Chapter 4

Black holes in AdS

4.1 Introduction

In the previous chapter, we studied the attractor mechanism for asymptotically flat black hole solutions in supergravity. We saw that the near horizon geometry played an important role in determining the attractor behavior. In this chapter, we give a small introduction to black holes in Anti de-Sitter spaces (AdS) with particular focus on the near horizon geometries of extremal black branes. Historically, black holes in AdS spaces gained attention when the positive energy theorem, which states that the energy of an asymptotically flat space time is non zero, was proven for asymptotically AdS spaces [108, 109]. This result was also extended to supergravities and gauged supergravities where one can often find AdS vacuum solutions [110, 111].

It is well known that a Schwarzschild black hole in asymptotically flat spacetime has negative specific heat and is thermodynamically unstable. However, in AdS spacetime the system undergoes a first order phase transition from a radiation dominated low temperature phase to a black hole dominated high temperature phase. Hence, the AdS schwarzschild black hole can exist with a positive specific heat and is thermodynamically stable at high temperature. This is the famous Hawking-Page transition [112]. In
the context of the $AdS/CFT$ correspondence [11], the Hawking page transition is equivalent to a confinement-deconfinement phase transition in a quark-gluon system in the dual theory [34].

Charged black branes play an important role in the correspondence as holographic duals to field theories at finite temperature and chemical potential. Extremal black branes, in particular, correspond to the zero temperature ground states of the dual field theory. Even at zero temperature, several systems in condensed matter theory display novel behaviour such as phase transitions due to quantum fluctuations [46]. Field theories which describe such systems often show a wide variety of phases while the corresponding dual black brane solutions are not as many. Also, many of the condensed matter systems have non-relativistic symmetry groups and it would be interesting to explore extremal black branes with such symmetries to map the study of quantum phase transitions to the gravity side.

Metrics which display symmetries of non-relativistic condensed matter systems such as Galilean [113] and Lifshitz [47] symmetries have been constructed, and can sometimes be embedded in string theory [15, 16, 114]. Interestingly, some charged dilatonic black branes with Lifshitz-like near horizon geometry and asymptotic AdS can also exhibit attractor behaviour [115, 116]. More recently, a large class of extremal homogeneous anisotropic black brane horizons have been extensively studied [7, 53]. These metrics have generalised translational symmetries which do not commute, as opposed to the usual translational symmetries along the brane directions. The generators of these symmetries form an algebra which is isomorphic to the three dimensional real Lie algebras given by the Bianchi classification. In this chapter, we review the construction of metrics with the Bianchi type symmetries. This will form a useful background for chapter 6, where we realise some of the Bianchi type metrics as generalised attractors. The most useful references for this chapter are [7, 53, 117–120].

The organisation of this chapter is as follows. In §4.2, we discuss the physics of the $AdS$ schwarzschild black hole and the $AdS$ Reissner Nordstrom black hole, followed by
description of black brane limits of these configurations and their near horizon geometries in §4.3. Taking the lead from the study of AdS Reissner-Nordstrom black brane, we study geometries with constant anholonomy coefficients and explore the connection with homogeneous spaces on §4.4. We then give a detailed description of five dimensional homogeneous extremal black brane horizons belonging to the Bianchi classification in §4.5. Then, we summarise the contents of this chapter in §4.6.

4.2 Schwarzschild and Reissner-Nordstrom black holes in AdS space

In this section, we will describe the Schwarzschild black hole in four dimensional AdS space followed by a discussion on the five dimensional AdS Reissner-Nordstrom black hole. First, we recall the definition of AdS spaces and describe some well known coordinate systems which will be useful later. AdS$_4$ space is defined as the hyperboloid,

$$-X_0^2 - X_4^2 + X_1^2 + X_2^2 + X_3^2 = -R^2, \tag{4.2.1}$$

embedded in a 4 + 1 dimensional flat space with the metric,

$$ds^2 = -dX_0^2 - dX_4^2 + dX_1^2 + dX_2^2 + dX_3^2. \tag{4.2.2}$$

It has the isometry group $SO(2,3)$ generated by the 10 Killing vectors,

$$J_{\alpha\beta} = X_\alpha \partial_\beta - X_\beta \partial_\alpha. \tag{4.2.3}$$
Using the following global coordinates,

\begin{align*}
X_0 &= R \cosh \rho \cos \tau, \quad X_4 = R \cosh \rho \sin \tau, \\
X_i &= R \sinh \rho \Omega_i, \quad \sum_{i=1}^{3} \Omega_i = 1,
\end{align*}

the metric (4.2.2) can be expressed as,

\[ ds^2 = R^2 (-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega^2). \]

(4.2.5)

where \( \rho \geq 0 \) and \( 0 \leq \tau \leq 2\pi \). This coordinate system is called a global coordinate since it covers the entire hyperboloid (4.2.1). Another commonly used set of coordinates are the Poincaré coordinates which cover one half of the hyperboloid. These coordinates are given by,

\begin{align*}
X_0 &= \frac{1}{2r} (1 + r^2 (R^2 + \vec{x}^2 - t^2)), \quad X_4 = R rt, \\
X_i &= R r x_i, \quad i = 1, 2, \\
X_3 &= \frac{1}{2r} (1 - r^2 (R^2 - \vec{x}^2 + t^2)),
\end{align*}

(4.2.6)

and the metric takes the form,

\[ ds^2 = R^2 (-r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2). \]

(4.2.7)

The AdS metric is a solution to Einstein’s equation with negative cosmological constant. There also exists another vacuum solution, the AdS Schwarzschild black hole which we discuss next.
4.2.1 AdS Schwarzschild Black hole

It is well known that the familiar Schwarzschild black hole in an asymptotically flat space time is thermodynamically unstable due to negative specific heat. However, black holes in AdS spaces have positive specific heat at high temperatures and thus thermodynamically stable [112]. The AdS schwarzschild black hole is a vacuum solution to Einstein’s equations with a negative cosmological constant. The black hole metric in four dimensions is given by,

\[
ds^2 = -V dt^2 + \frac{dr^2}{V} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( V = 1 - \frac{2M}{r} + \frac{\Lambda r^2}{3}, \) \( \Lambda \) is the cosmological constant \(^1\). We have also set the four dimensional Newtons constant \( G_4 = 1 \). For large \( r \) the black hole approaches the form,

\[
ds^2 = -(1 + \frac{\Lambda r^2}{3})dt^2 + \frac{dr^2}{(1 + \frac{\Lambda r^2}{3})} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{4.2.9}
\]

which is nothing but the \( AdS_4 \) metric, which can be obtained from \( (4.2.5) \) by the coordinate choice \( \tau = t \sqrt{\frac{\Lambda}{3}}, \sinh \rho = r \sqrt{\frac{\Lambda}{3}} \) and setting \( R = \sqrt{\frac{\Lambda}{3}} \). In the asymptotic limit, the schwarzschild \( AdS \) black hole approaches \( AdS \) space. The horizon of the black hole is located at \( r = r_h \), where \( r_h \) is the largest root of \( V(r) = 0 \).

