Chapter 4

Surface invariants

4.1 Introduction

Consider a $p$-form $\omega$ associated with a physical quantity which evolves under a flow field $u$. Then $\partial_t \omega + L_u \omega = 0$ means that the surface integral $\int_{C^p} \omega = \text{const.}$, where $C^p$ is a $p$-dimensional comoving surface. But in some cases $\partial_t \omega + L_u \omega$ need not be vanishing. So in this case we cannot say that $\int_{C^p} \omega$ is conserved for all $p$-dimensional comoving surfaces. Even then there may exist some $p$-dimensional comoving surfaces over which $\int_{C^p} \omega$ is a constant of the motion. Such invariant surfaces may be used for the qualitative study of the flow of continuous media in the absence of invariance of all comoving $p$-dimensional surfaces. In this chapter we are investigating what are the sufficient conditions for the existence of such invariant surfaces in continuous media using the language of differential forms and vector fields. Some illustrative examples are also given from hydrodynamics and magnetohydrodynamics.
4.2 Differential forms, vector fields and their invariance

In our classical vector calculus the flux preservation and line preservation are related to a vector field in the Euclidean space $\mathbb{R}^3$. But when we use the language of differential forms and vector fields, the flux preservation is identified with the integral invariance of differential forms and the line preservation is attributed to the vector fields [6, 24, 76]. This identification is significant for a general $n$-dimensional manifold. In introduction we have seen that in the usual Euclidean space $\mathbb{R}^3$ there is a one to one correspondence between vector fields and differential forms of degree one. Also a vector field $H$ can be identified as a two form $\omega^2$ by means of the interior product of $H$ with the volume form $\Omega = dx^1 \wedge dx^2 \wedge dx^3$, i.e., $\omega^2 = i_H \Omega$. Let $S$ be a surface such that $H \cdot dS$ vanishes at every point of the surface. Then if we consider the associated two form $\omega^2$; this means that on the surface $S$ the two form $\omega^2$ is annihilated or $S$ annuls $\omega^2$. In the Euclidean space $\mathbb{R}^3$, vector fields and differential forms of degree one or two may be used interchangeably. For a better analysis of topological properties of objects in the three dimensional space (or generally in an $n$-dimensional space) we should use the calculus of differential forms and vector fields. In the study of invariants in hydrodynamics this distinction gives rise to the classification of invariants as given in previous chapters and in [80].

We consider a general $p$-form $\omega^p$ and a flow field $\mathcal{P}$ in an $n$-dimensional manifold. Let $C$ be a $p$-dimensional surface moving with velocity $\mathcal{P}$, that is a $p$-dimensional comoving surface or simply a $p$-surface, then as given in chapter 1

$$\frac{d}{dt} \int_C \omega^p = \int_C \partial_t \omega^p + L_{\mathcal{P}} \omega^p$$ (4.1)

So, if $\partial_t \omega^p + L_{\mathcal{P}} \omega^p = 0$ then $\int_C \omega^p = \text{const.}$ for any $p$-surface $C$. 
4.3 Sequence of Lie derivatives

We consider a tensor field $T$ and a vector field $\mathcal{P}$ in an $n$-dimensional manifold. We define a sequence of tensor fields of the same type as that of $T$ recursively as follows:

\[ T^{(0)} = T, \]
\[ T^{(1)} = \partial_t T^{(0)} + L_\mathcal{P} T^{(0)} \text{ and} \]
\[ T^{(m+1)} = \partial_t T^{(m)} + L_\mathcal{P} T^{(m)}, \text{ for all } m \geq 1. \]

Here the operator $L_\mathcal{P}$ is the Lie derivative with respect to the vector field $\mathcal{P}$.

Let $\mathcal{T} = \{ T^{(0)}, T^{(1)}, T^{(2)}, \ldots, T^{(m)}, \ldots \}$, which is called the family of derived fields. The tensor field $T$ is said to be invariant under the flow if $T^{(1)} = 0$. The tensor fields which we are going to consider include differential forms and vector fields only.

