Chapter 2

Some geometrical aspects of p-adic integers and distribution of rational integers in the ring \( \Omega_p \)

2.1 Introduction

In this chapter studies have been confined to some geometrical aspects of p-adic integers, considering them as sequence with a non-Archimedean norm defined in terms of their weights. Partitioned \( \Omega_p \) [Definition 2.2.1] into equivalence classes and to visualized the behavior of the equivalence classes in it. Moreover a function \( \psi \) [Definition 2.2.7] has been defined which helps to determine the negative of a p-adic number readily. An effort has been made to visualize the distribution of rational integers in the ring of p-adic integers \( \Omega_p \).

Weight function \( \omega \) has been defined on the ring \( \Omega_p \) of p-adic integers considering them as sequences. The properties of \( \omega \) motivate us to the evaluation of a non-Archimedean norm in \( \Omega_p \) which coincide with the norm derived from the valuation function and subsequently a definition pertaining to an equivalence relation has been drawn in it. Visualization of geometric properties of the equivalence classes so obtained and illustration of the isosceles triangular equality, independence on the center of an open (close) ball in non-Archimedean analysis are the main objectives of this chapter. An operator \( \phi^{-1} \circ \psi \circ \phi \) has been employed to determine the negative of a p-adic integer. Finally an explanation has been provided in order to show how the rational integers are distributed in the ring \( \Omega_p \).
2.2 Preliminaries

Definition 2.2.1. We define a mapping \( \phi : \mathbb{Z}_p \rightarrow \Omega_p \) by,

\[
\phi(x) = \{x_i\} \quad \forall x = \sum_{i \geq 0} x_i p^i \in \mathbb{Z}_p
\]

\[.: \Omega_p = \{x = \{x_i\}_{i=0}^\infty | 0 \leq x_i \leq p-1, i = 0, 1, 2, \ldots\} \]

Definition 2.2.2. For any \( x, y \in \Omega_p \), we define their addition and multiplication by,

\( x + y = z \) where \( x_i + y_i + a_i = a_{i+1} + z_i \) \( \forall i = 0, 1, 2, \ldots \) with \( a_0 = 0 \)

\( xy = w \) where \( \sum_{j+k=i} x_j y_k + b_i = b_{i+1} + w_i \) \( \forall i = 0, 1, 2, \ldots \) with \( b_0 = 0 \)

The sequence \( a = \{a_i\} \) and \( b = \{b_i\} \) are said to be the carry-out sequence for the addition and multiplication of \( x, y \in \Omega_p \) and we denote them by \([x + y]\) and \([xy]\) respectively.

Definition 2.2.3. We define a function \( \omega : \Omega_p \rightarrow \mathbb{Z}^+ \cup \{\infty\} \) by,

\[
\omega(x) = \begin{cases} 
  n & \text{if } x_i = 0 \forall i < n \text{ and } x_n \neq 0 \\
  \infty & \text{if } x = 0
\end{cases}
\]

This function \( \omega \) is said to be the weight function, while we term \( \omega(x) \) to be the weight of the element \( x \) in \( \Omega_p \).

Remark 2.2.4. With the help of weight, the concept of the absolute value for an element \( x \in \Omega_p \) can be defined as follows:

\[
\|x\|_p = \frac{1}{\omega(x)}
\]

Definition 2.2.5. For \( k = 0, 1, 2, \ldots \) we define \( e_k = \{x_i\} \in \Omega_p \) by

\( x_k = 1 \) and \( x_i = 0 \) \( \forall i \neq k \)

Note: Clearly \( e_i e_j = e_{i+j} \) and \( \omega(e_k) = k \).

Definition 2.2.6. We define \( e = \{x_i\} \in \Omega_p \) by \( x_i = 1 \) \( \forall i = 0, 1, 2, \ldots \)

Clearly \( e = \sum_{i \geq 0} e_i \).
Definition 2.2.7. We define a function \( \psi : \Omega_p \rightarrow \Omega_p \) by,

\[
\psi(x) = y \Leftrightarrow x_i + y_i = \begin{cases} 
0 & \text{if } i < i_0 \\
p & \text{if } i = i_0 \\
p - 1 & \text{if } i > i_0 
\end{cases}
\]

provided \( x_{i_0} \) is the first non zero entry in \( x = \{x_i\} \in \Omega_p \).