Writing \( \tau = it \) and expanding the metric \( (4.2.8) \) near the horizon we find,

\[
ds^2 = (r - r_h)V'(r_h)d\tau^2 + \frac{dr^2}{(r - r_h)V'(r_h)} + r_h^2d\Omega^2_2. \tag{4.2.10}
\]

Rewriting \( r = r_h + \frac{V'(r_h)}{4}\rho^2 \) we get,

\[
ds^2 = \frac{V'(r_h)^2}{4}\rho^2 d\tau^2 + d\rho^2 + r_h^2d\Omega^2_2. \tag{4.2.11}
\]

\(^1\)For convenience we have chosen the conventions \( \Lambda > 0 \) for \( AdS \) spaces.
We see that the conical singularity at $r = r_h$ is resolved by regarding $r$ as an angular coordinate with a period,

$$\beta = \frac{4\pi}{V'(r_h)} = \frac{4\pi r_h}{(1 + \Lambda r_h^2)}.$$  \hspace{1cm} (4.2.12)

The temperature is the inverse of $\beta$ and has a minimum value,

$$T_{\text{min}} = \frac{\sqrt{\Lambda}}{2\pi},$$  \hspace{1cm} (4.2.13)

at $r_0 = 1/\sqrt{\Lambda}$. The mass of the black hole can be expressed in terms of the horizon radius $r_h$,

$$M = \frac{r_h}{2}\left(1 + \frac{r_h^2}{R^2}\right).$$  \hspace{1cm} (4.2.14)

As one can see, the temperature no longer decreases with the mass, but attains a minimum value $T_{\text{min}}$ below which only radiation exists. For $T > T_{\text{min}}$ there are two black hole solutions, one for $r_h < r_0$ and other for $r_h > r_0$. The former is called a small black hole, has negative specific heat and is thermodynamically unstable. While the black hole with $r_h > r_0$ has positive specific heat and is thermodynamically stable. The entropy of the AdS schwarzschild black hole calculated using euclidean path integral methods is given by,

$$S_{\text{BH}} = \pi r_h^2 = \frac{A_{\text{BH}}}{4},$$  \hspace{1cm} (4.2.15)

where $A_{\text{BH}}$ is the area of the black hole horizon.

### 4.2.2 AdS Reissner-Nordstrom Black hole

Another well known black hole solution in AdS space is the Reissner-Nordstrom black hole. This black hole solution is obtained from theories with gravity coupled to massless gauge fields. For the purpose of future reference, we will consider the five dimensional
black hole given by [7, 121, 122],

$$ds^2 = -V d\tilde{t}^2 + \frac{d\tilde{r}^2}{V} + \tilde{r}^2 d\Omega_3^2,$$

$$V = 1 + \frac{Q^2}{12\tilde{r}^4} + \frac{\tilde{r}^2 \Lambda}{12} - \frac{M}{\tilde{r}^2},$$

$$A = -Q \left( \frac{1}{2\tilde{r}^3} - \frac{1}{2\tilde{r}_h^3} \right) d\tilde{t}. \quad (4.2.16)$$

Where \(Q\) is the electric charge and \(M\) is the mass of the black hole. We have also set the five dimensional Newtons constant \(G_5 = 1\). As before the horizon radius \(\tilde{r}_h\) is determined by the largest root of \(V(\tilde{r}) = 0\). The temperature of the black hole is determined as before by euclidean rotation. We expand the metric near the horizon and resolve the conical singularity to get,

$$T = \frac{V'(\tilde{r}_h)}{4\pi} = \frac{12Mr_h^2 + r_h^4 \Lambda - 2Q^2}{24\pi r_h^5}. \quad (4.2.17)$$

From the above equation, we can see that the temperature vanishes when,

$$Q^2_c = 2r_h^6 \Lambda, \quad M_c = \frac{r_h^4 \Lambda}{4}, \quad (4.2.18)$$

Since for these values both \(V(\tilde{r}_h)\) and \(V'(\tilde{r}_h)\) vanish, the black hole becomes extremal. To understand the regime in which the extremal black hole is stable it is convenient to rewrite (4.2.17) as,

$$T = \frac{12 + 2\tilde{r}_h^2 \Lambda - \Phi^2}{24\pi \tilde{r}_h}, \quad \Phi = \frac{Q}{\tilde{r}_h}, \quad (4.2.19)$$

where we have used \(V(\tilde{r}_h) = 0\) for simplification. It is clear that \(\Phi\) plays the role of an electrostatic potential. In the large \(\tilde{r}_h\) regime, the temperature vanishes when \(\Phi \geq \sqrt{12}\) and,

$$\tilde{r}_h^2 = \frac{\Phi^2 - 12}{2\Lambda}. \quad (4.2.20)$$

Thus, the extremal black hole is also stable in the large \(\tilde{r}_h\) regime just as the Schwarzschild \(AdS\) black hole. For \(\Phi < \sqrt{12}\) in the small \(\tilde{r}_h \to 0\) regime the black hole is unstable and has negative specific heat. The entropy of the black hole can be calculated once again
using Euclidean path integral methods and we get,

\[ S_{BH} = \frac{2\pi^2 r_h^3}{4} = \frac{A}{4}. \quad (4.2.21) \]

## 4.3 Black branes and near horizon limits

In this section we will discuss black branes in \( AdS \). First we describe the Schwarzschild black brane and subsequently the Reissner-Nordstrom black brane in \( AdS \).