Let $\omega$ be a $p$-form and consider the derived space

\[ \mathcal{W} = \{ \omega^{(0)}, \omega^{(1)}, \ldots \omega^{(m)}, \ldots \} \]

Here if $\omega^{(1)} = 0$, then we can say that the $p$-form $\omega$ is invariantly transported under the flow field $\mathcal{P}$. In terms of integral it means that the surface integral $\int_C \omega = \text{const.}$, where $C$ is a $p$-surface in the $n$-dimensional space. In general $\omega^{(1)}$ need not be vanishing. In some cases it may be of the form $\omega^{(1)} = d\alpha$ where $\alpha$ is a $(p - 1)$-form. Then from the Stoke’s theorem it follows that

\[ \frac{d}{dt} \int_{\partial C} \omega = \frac{d}{dt} \int_C \omega = \int_C \partial_t \omega + L_\mathcal{P} \omega = \int_C d\omega^{(1)} = 0 \]

where $C$ is a $(p + 1)$-surface. This means that $\omega$ is a relative integral invariant. For the sequence of the derived space the relation (4.1) becomes

\[ \frac{d}{dt} \int_C \omega^{(m)} = \int_C \omega^{(m+1)} \quad (4.2) \]

Now for the sequence of derived forms we will prove the following theorem by
Theorem 1. For a fixed integer $r \geq 1$ and for some $q \geq 0$, if $\omega^{(q+r)} = \sum_{i=0}^{r-1} f_i \omega^{(q+i)}$, where $f_0, f_1, \ldots, f_{r-1}$ are scalar functions, then $\omega^{(m)} = \sum_{i=0}^{r-1} g_i \omega^{(q+i)}$, for all $m \geq q+r$, where $g_0, g_1, \ldots, g_{r-1}$ are also scalar functions.

Proof: The result is given to be true for $m = q+r$. So assume it is true for $m = k$, where $k \geq q+r$; i.e., $\omega^{(k)} = \sum_{i=0}^{r-1} h_i \omega^{(q+i)} = h_0 \omega^{(q)} + h_1 \omega^{(q+1)} + \ldots + h_{r-1} \omega^{(q+r-1)}$, where $h_i$'s are scalar functions. Then

$$\omega^{(k+1)} = \partial_t \omega^{(k)} + L_{\mathcal{P}} \omega^{(k)}$$

$$= \partial_t \left( \sum_{i=0}^{r-1} h_i \omega^{(q+i)} \right) + L_{\mathcal{P}} \left( \sum_{i=0}^{r-1} h_i \omega^{(q+i)} \right), \text{ by induction hypothesis}$$

$$= \sum_{i=0}^{r-1} \left[ \partial_t (h_i \omega^{(q+i)}) + L_{\mathcal{P}} (h_i \omega^{(q+i)}) \right], \text{ since Lie derivative is linear}$$

$$= \sum_{i=0}^{r-1} \left[ (\partial_t h_i + L_{\mathcal{P}} h_i) \omega^{(q+i)} + h_i \left( \partial_t \omega^{(q+i)} + L_{\mathcal{P}} \omega^{(q+i)} \right) \right]$$

$$= \sum_{i=0}^{r-1} \left[ l_i \omega^{(q+i)} + h_i \omega^{(q+i+1)} \right], \text{ where } l_i = \partial_t h_i + L_{\mathcal{P}} h_i$$

is a scalar function for all $i$

$$= \sum_{i=0}^{r-1} l_i \omega^{(q+i)} + \sum_{i=0}^{r-2} h_i \omega^{(q+i+1)} + h_{r-1} \omega^{(q+r)}$$

$$= \sum_{i=0}^{r-1} l_i \omega^{(q+i)} + \sum_{i=0}^{r-2} h_i \omega^{(q+i+1)} + h_{r-1} \sum_{i=0}^{r-1} f_i \omega^{(q+i)}, \text{ from the basis step}$$

