.3.14)

where the derivative is with respect to the scalars and $\phi_c, q_c$ are the critical points of the potential. The metrics $g_{i\bar{j}}, g_{XY}$ are positive definite and the term in the potential which can contribute to an $AdS$ vacuum is $P_{ij} P^{ij}$. For example, consider the case $n_Y = n_N = 0$ there is only the gravity multiplet with a single graviphoton. Since $K_{\chi}^i$ are zero, the prepotentials $P_{ij}$ are either zero or $SU(2)$ valued constants. If we choose to gauge a subgroup $U(1)_R \subset SU(2)_R$, then $P_{ij} = V \delta_{ij}$. Thus the potential (5.3.11) becomes,

$$\mathcal{V} = -4V^2,$$

(5.3.15)

which acts as a cosmological constant. The corresponding theory is referred in the literature as Anti de-Sitter supergravity [155]. The important point here is that one definitely need to gauge some of the R symmetry in the theory to get an $AdS$ vacuum. Of course, in general we could choose $P_{ij} = V_0 \delta_{ij}$. The parameters $V_0$ appear in the R symmetry gauging as $A_\mu(U(1)_R) = V_0 A_\mu^I$. In this case, we have only one graviphoton and we have chosen $V = V_0$. 

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we could simultaneously gauge the other symmetries simultaneously, as the terms $W^a$ and $N_{IA}$ in (5.3.11) can at the most change the shape of the critical point.

## 5.4 Gauged supergravity with one vector multiplet

In this section, we describe a simple gauged supergravity model in some detail [18, 19]. We will use this model for constructing generalised attractors in a later chapter. This gauged supergravity model consists of one vector multiplet ($A^I, I = 0$ corresponds to the graviphoton.) and two tensor multiplets. The field content is summarised as,

$$
\{ e^a_\mu, \psi^i_\mu, A^I_\mu, B^{M}_{\mu\nu}, \lambda^{\tilde{a}}, \phi^\delta \}.
$$

We use the same notations as in the previous section. The very special manifold, which is parametrised by the scalars in the vector and tensor multiplets has the coset structure given by,

$$
S = SO(1,1) \times \frac{SO(2,1)}{SO(2)}.
$$

This model is an example of a symmetric space discussed in the introduction of this chapter. In this model, the symmetry group of the scalar manifold is $G = SO(1,1) \times SO(2,1)$. The gauging we consider is an $SO(2) \subset SO(2,1)$ subgroup of the $SO(2,1)$ in $G$ and the gauging of the $U(1)_R \subset SU(2)_R$. The symmetries of the scalar manifold can be made manifest by going to a suitable basis such that the condition (5.2.3) is satisfied. Note that the index $I$ in (5.2.3) is replaced with $\tilde{I} = (I, M)$ to collectively label the scalars in the vector/tensor multiplets. The scalar constraint now reads as,

$$
N \equiv C_{IJK} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1.
$$
As always, we need to choose a suitable parametrisation to satisfy the above equation. We choose,

\[ h_I^0 = \sqrt{2} \xi^I \Bigg|_{N=1}, \quad h_I^1 = \frac{1}{\sqrt{6}} \frac{\partial}{\partial \xi^I} N \Bigg|_{N=1}, \tag{5.4.4} \]

such that the constraint takes the form,

\[ N(\xi) = \sqrt{2} \xi^0 [ (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 ] = 1, \tag{5.4.5} \]

where,

\[ \xi^0 = \frac{1}{\sqrt{2} ||\phi||^2}, \quad \xi^1 = \phi^1, \quad \xi^2 = \phi^2, \quad \xi^3 = \phi^3, \tag{5.4.6} \]

and,

\[ ||\phi||^2 = (\phi^1)^2 - (\phi^2)^2 - (\phi^3)^2, \tag{5.4.7} \]

is assumed to be positive so that \( a_{IJ} \) and \( g_{xy} \) are positive definite. The regions \( \phi^1 > 0 \) and \( \phi^1 < 0 \) are equivalent in the moduli space due to the above relation. However for our purposes we will stick to the region \( \phi^1 > 0 \) in the moduli space.