### 4.3.1 \( AdS \) Schwarzschild black brane

A black \( p \) brane is a generalisation of a black hole with additional translational symmetries along \( p \) spatial directions. In particular, this implies that for these objects the horizon does not have the spherical topology of black holes in the \( p \) brane directions. Instead, the topology of the horizon is planar. We illustrate this by considering the black brane limit of the \( AdS \) schwarzschild black hole (4.2.8) studied in the previous section. Consider the following rescaling of the co-ordinates [123],

\[ r = \left( \frac{2M}{R} \right)^{\frac{1}{3}} \rho, \quad t = \left( \frac{2M}{R} \right)^{-\frac{1}{3}} \tau. \quad (4.3.1) \]

The function \( V(r) \) takes the form,

\[ V(r) = 1 - \left( \frac{2M}{R} \right)^{\frac{2}{3}} \left[ \frac{\rho^2}{R^2} - \frac{R}{\rho} \right], \quad (4.3.2) \]

and the metric (4.2.8) looks like,

\[ ds^2 = -\left( \frac{2M}{R} \right)^{\frac{4}{3}} V(r) d\tau^2 + \left( \frac{2M}{R} \right)^{\frac{2}{3}} d\rho^2 + \left( \frac{2M}{R} \right)^{\frac{2}{3}} \rho^2 d\Omega^2. \quad (4.3.3) \]
In the limit $M \to \infty$ the radius of the $S^2$ becomes infinite and the sphere appears locally as $\mathbb{R}^2$. This is the familiar idea that a sphere $d\Omega = \sum_{i=1}^2 (dy^i)^2$, is just a plane with a point at infinity. By changing the sphere coordinates locally into $y^i = \left(\frac{2M}{R}\right)^{-\frac{1}{2}} x^i$ and considering the large $M$ limit we get,

$$ds^2 = -\left[\frac{\rho^2 - R}{\rho}\right]d\tau^2 + \frac{d\rho^2}{\left[\frac{\rho^2}{R^2} - \frac{R}{\rho}\right]} + \rho^2 (dx^i)^2,$$

(4.3.4)

we see that the (4.2.8) horizon has a planar topology in the black brane limit. The five dimensional analogue of the $AdS$ Schwarzschild black brane is realised as the near horizon geometry of extremal $D3$ branes in type IIB string theory [124].

### 4.3.2 Reissner-Nordstrom black brane

The black brane limit of the Reissner-Nordstrom solution (4.2.16) is obtained in the same way as in the Schwarzschild black brane and is given by,

$$ds^2 = -V(\tilde{r})d\tilde{t}^2 + \frac{d\tilde{r}^2}{V(\tilde{r})} + \tilde{r}^2 (d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2),$$

$$V(\tilde{r}) = \frac{Q^2}{12\tilde{r}^4} + \frac{\tilde{r}^2 \Lambda}{12} - \frac{M}{\tilde{r}^2},$$

(4.3.5)

We saw that in the extremal limit $V(\tilde{r}_h) = 0, V'(\tilde{r}_h) = 0$ and we get,

$$M = \frac{\tilde{r}_h^4 \Lambda}{4}, \quad Q = \sqrt{2}\tilde{r}_h \Lambda.$$

(4.3.6)

The function $V(\tilde{r})$ then takes the form,

$$V(\tilde{r}) = \frac{(\tilde{r} - \tilde{r}_h)^2 (\tilde{r} + \tilde{r}_h)^2 (\tilde{r}^2 + 2\tilde{r}_h^2) \Lambda}{12\tilde{r}^4}.$$  

(4.3.7)
We expand the metric near the horizon using the coordinates,
\[
\frac{\tilde{r} - \tilde{r}_h}{\tilde{r}_h} = r, \quad \tilde{t} = \frac{t}{\tilde{r}_h}, \quad \tilde{x}^i = \frac{x^i}{\tilde{r}_h},
\]
(4.3.8)
to get,
\[
ds^2 = -r^2 \Lambda dt^2 + \frac{dr^2}{\Lambda r^2} + (dx^2 + dy^2 + dz^2)
+ \Lambda \left[ \frac{7\Lambda}{3\tilde{r}_h} r^3 dt^2 + \frac{7}{3\tilde{r}_h} \frac{dr^2}{\Lambda r} + 2 \frac{r}{\tilde{r}_h} (dx^2 + dy^2 + dz^2) \right].
\]
(4.3.9)
The gauge field expands as,
\[
A = -r \sqrt{2\Lambda} dt.
\]
(4.3.10)
We get a one parameter \(\lambda\) worth of solutions, which looks locally like \(AdS_2 \times \mathbb{R}^3\),
\[
ds^2 = -r^2 \Lambda dt^2 + \frac{dr^2}{\Lambda r^2} + (dx^2 + dy^2 + dz^2),
\]
(4.3.11)
for the special value \(\lambda = 0\). On first thought, the value \(\lambda = 0\) appears to be singular. In the limit \(\tilde{r} \to \tilde{r}_h\) and \(\lambda \to 0\) such that \(r\) is kept fixed, the “near horizon” geometry of the full Reissner-Nordstrom black brane approaches a geometry which is isomorphic to \(AdS_2 \times \mathbb{R}^3\). It can be checked that the metric (4.3.11) itself is an independent solution of the Einstein equation with the gauge field (4.3.10) and is valid for any \(r\). It is a feature of extremal black holes that the “near horizon” geometry independently solves the equations of motion and is often easier to find than the full black hole solution itself.

We will now explore the symmetries preserved along the spatial directions of the \(AdS_2 \times \mathbb{R}^3\) metric. This will lead us into the discussion of homogeneous spaces and Bianchi classification. For this purpose we introduce the vielbein of the \(AdS_2 \times \mathbb{R}^3\) metric as \(^2\)
\[
e_0' = r\Lambda, \quad e_1' = \frac{1}{r\Lambda}, \quad e_2' = 1, \quad e_3' = 1, \quad e_4' = 1.
\]
(4.3.12)
\(^2\)The notations and conventions for tangent space are summarised in Appendix A.
The corresponding vector fields $\tilde{e}_a \equiv e_a^\mu \partial_\mu$ satisfy the algebra,

$$[\tilde{e}_a, \tilde{e}_b] = c_{ab}^c \tilde{e}_c,$$

(4.3.13)

with,

$$c_{10}^0 = \Lambda = -c_{01}^0,$$

(4.3.14)

being the only non vanishing anholonomy coefficients. Note that the anholonomy coefficients are constants independent of the spacetime coordinates. We also note that the sub-algebra generated by,

$$\tilde{e}_2 = \partial_x, \quad \tilde{e}_3 = \partial_y, \quad \tilde{e}_4 = \partial_z,$$

(4.3.15)

is isomorphic to the three dimensional Lie algebra,

$$[\tilde{e}_2, \tilde{e}_3] = 0, \quad [\tilde{e}_2, \tilde{e}_4] = 0, \quad [\tilde{e}_3, \tilde{e}_4] = 0,$$

(4.3.16)

which belongs to the Bianchi I class in the classification of real Lie algebras of dimension three [117–119]. In the next section, we explore the connection between constant anholonomy and homogeneous spaces.

### 4.4 Constant Anholonomy and Homogeneity

Towards the end of the previous section, we saw that the $\text{AdS}_2 \times \mathbb{R}^3$ solution has constant anholonomy coefficients and that the vector fields $\tilde{e}_a$ along the spatial directions are generators of a Lie algebra belonging to Bianchi Type I. Since we will be studying the attractors characterised by constant anholonomy in chapter 6, we would like to emphasise the relation between constant anholonomy and metrics with homogeneous symmetries.