$$= \sum_{i=0}^{r-1} (l_i + t_i) \omega^{(q+i)} + \sum_{i=0}^{r-2} h_i \omega^{(q+i+1)}, \text{ where } t_i = h_{r-1} f_i$$

$$= \sum_{i=0}^{r-1} g_i \omega^{(q+i)}, \text{ where } g_0 = l_0 + t_0 \text{ and } g_i = l_i + t_i + h_{i-1}$$

for $1 \leq i \leq r-1$

So the result is true for $m = k + 1$ also. Hence by induction it follows that the result is true for all $m \geq q+r$. Here note that if all the $f_i$'s are constants, then $g_i$'s
are also constant functions.

For $r = 1$, this result says that, if $\omega^{(q+1)} = f_0 \omega^{(q)}$ for some $q$ then $\omega^{(m)} = f_m \omega^{(q)}$, for all $m \geq q + 1$, where $f_m$ is a scalar function depending on $m$. Also assume $q = 0$, then this becomes: if $\omega^{(1)} = \alpha \omega$, then $\omega^{(m)} = \alpha_m \omega$ for all $m \geq 1$. When $r = 2$ and $q = 1$, this result is as follows: If $\omega^{(3)} = \alpha \omega^{(1)} + \beta \omega^{(2)}$, then $\omega^{(m)} = \alpha_m \omega^{(1)} + \beta_m \omega^{(2)}$, for all $m \geq 3$.

In continuous media we may assume that the surface integral $\int_C \omega$ is an analytic function of time, where $\omega$ is any $p$-form and $C$ is any $p$-surface. Let

$$\omega^{(q+r)} = \sum_{i=0}^{r-1} g_i \omega^{(q+i)} \quad (4.3)$$

for a $p$-form $\omega$. Also let over some $p$-surface $C$ the forms $\omega^{(q)}, \omega^{(q+1)}, \ldots, \omega^{(q+r-1)}$ are annihilated initially. Then $\phi(t) = \int_C \omega^{(q-1)}$ will be a constant of flow. This follows from theorem 1, and the equation $(4.2)$, using a Taylor's series expansion of $\phi(t)$. Also it follows that $\int_C \omega^{(q)} = 0$ throughout the flow. In general it follows that $\int_C \omega^{(m)} = 0$ throughout the flow for all $m \geq q$.

Again, let the condition $(4.3)$ is satisfied for a $p$-form $\omega$ with all $g_i$'s constant. Also, if $\int_C \omega^{(q+i)} (0 \leq i \leq r - 1)$ are vanishing initially for a $p$-surface $C$, then $\int_C \omega^{(q-1)}$ will be a constant throughout the flow. Here note that the condition $\omega^{(q+i)} (0 \leq i \leq r - 1)$ are annihilated initially need not be satisfied.

Let $\alpha$ is a $(p-1)$-form. Consider the derived fields $\omega^{(m)}$ of the $p$-form $\omega = d\alpha$ for all $m$. Then by induction it follows that $\omega^{(m)} = d\alpha^{(m)}$, for all $m \geq 1$. Then the following theorem is obvious:

**Theorem 2.** Consider the derived fields of the $p$-form $\omega = d\alpha$, where $\alpha$ is a $(p-1)$-form. If $\alpha^{(q+r)} = \sum_{i=0}^{r-1} f_i \alpha^{(q+i)}$, then $\omega^{(q+r)} = \sum_{i=0}^{r-1} f_i \omega^{(q+i)}$, provided $f_i$'s are constants.

Let $\alpha$ and $\beta$ are differential forms of degree $k$ and $p-k$ respectively, for some $p$. 
Then \( \omega = \alpha \wedge \beta \) is a \( p \)-form. Now we have the following theorem for the derived space \( \mathcal{W} = \{ \omega, \omega^{(1)}, \ldots, \omega^{(m)} \cdots \} \) of \( \omega \) by induction.