The function \( \psi \) so defined is said to be the **negative annihilator** on \( \Omega_p \).

Definition 2.2.8. For any \( \alpha \in B = \{0, 1, 2, 3, \ldots, p-1\} \) and for any \( x = \{x_i\} \in \Omega_p \) we define,

\[
\alpha x = y \Leftrightarrow \alpha x_i + b_i = b_{i+1} p + y_i \quad \forall i = 0, 1, 2, 3, \ldots \text{ with } \quad b_0 = 0.
\]

where \( b = \{b_i\} \) is the carry-out sequence \([\alpha x]\).

### 2.3 First section

Theorem 2.3.1. For any \( x, y \in \Omega_p \) and \( \alpha \in B = \{1, 2, \ldots, p-1\} \), following assertions are hold:

1. \( \omega(x \pm y) \geq \min\{\omega(x), \omega(y)\} \)
2. \( \omega(-x) = \omega(x) \)
3. \( \omega(\alpha x) = \omega(x) \)
4. \( \omega(x \pm y) = \omega(x) \) provided \( \omega(x) < \omega(y) \)
5. \( \omega(e_k) = k \) for any \( k = 0, 1, 2, \ldots \)
6. \( \omega(e_k x) = \omega(e_k) + \omega(x) \)

**Proof:** Suppose \( \omega(x) = m \) and \( \omega(y) = n \) so that,

\[
x_i = 0 \quad \forall i < m \text{ and } \quad x_m \neq 0 \quad \text{(2.3.1)}
\]

\[
y_i = 0 \quad \forall i < n \text{ and } \quad y_n \neq 0
\]
If we take \([x + y] = a = \{a_i\}\) then clearly \(a_i = 0\) \(\forall i \leq \min\{m, n\}\) and thus from (2.3.1) we obtain,

\[x_i + y_i = 0 \quad \forall i < \min\{m, n\} = \min\{\omega(x), \omega(y)\}\]

So, \(\omega(x + y) \geq \min\{\omega(x), \omega(y)\}\).

By the same argument it can be seen that \(\omega(x - y) \geq \min\{\omega(x), \omega(y)\}\).

The assertions (b) and (c) are follows trivially.

As \(\omega(x) < \omega(y)\), \(\min\{\omega(x), \omega(y)\}\) = \(\omega(x)\) and hence (d) follows from (a).

The assertion (e) follows directly from the definition of \(e_k\).

From the definition of product in \(\Omega_p\) it follows that

\[e_kx = \left\{ \frac{k \text{ numbers}}{0, 0, \ldots, 0, x_0, x_1, \ldots} \right\}\]

and hence \(\omega(e_kx) = \omega(e_k) + \omega(x)\).

**Corollary 2.3.2.** The absolute value defined in Remark 2.2.4 gives a non-Archimedean norm in \(\Omega_p\).

**Proof:** The non-negativity and positivity of \(\|\cdot\|_p\) are direct consequences of definitions. Let \(x, y \in \Omega_p\). It follows from (a) of Theorem 2.3.1 that

\[-\omega(x + y) \leq -\min\{\omega(x), \omega(y)\}\]

So,

\[\|x + y\|_p \leq p^{\max\{-\omega(x), -\omega(y)\}}\]

\[= \max \{p^{-\omega(x)}, p^{-\omega(y)}\}\]

\[= \max \{\|x\|_p, \|y\|_p\}\]

**Corollary 2.3.3.** The non-Archimedean norm \(\|\cdot\|_p\) so obtained in \(\Omega_p\) induced an equivalence relation, resulting a partition in \(\Omega_p\).
Proof: Let us define a relation ~ in $\Omega_p$ by,

$$x \sim y \iff ||x - y||_p \leq \frac{1}{p^n} \text{ for some } n \in \mathbb{N}$$

As $||x - x||_p = 0 \leq \frac{1}{p^n}$ for any $n \in \mathbb{N}$, $x \sim x \ \forall x \in \Omega_p$. The symmetry of ~ can be achieved from $\omega(-x) = \omega(x)$.