First, we explain the concept of homogeneous symmetries through a simple example.
Homogeneous symmetries are those which connect two different points on a manifold by a continuous transformation. In general, the generators of such symmetries do not commute which leads to a Lie algebraic structure [118]. For example, consider the following vector fields on a three dimensional euclidean space [7],

\[
\xi_1 = \partial_y, \quad \xi_2 = \partial_z, \quad \xi_3 = \partial_x + y\partial_z - z\partial_y.
\] (4.4.1)

This corresponds to the helical motion of a particle with translation along the \(x\) direction and rotations in the \((y,z)\) plane. The vector fields close to form an algebra,

\[
[\xi_1, \xi_2] = 0, \quad [\xi_1, \xi_3] = \xi_2, \quad [\xi_2, \xi_3] = -\xi_1.
\] (4.4.2)

Note the Lie algebraic structure and that the structure constants (anholonomy coefficients) are independent of space time coordinates. We will see later in §4.5 that this is isomorphic to the Bianchi VII Lie algebra. If there is a \(d\) dimensional metric with a subset of three Killing vectors that generate the above symmetries, the metric has a three dimensional homogeneous subspace with Bianchi VII symmetry.

We would also like to explain that homogeneity follows from the assumption of constant anholonomy. This will lead to an understanding of why metrics with homogeneous subspaces arise as generalised attractors characterised by constant anholonomy coefficients [14]. In this section, we will assume a generic ansatz for the metric belonging to Bianchi type I, impose constant anholonomy and then determine the restrictions it puts on the form of the metric. We will focus on five dimensional metrics with three dimensional homogeneous subspaces as we expect these geometries to be attractors in five dimensional gauged supergravity.

Let us consider a black brane metric of the form,

\[
ds^2 = -a(\tilde{r})^2d\tilde{t}^2 + \frac{d\tilde{r}^2}{b(\tilde{r})^2} + c(\tilde{r})^2d\tilde{x}^2 + d(\tilde{r})^2d\tilde{y}^2 + e(\tilde{r})^2d\tilde{z}^2.
\] (4.4.3)
where \( a(\tilde{r}), b(\tilde{r}), c(\tilde{r}), d(\tilde{r}) \) and \( e(\tilde{r}) \) are all functions of \( \tilde{r} \). The fünfbein for the metric are given by,

\[
e_0 = a(\tilde{r}), \quad e_1 = \frac{1}{b(\tilde{r})}, \quad e_2 = c(\tilde{r}), \quad e_3 = d(\tilde{r}), \quad e_4 = e(\tilde{r}).
\] (4.4.4)

The only independent non-vanishing anholonomy coefficients (A.0.2) are,

\[
c_{01} = b(\tilde{r}) \frac{a'(\tilde{r})}{a(\tilde{r})}, \quad c_{21} = b(\tilde{r}) \frac{c'(\tilde{r})}{c(\tilde{r})}, \quad c_{31} = b(\tilde{r}) \frac{d'(\tilde{r})}{d(\tilde{r})}, \quad c_{41} = b(\tilde{r}) \frac{e'(\tilde{r})}{e(\tilde{r})},
\] (4.4.5)

where the prime indicates derivative with respect to \( \tilde{r} \). Demanding constant anholonomy coefficients leads to the following equations,

\[
\frac{a'(\tilde{r})}{a(\tilde{r})} = \frac{C_0}{b(\tilde{r})}, \quad \frac{c'(\tilde{r})}{c(\tilde{r})} = \frac{C_2}{b(\tilde{r})}, \quad \frac{d'(\tilde{r})}{d(\tilde{r})} = \frac{C_3}{b(\tilde{r})}, \quad \frac{e'(\tilde{r})}{e(\tilde{r})} = \frac{C_4}{b(\tilde{r})},
\] (4.4.6)

where \( C_0, C_2, C_3, C_4 \) are the constant values of the anholonomy coefficients. Since we have assumed all the unknown functions to be pure functions of \( \tilde{r} \), we may treat the above partial differential equations as ordinary differential equations.

Let us consider some specific cases to simplify the problem. The first case, \( b(\tilde{r}) = a(\tilde{r}) \) leads to the near horizon geometry of the extremal AdS Reissner-Nordstrom black hole. 

\textit{case i}) \( b(\tilde{r}) = a(\tilde{r}) \) : The metric takes the following form,

\[
ds^2 = -C_0^2 r^2 dt^2 + \frac{dr^2}{C_0^2 r^2} + r^2 \frac{C_2}{C_0^2} dx^2 + r^2 \frac{C_3}{C_0^2} dy^2 + r^2 \frac{C_4}{C_0^2} dz^2.
\] (4.4.7)

where \( r = \tilde{r} + \frac{m}{\ell_0} \) and \((x, y, z) = (a_2 \tilde{x}, a_3 \tilde{y}, a_4 \tilde{z})\). Here all the \( a_i \) are integration constants.

The metric (4.4.7) is the near horizon geometry of the extremal Reissner-Nordstrom black brane (4.3.11) with the identifications \( C_0 = \sqrt{\Lambda}, C_2 = C_3 = C_4 = 0 \).

\textit{case ii}) \( b(\tilde{r}) = c(\tilde{r}) \) : Solving for the other functions, the metric takes the form,

\[
ds^2 = -r^2 \frac{C_0}{C_2} dt^2 + \frac{dr^2}{C_2 r^2} + r^2 C_2^2 dx^2 + r^2 \frac{C_3}{C_2} dy^2 + r^2 \frac{C_4}{C_2} dz^2.
\] (4.4.8)
where \( r = \tilde{r} + \frac{\phi}{L^2} \) and \((t, x, y, z) = (a_0 \tilde{t}, \tilde{x}, a_3 \tilde{y}, a_4 \tilde{z})\) and the \( a_i \)'s are all integration constants.