**Theorem 3.** If \( \omega = \alpha \wedge \beta \), then \( \omega^{(m)} = \sum_{r=0}^{m} \binom{m}{r} \alpha^{(r)} \wedge \beta^{(m-r)} \)

**Proof:** When \( m = 1 \),

\[
\omega^{(1)} = \alpha \wedge \beta^{(1)} + \alpha^{(1)} \wedge \beta
\]

So the result is true when \( m = 1 \). Let the result is true for \( m = k \). That is \( \omega^{(k)} = \sum_{r=0}^{k} \binom{k}{r} \alpha^{(r)} \wedge \beta^{(k-r)} \). Then

\[
\omega^{(k+1)} = \partial_t \omega^{(k)} + L_{\partial} \omega^{(k)}
\]

\[
= \sum_{r=0}^{k} \binom{k}{r} \{ \alpha^{(r)} \wedge \partial_t \beta^{(k-r)} + \partial_t \alpha^{(r)} \wedge \beta^{(k-r)} + \\
\alpha^{(r)} \wedge L_{\partial} \beta^{(k-r)} + L_{\partial} \alpha^{(r)} \wedge \beta^{(k-r)} \}
\]

\[
= \sum_{r=0}^{k} \binom{k}{r} \{ \alpha^{(r)} \wedge \beta^{(k-r+1)} + \alpha^{(r+1)} \wedge \beta^{(k-r)} \}
\]

\[
= \sum_{r=0}^{k} \binom{k}{r} \alpha^{(r)} \wedge \beta^{(k-r+1)} + \sum_{r=1}^{k+1} \binom{k}{r-1} \alpha^{(r)} \wedge \beta^{(k-r+1)}
\]

\[
= \alpha \wedge \beta^{(k+1)} + \sum_{r=1}^{k} \left\{ \binom{k}{r} \alpha^{(r)} \wedge \beta^{(k-r+1)} + \alpha^{(k+1)} \wedge \beta \right\}
\]

\[
= \alpha \wedge \beta^{(k+1)} + \sum_{r=1}^{k} \binom{k+1}{r} \alpha^{(r)} \wedge \beta^{(k+1-r)} + \alpha^{(k+1)} \wedge \beta
\]

\[
= \sum_{r=0}^{k+1} \binom{k+1}{r} \alpha^{(r)} \wedge \beta^{(k+1-r)}
\]

So the result is true for \( m = k + 1 \) also. Hence the theorem.

The following different cases of theorem 3 attract particular attention.

**Case I:** Let the derived fields of \( \alpha \) and \( \beta \) satisfy the conditions \( \alpha^{(q+1)} = \sum_{i=0}^{r-1} f_i \alpha^{(q+i)} \) and \( \beta^{(1)} = 0 \). then for all \( m \), \( \omega^{(m)} = \alpha^{(m)} \wedge \beta \) and \( \omega^{(q+r)} = \sum_{i=0}^{r-1} f_i \omega^{(q+i)} \).

**Case II:** Let the form \( \alpha \) satisfy the condition \( \alpha^{(r)} = \sum_{i=0}^{r-1} f_i \alpha^{(i)} \). Then, from theorem 3, it is clear that the expansion of \( \omega^{(m)} \) (for all \( m \)) contains \( \alpha^{(i)} \) \((0 \leq i \leq r - 1)\) as
a factor of each of the wedge product. So if \( \alpha^{(i)} \)'s are annihilated initially on any \( p \)-surface, then this \( p \)-surface will be an integral invariant surface for \( \omega \) and the integral is vanishing over all such surfaces. Here note that \( \beta \) can be any \( (p - k) \)-form.

**Case III:** Let \( \alpha \) and \( \beta \) satisfy the conditions \( \alpha^{(1)} = f\alpha \) and \( \beta^{(1)} = g\beta \). Then \( \omega^{(1)} = (f + g)\alpha \wedge \beta = h\omega \). If the \( k \)-form \( \alpha \) is vanishing on any \( p \)-surface initially or the \( (p - k) \)-form \( \beta \) is vanishing on any \( p \)-surface initially, then the \( p \)-surface will be an integral invariant surface for \( \omega \). Also if \( \omega \) is annihilated on any \( p \)-surface initially then such surfaces will be invariant surfaces for \( \omega \).