The \( h_I^I \) are related to the fields \( \phi \) in the Lagrangian through the following relations,

\[ h^0 = \frac{1}{\sqrt{3} ||\phi||^2}, \quad h^1 = \sqrt{\frac{2}{3}} \phi^1, \quad h^2 = \sqrt{\frac{2}{3}} \phi^2, \quad h^3 = \sqrt{\frac{2}{3}} \phi^3. \tag{5.4.8} \]

\[ h_0 = \frac{1}{\sqrt{3}} ||\phi||^2, \quad h_1 = \frac{2}{\sqrt{6}} ||\phi||^2, \quad h_2 = -\frac{2}{\sqrt{6}} ||\phi||^2, \quad h_3 = -\frac{2}{\sqrt{6}} ||\phi||^2. \tag{5.4.9} \]

Using the above relations and the scalar constraint (5.2.3) we can read off the non-vanishing \( C_{IJK} \) as,

\[ C_{011} = \frac{\sqrt{3}}{2}, \quad C_{022} = C_{033} = -\frac{\sqrt{3}}{2}. \tag{5.4.10} \]
The metric $a_{IJ}$ computed using the relation (5.2.4), (5.4.8) and (5.4.9) is given by,

$$a_{IJ} = \begin{pmatrix}
\|\phi\|^4 & 0 & 0 & 0 \\
0 & 2(\phi^1)^2\|\phi\|^4 - \|\phi\|^2 & -2\phi^1\phi^2\|\phi\|^4 & -2\phi^1\phi^3\|\phi\|^4 \\
0 & -2\phi^1\phi^2\|\phi\|^4 & 2(\phi^2)^2\|\phi\|^4 + \|\phi\|^2 & 2\phi^2\phi^3\|\phi\|^4 \\
0 & -2\phi^1\phi^3\|\phi\|^4 & 2\phi^2\phi^3\|\phi\|^4 & 2(\phi^3)^2\|\phi\|^4 + \|\phi\|^2
\end{pmatrix}.$$ (5.4.11)

The metric on the scalar manifold $g_{\tilde{x}\tilde{y}}$ is then computed using the completeness relations (5.2.5) and is given by,

$$g_{\tilde{x}\tilde{y}} = \begin{pmatrix}
4(\phi^1)^2\|\phi\|^4 - \|\phi\|^2 & -4\phi^1\phi^2\|\phi\|^4 & -4\phi^1\phi^3\|\phi\|^4 \\
-4\phi^1\phi^2\|\phi\|^4 & 4(\phi^2)^2\|\phi\|^4 + \|\phi\|^2 & 4\phi^2\phi^3\|\phi\|^4 \\
-4\phi^1\phi^3\|\phi\|^4 & 4\phi^2\phi^3\|\phi\|^4 & 4(\phi^3)^2\|\phi\|^4 + \|\phi\|^2
\end{pmatrix}.$$ (5.4.12)

Recollect that the scalar manifold has the symmetry group $G = SO(1,1) \times SO(2,1)$. We consider the gauging of a compact subgroup $SO(2)$ for our purposes. Since it is an abelian group and has only one generator, the Killing vector which generates this symmetry can couple to one vector field. In this case, the gauge field is the graviphoton $A_0^\mu$. The Killing vector that generates the $SO(2)$ symmetry is found by solving the Killing vector equation,

$$\tilde{\nabla}^\tilde{x}K_{0}^{\tilde{x}} + \tilde{\nabla}^{\tilde{y}}K_{0}^{\tilde{y}} = 0,$$ (5.4.13)

where the covariant derivative $\tilde{\nabla}$ is with respect to the metric $g_{\tilde{x}\tilde{y}}$ on the scalar manifold. It can be checked that the following vector,

$$K_{0}^{\tilde{x}} = \left\{ -\frac{\phi^1}{\|\phi\|^2}, \frac{\phi^2}{\|\phi\|^2}, \frac{\phi^3}{\|\phi\|^2} \right\},$$ (5.4.14)

is indeed a solution to the Killing equations. The $SO(2)$ symmetry which rotates the $\phi^2, \phi^3$ directions is manifest in (5.4.14).

The $U(1)_R$ gauging is done using a linear combination of gauge fields in the theory given.
by,
\[ A_\mu(U(1)_R) = V_I A'_\mu, \quad I = 0, 1, \]  
(5.4.15)
where \( V_I \) are constant parameters. For a general gauging of a non-abelian \( R \) symmetry the parameters \( V_I \) are constrained by the relation,
\[ V_I f_{JK} = 0, \]  
(5.4.16)
where \( f_{JK} \) are structure constants of the gauge group. For abelian gauging such as the one considered here, the parameters \( V_I \) are free since the structure constants vanish.