The metric (4.4.8) is called the anisotropic Lifshitz metric [125, 126] and can be put in a familiar form by choosing \( C_0 = \frac{u_0}{L}, C_2 = \frac{1}{L}, C_3 = \frac{u_1}{L}, C_4 = \frac{u_2}{L} \) and \( a_0 = \frac{1}{L^{1/6}}, a_3 = \frac{1}{L^{1/3}}, a_4 = \frac{1}{L^{2/3}} \) to get,

\[
\begin{align*}
\text{ds}^2 &= \frac{L^2}{-\tilde{r}^{2m} d\tilde{r}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2} + \tilde{r}^2 d\tilde{x}^2 + \tilde{r}^{2m} d\tilde{y}^2 + \tilde{r}^{2m} d\tilde{z}^2}. 
\end{align*}
\tag{4.4.9}
\]

The isotropic Lifshitz metric [47] can be obtained by choosing \( C_0 = \frac{u_0}{L}, C_2 = C_3 = C_4 = \frac{1}{L} \) and \( a_0 = \frac{1}{L^{1/6}}, a_3 = a_4 = \frac{1}{L}, \) where \( L \) is the size of the spacetime. Redefining \( \tilde{t} = Lt, (\tilde{r}, \tilde{x}, \tilde{y}, \tilde{z}) = \frac{1}{L}(r, x, y, z), \) one gets the standard Lifshitz metric,

\[
\begin{align*}
\text{ds}^2 &= L^2 \left[ -\tilde{r}^{2m} d\tilde{r}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2} + \tilde{r}^2 (d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2) \right]. 
\end{align*}
\tag{4.4.10}
\]

Thus, one can see that constant anholonomy requires the extremal black brane metric (4.4.3) to have a specific form such as (4.4.7) or (4.4.8). We now argue that metrics with constant holonomy are homogeneous spaces.

The hypersurfaces on which the algebra of vectors \( \tilde{e}_i, i = 1, 2, 3, \) (see Appendix A) have constant anholonomy coefficients are called surfaces of transitivity and the vectors \( \tilde{e}_i \) generate a simply transitive group. It is known that for homogeneous spacetimes with space-like hypersurfaces of dimension three, there exists Lie groups of symmetries that act simply transitively on the surfaces [127]. Thus the algebra of the invariant vectors (A.0.2) can be shown to be isomorphic to the real Lie algebras of dimension three, which were classified by Bianchi [117]. The three dimensional real Lie algebras are of nine types labelled Bianchi I-IX. These symmetries are realised in homogeneous spaces where the Killing vectors generate an isomorphic Lie algebra.

Consider a basis of Killing vectors that generate a simply transitive group of dimension three. These Killing vectors have the algebra,

\[
\begin{align*}
\left[ \xi_\mu, \xi_\nu \right] &= \tilde{C}_{\mu \nu}^\lambda \xi_\lambda . 
\end{align*}
\tag{4.4.11}
\]
For each of the Bianchi classes, one can go to a suitable basis and construct invariant vector fields $\tilde{e}_i$ that commute with the Killing vectors,

$$[\xi_\mu, \tilde{e}_i] = 0.$$  \hspace{1cm} (4.4.12)

Now, the Jacobi identity between $(\tilde{e}_i, \xi_\mu, \xi_\nu)$ implies $\tilde{C}_{\mu\nu}^\lambda$ are constants in spacetime. These are the structure constants of the three dimensional real Lie algebras given by the Bianchi classification. The Jacobi identity between $(\tilde{e}_i, \tilde{e}_j, \xi_\mu)$ together with (4.4.12) imply that the anholonomy coefficients $c_{ij}^k$ are constants on the surface of transitivity.

Alternatively, given that the invariant one form have an algebra (A.0.2) with constant anholonomy coefficients, [119] have shown that (4.4.12) is satisfied by three independent Killing vectors, provided the following conditions are satisfied:

$$c_{0i}^0 = c_{ij}^0 = 0.$$  \hspace{1cm} (4.4.13)

A quick look at the metric (4.4.3), its vielbeins and non-vanishing anholonomy coefficients shows that both the conditions hold good for all $i, j = 1, 2, 3$. This implies (4.4.12) is satisfied for the spatial directions $(x, y, z)$, which means that these directions are homogeneous. We have used a simple class belonging to Bianchi type I to illustrate the connection between constant anholonomy coefficients and homogeneous spaces. This argument equally applies to all the Bianchi classes. In the next section, we list the various Bianchi type algebras, their structure constants, and briefly give an overview of the construction of metrics with these symmetry groups along the spatial directions [7].

### 4.5 Bianchi classification

In this section, we illustrate the construction of the five dimensional black brane horizons with homogeneous symmetries in the spatial directions [7]. To ensure that the metric
has the required symmetries it is written in terms of invariant one forms $\omega^j$ dual to the invariant vectors $\tilde{e}_i$. The invariant one forms satisfy the relation,

$$d\omega^k = \frac{1}{2} c^k_{ij} \omega^i \wedge \omega^j.$$  \hspace{1cm} (4.5.1)

The rest of the metric has to be fixed by demanding additional symmetries. Assuming time translational symmetries requires the metric to be time independent and requiring scaling symmetries of the form,

$$\hat{r} \to \lambda \hat{r}, \quad \hat{t} \to \lambda^{-u_0} \hat{t}, \quad \omega^i \to \lambda^{-u_i} \omega^i,$$  \hspace{1cm} (4.5.2)

fixes the metric to be of the form,

$$ds^2 = L^2 \left[ -\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2(u_i+u_j)} \eta_{ij} \omega^i \otimes \omega^j \right],$$  \hspace{1cm} (4.5.3)

where $u_0, u_i$ are positive in order to have a regular horizon, $i = 1, 2, 3$ corresponds to the $\hat{x}, \hat{y}, \hat{z}$ directions and $\eta_{ij}$ is a constant diagonal metric independent of spacetime coordinates. Note that the scaling symmetries in $\omega^j$ are determined by scaling $(\hat{x}, \hat{y}, \hat{z})$. The nature of the one forms will dictate what powers of $\hat{r}$ that will appear to have the required scale invariance of the metric. In fact, this can be determined just by looking at the Killing vectors that generate the homogeneous symmetries and we explain this below.

**Bianchi I**

We have already discussed this class in the previous section, here we get the general form of the metric (4.4.7) from symmetry considerations. The symmetry group of the Bianchi I class is isomorphic to the three dimensional translational group. It is also the symmetry group of the flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe. The Bianchi I class is also the most simplest among the Bianchi classes and is generated by the Killing
vectors $\xi^i$ which commute with each other. The Killing vectors, invariant vector fields, invariant one forms and structure constants are,

$$c_{ij}^k = 0, \quad d\omega^j = 0,$$

$$\xi_1 = \partial_{\hat{x}} = \tilde{e}_1, \quad \omega^1 = d\hat{x},$$

$$\xi_2 = \partial_{\hat{y}} = \tilde{e}_2, \quad \omega^2 = d\hat{y},$$

$$\xi_3 = \partial_{\hat{z}} = \tilde{e}_3, \quad \omega^3 = d\hat{z}. \quad (4.5.4)$$