For an arbitrary vector field \( J \) in an \( n \)-dimensional space, consider the derived space \( \mathcal{J} = \{ J(0), J(1), \ldots, J(m), \ldots \} \). Then the theorem 1 is true for this family of vector fields also, which can be stated as follows:

**Theorem 4.** Let \( J \) be a vector field in an \( n \)-dimensional space. For \( r \geq 1 \) and \( q \geq 0 \), if \( J^{(q+r)} = f_0 J^{(q)} + f_1 J^{(q+1)} + \ldots + f_{r-1} J^{(q+r-1)} \) then for all \( m \geq q + r \), \( J^{(m)} = g_0 J^{(q)} + g_1 J^{(q+1)} \ldots + g_{r-1} J^{(q+r-1)} \), where \( f_i \)'s and \( g_i \)'s are scalar functions.

Let \( \omega \) be a \( (p + 1) \)-form. Then consider the \( p \)-form \( \alpha = i_J \omega \), where \( J \) is a vector field. Then as theorem 3 we can prove the following theorem.

**Theorem 5.** If \( \alpha = i_J \omega \), where \( \omega \) is a \( (p + 1) \)-form and \( J \) is a vector field in an \( n \)-dimensional space, then \( \alpha^{(m)} = \sum_{r=0}^{m} \binom{m}{r} i_{J^{(r)}} \omega^{(m-r)} \)

The following different cases of theorem 5 are interesting:

**Case I:** If \( \omega^{(1)} = 0 \), then \( \alpha^{(m)} = i_{J(m)} \omega \). Also let \( J^{(q+r)} = \sum_{i=0}^{r-1} f_i J^{(q+i)} \). Then we get \( \alpha^{(q+r)} = \sum_{i=0}^{r-1} f_i \alpha^{(q+i)} \).

**Case II:** When \( J^{(1)} = 0 \) and \( \omega^{(q+r)} = \sum_{i=0}^{r-1} f_i \omega^{(q+i)} \), we get \( \alpha^{(m)} = i_{J^{(m)}} \omega^{(m)} \) and \( \alpha^{(q+r)} = \sum_{i=0}^{r-1} f_i \alpha^{(q+i)} \).

**Case III:** Here if \( \omega^{(1)} = f\omega \) and \( J^{(1)} = gJ \), then we have \( \alpha^{(1)} = h\alpha \), where \( h = f + g \).

We will use the above results in the coming sections to characterize the different types of surface invariants in three dimensional continuous media. We shall discuss line preservation and surface preservation of vector fields also.
4.4 Surface invariants related to one forms

Let $\omega$ be a one form associated with a geometrical object in a three dimensional flow with velocity vector $u$. Consider the derived space of this one form. We will discuss the following three cases.

**Case I:** Here we consider the case where $\omega^{(1)} = 0$. Then it follows that the integral $\int_C \omega$ over any 1-surface $C$ (that is, over a comoving curve $C$) is a constant of the flow.

**Case II:** If $\omega^{(1)}$ is not vanishing and $\omega^{(1)} = f\omega$, where $f$ is a scalar function, then from theorem 1 it follows that $\omega^{(m)} = f_m\omega$ for all $m \geq 1$. So if there exists a curve over which $\omega$ is vanishing initially, then $\int_C \omega = 0$ throughout the flow. So these curves over which $\omega$ vanishes initially will constitute the invariant curves of the flow. Also it follows that the line integrals $\int_C \omega^{(m)}$ are vanishing throughout the flow for all $m \geq 1$.