Finally let, $x \sim y$ and $y \sim z$.

$$||x - z||_p = ||(x - y) + (y - z)||_p$$

$$\leq \max\{||x - y||_p, ||y - z||_p\}$$

$$\leq \left\{ \frac{1}{p^m}, \frac{1}{p^n} \right\} \text{ for some } m, n \in \mathbb{N}$$

$$= \frac{1}{p^k} \text{ where } k = \min\{m, n\}$$

This proves the transitivity of the relation ~ in $\Omega_p$.

Corollary 2.3.4. It is possible to partition $\Omega_p$ into a countable collection of equi-norm equivalence classes $\{C_i\}_{i=0}^{\infty}$ of equal cardinality $i \neq \infty$ such that,

$$x \pm C_n \subset C_m \ \forall x \in C_m, \ \forall \ n > m \text{ and } n \neq \infty$$

Proof: The equivalence relation $x \sim y \iff ||x||_p = \frac{1}{p^n} = ||y||_p$ in $\Omega_p$ will produce the equivalence classes,

$$C_i = \left\{ x \in \Omega_p : ||x||_p = \frac{1}{p^i} \right\} = \{ x \in \Omega_p : \omega(x) = i \}$$

In other words, $C_i$ is a subset of $\Omega_p$ containing of those sequences whose first nonzero term is preceded by $i$ number of zeros. Let us choose $C_m$ and $C_n$ with $m < n < \infty$. We now consider the map $T : C_m \longrightarrow C_n$ defined by

$$T(x) = e_{n-m}x$$

It can easily be seen that T is a bijection.

$$\therefore \#(C_m) = \#(C_n)$$
Note that $\#(C_\infty) = 1$.

Lastly we take $x \pm y \in x \pm C_n$ with $x \in C_m$.

From (a) of Theorem 2.3.1 it follows that

$$\omega(x \pm y) = \omega(x).$$

Thus $x \pm y \in C_m$ and hence $x \pm C_n \subset C_m$.

**Theorem 2.3.5.** An open ball $B(a, r)$ in $\Omega_p$ is either a singleton $\{a\}$ or expressible as $\bigcup_{i \geq n} C_i$ for some suitably chosen $n \in \mathbb{N}$.

**Proof:** Let us consider an open ball $B(a, r)$ in $\Omega_p$ with $a \in C_m$ for some $m \in \mathbb{N}$. We now consider the following two cases:

**Case I:** Suppose that $0 < r \leq \|a\|_p$. We want to show that $B(a, r) = \{a\}$.

If possible, let us assume that there exists $x \in \Omega_p$ such that $x \neq a$ but $x \in B(a, r)$. As $x \in B(a, r)$ and $a \in C_m$, we must have,

$$x \in C_i \quad \text{for some} \quad i \geq m.$$

Otherwise for an $x \in C_i$ with $i < m$, $x - a \in C_i$ as consequence of which

$$\|x - a\|_p = \frac{1}{p^i} > \frac{1}{p^m} = \|a\|_p \geq r.$$

But this contradicts our assumption that $x \in B(a, r)$. Thus we can conclude that $x - a \in C_m$ and hence

$$\|x - a\|_p = \|a\|_p \geq r.$$

This is again a contradiction, and leads to the conclusion that

$$B(a, r) = \{a\}.$$

**Case II:** Suppose that $0 < \|a\|_p < r$. It is always possible to choose $n \in \mathbb{N}$ such that

$$\frac{1}{p^n} < r \leq \frac{1}{p^{n-1}}.$$

As $a \in C_m$, $\frac{1}{p^m} = \|a\|_p < r$, that is, $\frac{1}{p^m} \leq \frac{1}{p^n}$ and hence $m \geq n$.