As one can see from above, demanding scale invariance in the directions is possible with the weights,

$$(\hat{x}, \hat{y}, \hat{z}) \to (\lambda^{-u_1} \hat{x}, \lambda^{-u_2} \hat{y}, \lambda^{-u_3} \hat{z}) \quad (4.5.5)$$

and the one forms scale as,

$$(\omega^1, \omega^2, \omega^3) \to (\lambda^{-u_1} \omega^1, \lambda^{-u_2} \omega^2, \lambda^{-u_3} \omega^3). \quad (4.5.6)$$

Hence the most general metric of Bianchi type I with the scale invariance (4.5.2) along all the directions is given by,

$$ds^2 = L^2 \left[ - \hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2u_1} (\omega^1)^2 + \hat{r}^{2u_2} (\omega^2)^2 + \hat{r}^{2u_3} (\omega^3)^2 \right]. \quad (4.5.7)$$

We see that symmetries are quite powerful and the most general form of the Bianchi I type has been determined by requiring homogeneous symmetries and scale invariance. The $AdS$ metric is a special example of this type, with $u_0 = u_1 = u_2 = u_3 = 1$. We also saw earlier that the Lifshitz and $AdS_2 \times \mathbb{R}^3$ are examples of this class.
Bianchi II

The Bianchi II group is called the Heisenberg group of symmetries. The Killing vectors, invariant vector fields, invariant one forms and structure constants are,

\[ c_{23}^1 = 1 = -c_{32}^1, \]
\[ \xi_1 = \partial_{\hat{y}}, \quad \tilde{e}_1 = \partial_{\hat{y}}, \quad \omega^1 = d\hat{y} - \hat{x}d\hat{z}, \quad d\omega^1 = \omega^2 \wedge \omega^3, \]
\[ \xi_2 = \partial_{\hat{z}}, \quad \tilde{e}_2 = \hat{x}\partial_{\hat{y}} + \partial_{\hat{z}}, \quad \omega^2 = d\hat{z}, \quad d\omega^2 = 0, \]
\[ \xi_3 = \partial_{\hat{x}} + \hat{z}\partial_{\hat{y}}, \quad \tilde{e}_3 = \partial_{\hat{x}}, \quad \omega^3 = d\hat{x}, \quad d\omega^3 = 0. \] (4.5.8)

This time we see that the scale invariance is different. The coordinates scale as,

\[ (\hat{x}, \hat{y}, \hat{z}) \rightarrow (\lambda^{-u_1}\hat{x}, \lambda^{-(u_1+u_3)}\hat{y}, \lambda^{-u_3}\hat{z}), \] (4.5.9)

so that the invariant one forms scale as,

\[ (\omega^1, \omega^2, \omega^3) \rightarrow (\lambda^{-(u_1+u_3)}\omega^1, \lambda^{-u_3}\omega^2, \lambda^{-u_1}\omega^3), \] (4.5.10)

which fixes the form of the metric to be,

\[ ds^2 = L^2 \left[ -\hat{r}^2(\omega^1)^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2(\omega^2)^2 + \hat{r}^2(\omega^3)^2 \right]. \] (4.5.11)

Bianchi VI\(_h\), V and III

We discuss the Bianchi VI\(_h\) group of symmetries in detail, the Bianchi VI\(_h\) algebra is labelled by one arbitrary unfixed parameter \( h \neq 0, 1 \), which is constant and independent of the coordinates. The Bianchi III algebra is recovered when \( h = 0 \) and Bianchi V is recovered when \( h = 1 \). The Killing vectors, invariant vector fields, invariant one forms
and structure constants are,

\[ c_{13}^1 = 1, \quad c_{23}^2 = h, \]

\[ \xi_1 = \partial_{\hat{\eta}} \]
\[ \tilde{e}_1 = e^{\hat{\xi}} \partial_{\hat{\eta}} \]
\[ \omega^1 = e^{-\hat{\xi}} d\hat{y} \]
\[ d\omega^1 = \omega^1 \wedge \omega^3, \]

\[ \xi_2 = \partial_{\hat{\zeta}} \]
\[ \tilde{e}_2 = e^{h \hat{\zeta}} \partial_{\hat{\zeta}} \]
\[ \omega^2 = e^{-h \hat{\zeta}} d\hat{z} \]
\[ d\omega^2 = h \omega^2 \wedge \omega^3, \]

\[ \xi_3 = \partial_{\hat{x}} + \hat{y} \partial_{\hat{y}} + h \hat{z} \partial_{\hat{z}} \]
\[ \tilde{e}_3 = \partial_{\hat{x}} \]
\[ \omega^3 = d\hat{x} \]
\[ d\omega^3 = 0. \]  (4.5.12)

As one can see from above, there is no scaling possible in the \( x \) direction for all the three classes. The scaling in the other directions are,

\[ (\hat{x}, \hat{y}, \hat{z}) \rightarrow (\hat{x}, \lambda^{-u_2} \hat{y}, \lambda^{-u_3} \hat{z}) \]  (4.5.13)

and the one forms scale as,

\[ (\omega^1, \omega^2, \omega^3) \rightarrow (\lambda^{-u_2} \omega^1, \lambda^{-u_3} \omega^2, \omega^3) \]  (4.5.14)

which fix the metric to be of the form,

\[ ds^2 = L^2 \left[ - r^{2u_2} d\hat{r}^2 + \frac{dr^2}{r^2} + r^{2u_2} (\omega^1)^2 + r^{2u_3} (\omega^2)^2 + (\omega^3)^2 \right]. \]  (4.5.15)

The Bianchi III and V classes are recovered by taking \( h = 0 \) and \( h = 1 \) respectively. The Bianchi V class has a cosmological significance, it is the symmetry group of an open FLRW universe.

**Bianchi IV**

This is yet another class where scale invariance is not present in the \( \hat{x} \) direction. The structure constants are given by,

\[ c_{13}^1 = c_{23}^1 = c_{23}^2 = 1. \]  (4.5.16)
The Killing vectors and the invariant vector fields are,

\[ \xi_1 = \partial_{\hat{y}}, \quad \tilde{\xi}_1 = e^\hat{x} \partial_{\hat{y}}, \]
\[ \xi_2 = \partial_{\hat{z}}, \quad \tilde{\xi}_2 = \hat{x} e^\hat{y} \partial_{\hat{y}} + e^\hat{x} \partial_{\hat{z}}, \]
\[ \xi_3 = \partial_{\hat{x}} + (\hat{y} + \hat{z}) \partial_{\hat{y}} + \hat{z} \partial_{\hat{z}}, \quad \tilde{\xi}_3 = \partial_{\hat{x}}. \] (4.5.17)

The invariant one forms are,

\[ \omega_1 = e^{-\hat{x}} d\hat{y} - \hat{x} e^{-\hat{x}} d\hat{z}, \quad d\omega_1 = \omega_1 \wedge \omega^3 + \omega_2 \wedge \omega^3, \]
\[ \omega_2 = e^{-\hat{x}} d\hat{z}, \quad d\omega_2 = \omega_2 \wedge \omega^3, \]
\[ \omega_3 = d\hat{x}, \quad d\omega_3 = 0. \] (4.5.18)