Now, we can calculate the scalar potential using (5.3.11) by setting \( N_i A \) to zero since there are no hypermultiplets. We also have \( P_{ij} = V_I \delta_{ij} \) for \( U(1)_R \) gauging, as discussed in the previous section. Using the relations (5.3.12) the potential (5.3.11) can be expressed as,
\[ V = \frac{3}{16} g^2 \Omega^{MN} \Omega^{PQ} h_N h_P h_Q h_R - g^2 \left[ 4 h^I h^J V_I V_J - 2 g^2 \right], \]  
(5.4.17)
where \( h^I \) are as defined in (5.2.9), and the conventions for \( \Omega_{MN} \) are,
\[ \Omega_{MN} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]  
(5.4.18)
After some simplifications using (5.4.8) and (5.4.9) the scalar potential of this model can be written as,
\[ V(\phi) = \frac{g^2}{8} \left[ \left( \frac{\phi^2}{\phi} \right)^2 + \left( \phi^3 \right)^2 \right] - 2 g^2 \left[ 2 \sqrt{2} \frac{\phi^1}{||\phi||^2} V_0 V_1 + ||\phi||^2 \right]. \]  
(5.4.19)
The critical points of this potential have been investigated in great detail in [19]. We will consider the case where the \( AdS \) vacuum preserves \( N = 2 \) supersymmetry. This requires the condition [13],
\[ W_{\alpha | \alpha}^i = P_{ij}^\alpha | \alpha = 0. \]  
(5.4.20)
We will derive this condition in the next chapter by studying the Killing spinor integrability conditions for generalised attractors. The critical point which satisfies (5.4.20) is given by,

\[ \phi^2 = 0, \quad \phi^3 = 0, \quad \phi_1^c = \left( \sqrt{2} \frac{V_0}{V_1} \right)^{\frac{1}{3}} , \quad (5.4.21) \]

together with the constraints,

\[ V_0 V_1 > 0, \quad 32 \frac{g^2}{g^2} V_0^2 \leq 1 . \quad (5.4.22) \]

These constraints determine the nature of the critical point. In this case, the critical point is a saddle point with a maximum in the \( \phi^1 \) direction and minima in \( \phi^2, \phi^3 \) directions. The value of the potential (5.4.19) at the critical point (5.4.21),

\[ \mathcal{V}(\phi_c) = \Lambda = -6g^2(\phi_1^c)^2 V_1^2 , \quad (5.4.23) \]

is the value of the \( AdS \) cosmological constant of the theory.

**5.5 Summary**

In this chapter, we have provided some background material in five dimensional \( N = 2 \) gauged supergravity. First we studied the global symmetries of ungauged supergravity. We saw that the scalars in the vector multiplet parametrise a very special manifold and the scalars in the hypermultiplet parametrise a quaternionic Kähler manifold. We also studied the \( N = 2 \) superconformal algebra and discussed global \( SU(2)_R \) symmetry group.

We then discussed the gauging of a subgroup of the global symmetries of the ungauged theory. We saw that the symmetries can be gauged by introducing Killing vectors acting on the scalar manifold. These Killing vectors generate the group of symmetries which are to be gauged. The various gauge fields in the theory couple to the Killing vectors and
the ordinary derivatives get replaced by gauge covariant derivatives. Due to the additional terms that appear because of the gauging, supersymmetric closure requires the existence of a potential term in gauged supergravity. We saw that the value of the potential at its critical point gives the cosmological constant of the theory. In particular, we saw that a subgroup of the $R$ symmetry group has to be gauged to get a negative cosmological constant and hence $AdS$ vacuum.

Then, we studied an example of a simple gauged supergravity with one vector multiplet. We demonstrated that by choosing an explicit parametrisation for the scalar constraint (5.2.3) the symmetries of the manifold can be made manifest. In particular, we showed that the metric on the scalar manifold can be written down explicitly and constructed the Killing vector which generates an abelian symmetry group. We also discussed the gauging of a $U(1)$ component of the $SU(2)_R$ symmetry group and explained that the parameters involved in the gauging are unconstrained for abelian gauging of the $R$ symmetry group. We saw that the potential in this model has critical points which gave rise to an $AdS$ vacuum.