For any point $x \in B(a, r)$, we have
\[ \|x\|_p \leq \max\{\|x - a\|_p, \|a\|_p\} \]
\[ \leq \max\left\{ \frac{1}{p^n}, \frac{1}{p^m} \right\} \]
\[ = \frac{1}{p^n} \]

It follows that

\[ B(a, r) \subset \bigcup_{i \geq n} C_i \]

On the other way we proceed as

\[ y \in \bigcup_{i \geq n} C_i \Rightarrow y \in C_i \text{ for some } i \geq n \]
\[ \Rightarrow y - a \in C_m \text{ or } C_i \text{ according as } i \geq m \text{ or } m \geq i \]
\[ \Rightarrow \|y - a\|_p = \frac{1}{p^m} \text{ or } \frac{1}{p^i} \leq \frac{1}{p^n} < r \]
\[ \Rightarrow y \in B(a, r) \]

This shows that \( \bigcup_{i \geq n} C_i \subset B(a, r) \). Hence \( B(a, r) = \bigcup_{i \geq n} C_i \).

**Theorem 2.3.6.** The function \( \phi \) defined in Definition 2.2.1 satisfies the following:

(a) It is a bijection

(b) It is isometric i.e. \( \|\phi(x)\|_p = \|x\|_p \) \( \forall x \in \mathbb{Z}_p \)

(c) There exist uncountable number of \( p \)-adic integers of equal norm.

In other words \( \#(C_i) > \aleph_0 \quad \forall 0 \leq i < \infty \).

**Proof:** The assertions (a) and (b) are immediate consequences of the definition of \( \phi \). Suppose that (c) does not hold. Then there exists an \( i(< \infty) \) such that \( C_i \) is countable. As \( C_i \) are of equal cardinality \( [i \neq \infty, \text{Corollary 2.3.4}] \) and \( \Omega_p = \bigcup_{i \geq 0} C_i \), therefore \( \Omega_p \) is countable. But this contradicts the fact that \( \phi \) is a bijection and \( \mathbb{Z}_p \) is uncountable.
Theorem 2.3.7. The negative-annihilator \( \psi \) defined in Definition 2.2.7 satisfies the following:

(a) It is isometric i.e. \( \|\psi(x)\|_p = \|x\|_p \) \( \forall x \in \Omega_p \)

(b) It is bijective and an involution

(c) It has only one fixed point, namely zero.

(d) \( (\phi^{-1} \circ \psi \circ \phi)(x) = -x \ \forall x \in \mathbb{Z}_p \)

Proof: For \( x = 0 \), (a) trivially hold good. Let us choose a nonzero \( x = \{x_i\} \in \Omega_p \) and let \( x_{i_0} \) be the first nonzero entry in \( x \). If \( \psi(x) = y = \{y_i\} \) then by Definition 2.2.7 \( y_{i_0} = p - x_{i_0} \neq 0 \) and \( y_i = 0 \ \forall i < i_0 \). Subsequently

\[ \omega(x) = i_0 = \omega(y) \]

and hence \( \|x\|_p = ||\psi(x)||_p \).

This proves (a).

Suppose that \( x, y \in \Omega_p \) such that \( \psi(x) = \psi(y) \). If we take \( \psi(x) = x' \) and \( \psi(y) = y' \) then from (a) it follows that,

\[ \omega(x) = \omega(x') = \omega(y) = \omega(y') = i_0 \text{(say)}. \]

So \( x_i = 0 = y_i \ \forall i < i_0 \)

Also from the Definition 2.2.7 it follows that

\[ x_{i_0} + x_{i_0}' = y_{i_0} + y_{i_0}' = p \text{ and } x_i + x_i' = y_i + y_i' = (p - 1) \ \forall i > i_0 \quad (2.3.2) \]

As \( x' = y' \), from the equation 2.3.2 it follows that \( x = y \). This proves that \( \psi \) is one-one.

The ontoness of \( \psi \) is obvious and consequently \( \psi \) is a bijection.