Now consider the case where $\omega^{(2)} = f\omega^{(1)}$ for some scalar function $f$, then any curve $C$ over which $\omega^{(1)}$ vanishes initially will be an integral invariant 1-surface for the one form $\omega$. This line integral of $\omega$ over $C$ need not be zero. Let $S$ be a 2-dimensional surface over which $\omega^{(1)}$ vanishes initially. Then for any curve $C$ on this surface, $\int_C \omega$ is a constant. But by Frobenius integrability condition a family of integral surfaces for the one form $\omega^{(1)}$ exists only if $\omega^{(1)} \wedge d\omega^{(1)} = 0$. So when this condition is satisfied initially, then for all curves $C$ on this family of surfaces $\int_C \omega$ is a constant of the flow, provided $\omega^{(2)} = f\omega^{(1)}$. Even though the Frobenius integrability condition is not satisfied there may exist many curves over which the integral is a constant of the flow.

For example, let us consider the vector field associated with the one form $\omega^{(1)}$ in the Euclidean space $\mathbb{R}^3$. Then consider the vector tubes of this field. If this tube has the topology of a cylinder initially, then for any curve of intersection of this tube with a plane perpendicular to the axis of the tube will be an integral invariant curve for the one form $\omega$. Let $A$ be the vector field associated with $\omega^{(1)}$ and $B$ be an arbitrary vector field not collinear with $A$. Then initially on the vector lines of the vector field $A \times B$, $\omega^{(1)}$ is annihilated. So all these vector lines form a family of invariant 1-surface for $\omega$. 
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In general, if \( \omega^{(q+1)} = f \omega^{(q)} \), then the curves \( C \) on which \( \omega^{(q)} \) vanishes initially constitute the invariant curves for the one form \( \omega^{(q-1)} \). If there exists a family of integral surfaces for the one form \( \omega^{(q)} \) initially, then for all curves on this surface \( \int_C \omega^{(q-1)} \) is an invariant. The condition for the existence of such a family of integral surface initially is that \( \omega^{(q)} \wedge d\omega^{(q)} \) vanishes initially. Also note that for all such invariant curves \( C \) of \( \omega^{(q-1)} \), the line integral \( \int_C \omega^{(m)} \) is an integral invariant which vanishes, for all \( m \geq q \).

**Case III:** Here we assume that \( \omega^{(1)} \) and \( \omega^{(2)} \) are independent forms, but \( \omega^{(3)} = f \omega^{(1)} + g \omega^{(2)} \) where \( f \) and \( g \) are scalar functions. Then, if there exists a curve \( C \) over which both \( \omega^{(1)} \) and \( \omega^{(2)} \) are annihilated, then \( \int_C \omega \) is a constant of the flow. If \( \omega^{(2)} = f \omega + g \omega^{(1)} \), then \( \int_C \omega = 0 \) over the curves \( C \) on which \( \omega \) and \( \omega^{(1)} \) are annihilated. In general if \( \omega^{(q+2)} = f \omega^{(q)} + g \omega^{(q+1)} \), and if there exists a curve on which \( \omega^{(q)} \) and \( \omega^{(q+1)} \) are annihilated, then \( \int_C \omega^{(q-1)} \) is a constant. Also consider \( \alpha = \omega^{(q)} \wedge \omega^{(q+1)} \) which is a two form. Then

\[
(\omega^{(q)} \wedge \omega^{(q+1)})^{(1)} = \partial_t (\omega^{(q)} \wedge \omega^{(q+1)}) + L_u (\omega^{(q)} \wedge \omega^{(q+1)})
\]

\[
= \omega^{(q+1)} \wedge \omega^{(q+1)} + \omega^{(q)} \wedge \omega^{(q+2)}
\]

\[
= \omega^{(q)} \wedge \omega^{(q+2)}
\]

\[
= \omega^{(q)} \wedge (f \omega^{(q)} + g \omega^{(q+1)})
\]

\[
= g \omega^{(q)} \wedge \omega^{(q+1)}
\]