The scaling symmetries in the other directions are,

\[ (\hat{x}, \hat{y}, \hat{z}) \rightarrow (\hat{x}, \lambda^{-1}\lambda^2\hat{y}, \lambda^{-1}\lambda^2\hat{z}), \] (4.5.19)

and the one forms scale as,

\[ (\omega_1, \omega_2, \omega_3) \rightarrow (\lambda^{-1}\lambda^2\omega_1, \lambda^{-1}\lambda^2\omega_2, \omega_3), \] (4.5.20)

which fix the metric to be of the form,

\[ ds^2 = L^2 \left[ - \hat{r}^2 + \frac{dr^2}{\hat{r}^2} + \hat{r}^2((\omega_1)^2 + (\omega_2)^2) + (\omega_3)^2 \right]. \] (4.5.21)

**Bianchi VII\_0**

This class is a favourite example since its symmetries have a nice physical description. The algebra of the Bianchi VII\_h class has an arbitrary unfixed constant parameter as in the
previous case. We consider the $h = 0$ case in this section. The structure constants are,

$$c_{32}^1 = c_{13}^2 = 1.$$ \hspace{1cm} (4.5.22)

The Killing vectors and invariant vector fields are given by,

$$\xi_1 = \partial_{\hat{y}}, \hspace{1cm} \tilde{\xi}_1 = \cos(\hat{x})\partial_{\hat{y}} + \sin(\hat{x})\partial_{\hat{z}},$$

$$\xi_2 = \partial_{\hat{z}}, \hspace{1cm} \tilde{\xi}_2 = -\sin(\hat{x})\partial_{\hat{y}} + \cos(\hat{x})\partial_{\hat{z}},$$

$$\xi_3 = \partial_{\hat{x}} - \hat{z}\partial_{\hat{y}} + \hat{y}\partial_{\hat{z}}, \hspace{1cm} \tilde{\xi}_3 = \partial_{\hat{x}}.$$ \hspace{1cm} (4.5.23)

The invariant one forms are given by,

$$\omega^1 = \cos(\hat{x})d\hat{y} + \sin(\hat{x})d\hat{z}, \hspace{1cm} d\omega^1 = -\omega^2 \wedge \omega^3,$$

$$\omega^2 = -\sin(\hat{x})d\hat{y} + \cos(\hat{x})d\hat{z}, \hspace{1cm} d\omega^2 = \omega^1 \wedge \omega^3,$$

$$\omega^3 = d\hat{x}, \hspace{1cm} d\omega^3 = 0.$$ \hspace{1cm} (4.5.24)

We see again that there is no scaling in the $\hat{x}$ direction and the $\hat{y}, \hat{z}$ directions scale uniformly as,

$$(\hat{x}, \hat{y}, \hat{z}) \rightarrow (\hat{x}, \lambda^{-u_2}\hat{y}, \lambda^{-u_2}\hat{z}),$$ \hspace{1cm} (4.5.25)

and the one forms scale as,

$$(\omega^1, \omega^2, \omega^3) \rightarrow (\lambda^{-u_2}\omega^1, \lambda^{-u_2}\omega^2, \omega^3),$$ \hspace{1cm} (4.5.26)

which fix the metric to be of the form,

$$ds^2 = L^2 \left[ -\hat{r}^2 d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2(\alpha(\omega^1)^2 + (\omega^2)^2) + (\omega^3)^2 \right].$$ \hspace{1cm} (4.5.27)

We have put an arbitrary constant parameter $\alpha \neq 0$ just to highlight the difference with the Bianchi type I. As one can see from the invariant one forms (4.5.24), $(\omega^1)^2 + (\omega^2)^2 =
\( d\tilde{y}^2 + d\tilde{z}^2 \) and when \( \alpha = 1 \), this becomes a special case of the Bianchi I types (4.5.7). The physical description of the symmetry group is as follows. The Killing vector \( \xi_3 \) generates translations along \( \hat{x} \) and rotations along the \((\hat{y}, \hat{z})\) plane. One full rotation in the \((\hat{y}, \hat{z})\) plane corresponds to a translation of \( 2\pi L \) along the \( \hat{x} \) direction giving rise to a helical motion. In this case, the size of the spacetime \( L \) has a physical description as the pitch of the helix.

**Bianchi VIII**

This is the first class that we deal with which gives rise to a metric with no scaling symmetry along all the three spatial directions! The structure constants are,

\[
c_{32}^1 = c_{31}^2 = c_{12}^3 = 1 .
\]

The Killing vectors and the invariant vector fields are given by,

\[
\begin{align*}
\xi_1 &= \frac{1}{2} e^{-y} \partial_{\tilde{x}} + \frac{1}{2} (e^{y} - \hat{y}^2 e^{-y}) \partial_{\tilde{y}} - \hat{y} e^{-y} \partial_{\tilde{z}} , & \tilde{e}_1 &= \frac{1}{2} (1 + \hat{x}^2) \partial_{\tilde{x}} + \frac{1}{2} (1 - 2\hat{\tilde{x}}\hat{y}) \partial_{\tilde{y}} - \hat{\tilde{x}} \partial_{\tilde{z}} , \\
\xi_2 &= \partial_{\tilde{z}} , & \tilde{e}_2 &= -\hat{\tilde{x}} \partial_{\tilde{x}} + \hat{\tilde{y}} \partial_{\tilde{y}} + \partial_{\tilde{z}} , \\
\xi_3 &= \frac{1}{2} e^{-y} \partial_{\tilde{x}} - \frac{1}{2} (e^{y} + \hat{y}^2 e^{-y}) \partial_{\tilde{y}} - \hat{y} e^{-y} \partial_{\tilde{z}} , & \tilde{e}_3 &= \frac{1}{2} (1 - \hat{x}^2) \partial_{\tilde{x}} + \frac{1}{2} (-1 + 2\hat{\tilde{x}}\hat{y}) \partial_{\tilde{y}} + \hat{\tilde{x}} \partial_{\tilde{z}} .
\end{align*}
\]

(4.5.28)