Suppose that \( \psi(x) = x' \) and \( \psi(x') = x'' \). By the assertion (a), \( \omega(x) = \omega(x') = \omega(x'') \).

So \( x_i = 0 = x_i'' \ \forall i < i_0 \).

Also from the definition of the negative annihilator \( \psi \) we get,

\[ x_{i_0} + x_{i_0}' = p = x_{i_0} + x_{i_0}' \]

\[ x_i + x_i' = p - 1 = x_i' + x_i'' \ \forall i > i_0 \]
Hence $x = x''$ and consequently $\psi^2 = I$.

This proves that $\psi$ is an involution.

Obviously $\psi(0) = 0$. If possible, let us choose a nonzero element $x \in \Omega_p$ such that $\psi(x) = x$. If $x_{i_0}$ is the first nonzero entry in $x$ then $x_{i_0} + x_{i_0} = p$ and $x_i + x_i = p - 1$ for all $i > i_0$; which contradicts each other as $x_i \in \{0, 1, \ldots, p - 1\}$ and hence 0 is the only fixed point of $\psi$.

Clearly $\phi^{-1} \circ \psi \circ \phi$ is an operator on $\mathbb{Z}_p$.

For $x = \sum_{i \geq 0} x_ip^i$ we take

$$\phi(x) = x' \text{ and } \psi(\phi(x)) = \psi(x') = y.$$ 

If $x_{i_0}$ is the first nonzero entry in $x' = \{x_i\}$ then by the definition of $\psi$ we get,

$$x_i + y_i = \begin{cases} 
0 & \text{if } i < i_0 \\
p & \text{if } i = i_0 \\
p - 1 & \text{if } i > i_0 
\end{cases}$$

But $\phi^{-1}(\psi(\phi(x))) = \sum_{i \geq 0} y_ip^i$ and hence,
\[
x + \phi^{-1}(\psi(\phi(x))) = \sum_{i \geq 0} x_i p^i + \sum_{i \geq 0} y_i p^i
\]
\[
= (x_0 + y_0)p^0 + \sum_{i \geq 0+1} (x_i + y_i)p^i
\]
\[
= 0
\]
It follows that \( \phi^{-1}(\psi(\phi(x))) \) is the negative of \( x \in \mathbb{Z}_p \) and \( \psi(x) \) is the negative of \( x \in \Omega_p \).

2.4 Second section

In this section a geometrical interpretation has been given to the equivalence classes \( \{C_i\}_{i \geq 0} \) of equi-norm elements, isosceles triangular equality and the independence of the open (or close) balls on their center, in non-Archimedean analysis. Moreover, an effort has been made to locate the rational integers in the ring \( \Omega_p \). This provides us an opportunity to identify the area of highest density of the non-rational p-adic integers in \( \Omega_p \).

2.4.1 Geometrical interpretation of isosceles triangular equality

Let us explain the isosceles triangular equality in the ring \( \Omega_p \) with the help of equi-norm equivalence classes \( \{C_i\}_{i \geq 0} \). Geometrically we consider each equi-norm equivalence class \( C_i \) as a circle with center at 0 and of radius \( \frac{1}{p^i} \), \( i = 0, 1, 2, \ldots \). In fact \( C_\infty = \{0\} \) will be the center of \( C_i \) \( i \geq 0 \), when we consider them as circles.

For the sake of simplicity let us take a point \( A(x) \) on the circle \( C_0 \) and a point \( B(y) \) on the circle \( C_1 \). It follows from the Corollary 2.3.4 that \( x \pm C_1 \subset C_0 \) and hence \( x \pm y \) will lie on the circle \( C_0 \). Let \( P \) and \( Q \) be the points on the circle \( C_0 \) corresponding to \( x + y \) and \( x - y \) respectively. As \( \|(x \pm y) - x\|_p = \frac{1}{p} \) and \( \|y\|_p = \frac{1}{p} \) the positions of the points \( P \) and \( Q \) on the circle \( C_0 \) will be such that \( PA = \|y\|_p = QA \).