So \( \alpha^{(1)} = g \alpha \) for the two form \( \alpha = \omega^{(q)} \wedge \omega^{(q+1)} \) when \( \omega^{(q+2)} = f \omega^{(q)} + g \omega^{(q+1)} \). Hence if initially \( \omega^{(q)} \) (or \( \omega^{(q+1)} \)) is a surface forming one form, i.e., \( \omega^{(q)} \wedge d\omega^{(q)} = 0 \) (or \( \omega^{(q+1)} \wedge d\omega^{(q+1)} = 0 \)), then over these family of surfaces the two form \( \alpha \) is annihilated and hence these surfaces will be invariant surfaces of the two form \( \alpha \). That is, for all such 2-surfaces \( C \), \( \int_C \alpha \) is a constant, which is zero. For example when \( q = 0 \), \( \omega^{(2)} = f \omega + g \omega^{(1)} \), then \( \alpha = \omega \wedge \omega^{(1)} \) is having such invariant surfaces provided \( \omega \wedge d\omega = 0 \) or \( \omega^{(1)} \wedge d\omega^{(1)} = 0 \), or both.
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Now we will discuss the three special cases of theorem 3 as given in section 4.3, when \( \alpha \) and \( \beta \) are 1-forms. In the first case let \( \alpha^{(2)} = f\alpha^{(1)} \) and \( \beta^{(1)} = 0 \). Then \( \omega^{(2)} = f\omega^{(1)} \), where \( \omega = \alpha \wedge \beta \). If initially \( \alpha^{(1)} \wedge d\alpha^{(1)} = 0 \), then on the integral surfaces of \( \alpha^{(1)} \), \( \omega^{(1)} \) is annihilated initially as well as \( \alpha^{(1)} \). So \( \int_C \omega \) will be a constant of motion for any 2-surface \( C \) on this family of integral surfaces.

In the second case, let \( \alpha^{(1)} = f\alpha \) and \( \beta \) is any 1-form. Then \( \omega = \alpha \wedge \beta \) is such that each term in \( \omega^{(m)} \) contains \( \alpha \) as a factor in the wedge product. So if there exist integral surfaces for \( \alpha \) initially (that is, \( \alpha \wedge d\alpha = 0 \) initially), then any 2-surface on this family of integral surfaces will be an integral invariant surface for \( \omega \). This is also true when \( \alpha^{(1)} = 0 \).

In the last case, let \( \alpha \) and \( \beta \) be 1-forms satisfying \( \alpha^{(1)} = f\alpha \) and \( \beta^{(1)} = g\beta \). Then \( \omega^{(1)} = h\omega \). So if \( \alpha \wedge d\alpha = 0 \) (or, \( \beta \wedge d\beta = 0 \)), then any 2-surface lying on the family of integral surfaces of \( \alpha \) (or, \( \beta \)) will be an integral invariant surface for \( \omega \).

4.5 Line and surface preservation of vector fields

Let \( \mathbf{J} \) be a vector field in the flow of a continuous media. Then consider the sequence of derived vector fields of same type. We will distinguish the following different cases:

**Case I:** If \( \mathbf{J}^{(1)} = 0 \), then the \( \mathbf{J} \)-lines are said to be frozen into the medium. In other words, if this condition is satisfied, then the vector lines of \( \mathbf{J} \) are said to be material lines. The vorticity field in the ideal incompressible hydrodynamic flow and magnetic field in the ideal incompressible MHD flows are well known examples of frozen in fields.

**Case II:** Let \( \mathbf{J}^{(1)} = \lambda \mathbf{J} \) for some scalar function \( \lambda \). In this case also the vector lines are said to be preserved, that is, the vector lines of \( \mathbf{J} \) are material lines. Also from theorem 4 it follows that all the derived vector fields \( \mathbf{J}^{(m)} \) satisfy the equation \( \mathbf{J}^{(m)} = \lambda_m \mathbf{J} \). That is, the vector lines of each of these derived fields \( \mathbf{J}^{(m)} \) coincide with that of \( \mathbf{J} \). So the vector lines of the derived fields are also material lines.
Case III: Let

\[ J^{(2)} = fJ + gJ^{(1)} \]  