The invariant one forms are given by,

\[
\begin{align*}
\omega^1 &= d\tilde{x} + (1 + \hat{y}^2) d\tilde{y}^2 + (\hat{x} - \hat{\tilde{y}} - \hat{x}^2 \hat{y}) d\tilde{z} , & d\omega^1 &= -\omega^2 \wedge \omega^3 , \\
\omega^2 &= 2\hat{x} d\tilde{y} + (1 - 2\hat{\tilde{y}}) d\tilde{z} , & d\omega^2 &= \omega^3 \wedge \omega^1 , \\
\omega^3 &= d\tilde{x} + (-1 + \hat{x}^2) d\tilde{y}^2 + (\hat{x} + \hat{\tilde{y}} - \hat{x}^2 \hat{y}) d\tilde{z} , & d\omega^3 &= \omega^1 \wedge \omega^2 .
\end{align*}
\]

(4.5.29)
This class has no scaling along any of the $\hat{x}, \hat{y}, \hat{z}$ directions with homogeneous symmetries. The metric then takes the form,

$$ds^2 = \left[ -\hat{r}^2 d\hat{r}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 \right]. \quad (4.5.30)$$

The interesting thing about this metric is it factorises as $\text{Lif}_2(u_0) \times M_{VIII}$, where $\text{Lif}_2(u_0)$ is the two dimensional Lifshitz metric and $M_{VIII}$ is the metric written in terms of invariant one forms respecting the symmetry group of Bianchi VIII.

**Bianchi IX**

The symmetry group of this class is isomorphic to the three dimensional rotational group $SO(3, \mathbb{R})$. The structure constants are,

$$c^{1}_{23} = c^{2}_{31} = c^{3}_{12} = 1. \quad (4.5.31)$$

The Killing vectors and invariant vector fields are given by,

$$\xi_1 = \partial_{\hat{z}}, \quad \tilde{e}_1 = -\sin(\hat{z})\partial_{\hat{x}} + \frac{\cos(\hat{z})}{\sin(\hat{x})}\partial_{\hat{y}} - \cot(\hat{x})\cos(\hat{z})\partial_{\hat{z}},$$

$$\xi_2 = \cos(\hat{y})\partial_{\hat{z}} - \cot(\hat{x})\sin(\hat{y})\partial_{\hat{x}} + \frac{\sin(\hat{y})}{\sin(\hat{x})}\partial_{\hat{z}}, \quad \tilde{e}_2 = \cos(\hat{z})\partial_{\hat{x}} + \frac{\sin(\hat{z})}{\sin(\hat{x})}\partial_{\hat{y}} - \sin(\hat{z})\cot(\hat{x})\partial_{\hat{z}},$$

$$\xi_3 = -\sin(\hat{y})\partial_{\hat{x}} - \cot(\hat{y})\cos(\hat{y})\partial_{\hat{x}} + \frac{\cos(\hat{y})}{\sin(\hat{x})}\partial_{\hat{z}}, \quad \tilde{e}_3 = \partial_{\hat{z}}. \quad (4.5.32)$$

The invariant one forms are given by,

$$\omega^1 = -\sin(\hat{z})d\hat{r} + \sin(\hat{x})\cos(\hat{z})d\hat{y}, \quad d\omega^1 = \omega^2 \land \omega^3,$$

$$\omega^2 = \cos(\hat{z})d\hat{r} + \sin(\hat{x})\sin(\hat{z})d\hat{y}, \quad d\omega^2 = \omega^3 \land \omega^1,$$

$$\omega^3 = \cos(\hat{x})d\hat{y} + d\hat{z}, \quad d\omega^3 = \omega^1 \land \omega^2. \quad (4.5.33)$$
This time again there are no scaling symmetries in the \((\hat{x}, \hat{y}, \hat{z})\) directions and the metric factorises into \(Li_f u_0(2) \times M_{IX}\),

\[
    ds^2 = -\hat{r}^2 u_0 d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + (\omega_1)^2 + (\omega_2)^2 + (\omega_3)^2.
\]  

(4.5.34)

Metrics with Bianchi IX symmetries have the symmetry group \(SO(3, R)\). In 3 + 1 dimensions metrics of such type have been extensively studied in cosmology. In fact, the closed FRW universe has the Bianchi IX symmetry and describes an anisotropic universe with rotating matter.

So far, we discussed the various symmetry classes in the Bianchi classification and used simple scaling symmetry requirements to arrive at metrics which respect these symmetries. It is well known that the near horizon geometries of black holes independently solve the field equations. It has been shown in [7] that many of these solutions can be recovered from simple matter systems like gravity coupled to massive gauge fields. In chapter 6, we will use this information to construct some of the Bianchi type solutions from gauged supergravity.

We end this chapter with an observation. If we did not demand scale invariance along the \((\hat{x}, \hat{y}, \hat{z})\) directions, but keep the scale invariance along \((\hat{r}, \hat{t})\) directions the metric (4.5.3) splits into a direct product form as,

\[
    ds^2 = L^2 \left[ -\hat{r}^2 u_0 d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \eta_{ij} \omega^i \otimes \omega^j \right].
\]  

(4.5.35)

We call this subset of metrics as \(Li_f u_0(2) \otimes M\), where \(Li_f u_0(2)\) is the two dimensional Lifshitz metric and \(M\) corresponds to the spatial part of the metrics that display homogeneous symmetries labelled as \(M_I, M_{II}, \ldots M_{IX}\).
4.6 Summary

In this chapter, we studied black holes in Anti de-Sitter space. We started by studying the Schwarzschild and Reissner-Nordstrom black holes in this background. We then studied the black brane limit of the Schwarzschild black hole and the near horizon geometry of the extremal Reissner-Nordstrom black brane. We also observed that the near horizon geometry of the extremal Reissner-Nordstrom black brane takes the form $AdS_2 \times \mathbb{R}^3$, and in addition has constant anholonomy coefficients.

We then studied some of the properties of the near horizon geometry. Especially we observed starting with the assumption of constant anholonomy in a simple example and obtained the most general metric type consistent with scale invariance and the symmetries of Bianchi I class. We then explained the relation between constant anholonomy and homogeneous spaces. We saw that the various Bianchi classes indeed have constant anholonomy coefficients and discussed the homogeneous extremal black brane horizons classified by the Bianchi classification in detail.

We saw that a simple requirement demanding scale invariance and homogeneity along various directions was sufficient to fix the most general form of the metric. We observed that when we demand scale invariance only along the radial and time directions, the most general metric consistent with these symmetries split into a direct product form which is reminiscent of near horizon geometries of extremal black holes.

As an end note, we feel it is important to mention that not all of the Bianchi class metrics have been realised as near horizon geometries as extremal Black branes. Numerically, solutions interpolating between one of the Bianchi classes and $AdS_5$ have been found in [7]. An analytic interpolating black hole solution is still lacking except for familiar cases like $AdS_2 \times \mathbb{R}^3$. Such interpolating solutions would be very valuable both for understanding the attractor mechanism in $AdS$ spaces as well as for studying field theory renormalisation group flows in the dual gravity side.