We now draw the triangles \( OPA \) and \( OQA \).

Clearly \( OA = \|x\|_p = \|x + y\|_p = OP \) and \( PA = \|y\|_p \).

Also \( OA = \|x\|_p = \|x - y\|_p = OQ \) and \( QA = \|y\|_p \).
It follows that the triangles $OPA$ and $OQA$ are isosceles with two sides of lengths equal to $||x||_p$ and $||y||_p$.
This gives us the geometrical explanation of the isosceles triangular equality in non-Archimedean analysis.

2.4.2 Geometrical interpretation of the independence of an open (or close) ball on its center

In case $0 < r \leq ||a||_p$, $B(a, r) = \{a\}$ [case I Theorem 2.3.5] and hence we have nothing to talk about the independence of the centre. So let us consider an open ball $B(a, r)$ with the condition $0 < ||a||_p < r$, $a \in C_m$ and suppose that $b \in B(a, r)$.
As $a \in C_m$ and $b \in B(a, r)$, $b \in C_i$ for some $i \geq m$. Hence $0 < ||b||_p \leq ||a||_p < r$.

It follows from case II of Theorem 2.3.5 that $B(a, r) = \bigcup_{i \geq n} C_i = B(b, r)$.

**Observations:** (i) The cardinality of any two equi-norm equivalence classes $C_i$ can be geometrically interpreted from the fact that any line starting from $C_\infty = \{0\}$ meets each
$C_t(i \neq \infty)$ at an unique point. This gives a one to one correspondence between any two equi-norm equivalence classes $C_m$ and $C_n(m \neq n)$.

(ii) For any $x \in \Omega_p$, as weight $\omega(x)$ increases, the corresponding norm $\|x\|_p$ decreases. So in the ring $\Omega_p$ an element sinks deeper and deeper as its weights $\omega(x)$ increases and norm $\|x\|_p$ decreases.

### 2.5 Distribution of rational integers in the ring $\Omega_p$

Here we want to have a look how the rational integers are distributed in the ring of $p$-adic integers $\Omega_p$. For this purpose we note the following steps:

**Step I:** We consider the family of concentric circles $C_t$ with center at 0 and of radius $\frac{1}{p^i}, i = 0, 1, 2, 3, \ldots$

**Step II:** We take the set $B_i = \{e_1, 2e_1, 3e_1, \ldots, (p-1)e_1\}$ as the set of basic points for the circle $C_t, t = 0, 1, 2, 3, \ldots$

**Step III:** We consider the $(p-1)$ numbers of diameters $A_1A'_1, \ldots, A_{p-1}A'_{p-1}$ of the circle $C_0$ which are joints of the basic points $1, 2, 3, \ldots, (p-1)$ for $C_0$ with their negatives, what we call the set of basic diameters. The same set of diameters will serve as the basic diameters for the rest of the family of circles $C_t, t = 1, 2, 3, \ldots$

**Step IV:** We now consider the equivalence classes $[0], [1], \ldots, [p-1]$ associated with the equivalence relation $mod(p)$ in the set of rational integers $\mathbb{Z}$.

**Step V:** As we know every positive rational integer $m$ is a recurring $p$-adic integer of finite degree with the recurred set $R=(0)$, therefore,

$$m = \alpha_0 e_{i_0} + \alpha_1 e_{i_1} + \ldots + \alpha_{i_k-1} e_{i_k-1} + \alpha_{i_k} e_{i_k} \quad (2.5.1)$$

where $i_0 < i_1 < \ldots < i_k$ and $\alpha_{i_0}, \alpha_{i_1}, \ldots, \alpha_{i_k} \in B$

Such an integer $m$ will lie on the circle $C_{i_0}$, on the left of the basic point $\alpha_{i_k} e_{i_0}$ in anticlockwise direction. In loose sense in the $p$-adic expansion (2.5.1) of $m$, $\alpha_{i_k} e_{i_k}$ will serve as the integral part of $m$, while the remaining part (2.5.1) will serve as the decimal part of
For the decimal part we consider the following transformation:

\[ \alpha_{e_0} e_0 + \alpha_{e_1} e_1 + \ldots + \alpha_{e_{k-1}} e_{k-1} = \left( \frac{\alpha_{e_0}}{p_{n_k-e_0}} + \frac{\alpha_{e_1}}{p_{n_{k-1}}-e_1} + \ldots + \frac{\alpha_{e_{k-1}}}{p_{n_{k-1}}-e_{k-1}} \right) e_{k} \]

This transformation will determine the exact location of the rational integer \( m \) on the circle \( C_{e_0} \).

In this process we always determine the locations of a set of \((p-1)\) rational integers at a time. This set of \((p-1)\) rational integers either belongs to \([0]\) (in this case they are not necessarily consecutive integers) or taking one each from \([1], [2], \ldots, [p - 1] \). Whenever we write \( m_1, m_2, \ldots, m_{p-1} \rightarrow \frac{(\alpha e_i, \beta)}{p^k} \), we mean that this set of rational integers will occupy consecutive positions on the circle \( C_i \) after the rational integer \( \beta \) (position of which is already identified) when we make \( p^k \) equal parts of \( C_i \) in between the basic points. If we write \( m_1, m_2, \ldots, m_{p-1} \rightarrow \frac{\alpha e_i}{p^k} \), we mean that the set of rational integers will occupy the consecutive positions just after the basic points \( \alpha e_i \) after making \( p^k \) equal parts between them.

For the sake of simplicity we now illustrate the above procedure for the distribution of rational integers in the ring of \( p \)-adic integer \( \Omega_5 \).

Here \( B_i = \{ e_i, 2e_i, 3e_i, 4e_i \} \), \( i = 0, 1, 2, \ldots \).

Let us consider the set of rational integers \( \{6, 7, 8, 9\} \) taking one each from \([1], [2], [3]\) and \([4]\).

As \( 6 = e_0 + e_1 = \frac{1}{5} e_1 + e_1, 7 = \frac{2}{5} e_1 + e_1, 8 = \frac{3}{5} e_1 + e_1, 9 = \frac{4}{5} e_1 + e_1 \).

\[ \therefore 6, 7, 8, 9 \rightarrow \frac{e_0}{5} \]

Similarly it can be seen that,

\[ 11, 12, 13, 14 \rightarrow \frac{2e_0}{5} \]
\[ 16, 17, 18, 19 \rightarrow \frac{3e_0}{5} \]
\[ 21, 22, 23, 24 \rightarrow \frac{4e_0}{5} \]

Let us consider the set of rational integers \( \{31, 32, 33, 34\} \).
Figure 2.3: Distribution of the rational integers in the ring $\Omega_5$

As, $31 = e_0 + e_1 + e_2 = \frac{6}{5^2} e_2 + e_2$, $32 = \frac{7}{5^2} e_2 + e_2$, $33 = \frac{8}{5^2} e_2 + e_2$, $34 = \frac{9}{5^2} e_2 + e_2$.

\[
\therefore 31, 32, 33, 34 \rightarrow \frac{(e_0, 6)}{5^2}
\]

Fig 2.3 and Fig 2.4 illustrates the distribution of rational integers in the ring $\Omega_5$.

In the light of the above discussion we can conclude that the area $BOA'$ is free from rational integers and this is the area of highest density for the non-rational $p$-adic integers. We
Figure 2.4: Group-wise distribution of rational integers in the ring $\Omega_5$
now put forward the following question:

*What is the p-adic integer corresponding to the point B?*

If we take the p-adic integer $e$ corresponds to $B$ then $(p - 1)e = -e_0$ will correspond to $A_1$, so that $p^1e, 2p^1e, \ldots,(p - 1)p^1e$ will be the basic points for the circles $C_i \forall i = 0, 1, 2, \ldots$ to occupy the remaining parts.

We conclude this paper with the following open questions:

- Can we locate any non-rational p-adic integer in the area $A_1OB$?
- What will be the form of a non-rational p-adic integer which lies in the area $A_1OB$?