(4.4)

where \( f \) and \( g \) are some scalar functions. Then for any \( m \geq 2 \), \( J^{(m)} = f_mJ + g_mJ^{(1)} \), from theorem 1. Also let there exist a family of integral surface for \( J \) and \( J^{(1)} \). The necessary and sufficient condition for the existence of such a family of integral surfaces for these vector fields is that their Lie bracket is a linear combination of themselves (Frobenius theorem). That is

\[ [J, J^{(1)}] = f'J + g'J^{(1)} \]  

(4.5)

where \( f' \) and \( g' \) are scalar functions. So when equations (4.4) and (4.5) are satisfied simultaneously, then on the integral surfaces of \( J \) and \( J^{(1)} \) lie all \( J^{(m)} \), for \( m \geq 0 \).

Now we are going to show that if the condition (4.5) is satisfied initially, that is there exist a family of integral surfaces for \( J \) and \( J^{(1)} \) initially, then these surfaces will remain as integral surfaces for these vector fields throughout the flow. Consider the one form \( \omega = i_{J^{(1)}}i_J \Omega \), where \( \Omega = dx^1 \wedge dx^2 \wedge dx^3 \) is the volume element form. Clearly the vector field corresponding to this one form is \( J \times J^{(1)} \). Then the integral surfaces of \( J \) and \( J^{(1)} \), if they exist, annuls the one form \( \omega \) and they are the integral surfaces of \( \omega \) also.

Remark: Here integral surfaces for \( \omega \) exists if and only if \( \omega \wedge d\omega = 0 \). But \( \omega \wedge d\omega = 0 \iff (J \times J^{(1)}) \cdot \nabla \times (J \times J^{(1)}) = 0 \iff (J \times J^{(1)}) \cdot \{[J, J^{(1)}] - (\nabla \cdot J)J^{(1)} + (\nabla \cdot J^{(1)})J\} = 0 \iff [J, J^{(1)}] = f'J + g'J^{(1)} \). So the integral surfaces for \( \omega \) exist if and only if there exist integral surfaces for \( J \) and \( J^{(1)} \).

For the above one form \( \omega \),

\[ \omega^{(1)} = \partial_t \omega + L_u \omega \]

\[ = \partial_t(i_{J^{(1)}}i_J \Omega) + L_u(i_{J^{(1)}}i_J \Omega) \]
\[ \{i_{\partial_i J(1)} + i_{J(1)} \partial_t \} i_J \Omega + \{i_{[u,J(1)]} + i_{J(1)} L_u \} i_J \Omega \]

(since \( \partial_i i_X - i_X \partial_i = i_{\partial X} \) and using properties of Lie derivative)

\[ = \{i_{\partial_i J(1)} + [u,J(1)] + i_{J(1)} \partial_t + i_{J(1)} L_u \} \Omega \]

\[ = i_{J(t)} i_J \Omega + i_{J(1)} \{ \partial_t i_J + L_u i_J \} \Omega \]

\[ = i_{J(t)} i_J \Omega + i_{J(1)} \left[ i_{\partial_t J + [u,J]} + i_J (\partial_t + L_u) \right] \Omega \]

\[ = i_{J(t)} i_J \Omega + i_{J(1)} \left[ i_{J(1)} + i_J (\partial_t + L_u) \right] \Omega \]

\[ = i_{J(t)} i_J \Omega + i_{J(1)} i_J (\partial_t + L_u) \Omega, \text{ since } i_{J(1)} i_J \Omega = 0 \]

\[ = i_{J(t)} i_J \Omega + i_{J(1)} i_J (\nabla \cdot u) \Omega, \text{ since } \partial_t \Omega = 0 \text{ and } L_u \Omega = (\nabla \cdot u) \Omega \]

\[ = i_{J(t)} i_J \Omega + (\nabla \cdot u) \Omega, \text{ since } i_X f \Omega = f i_X \Omega \text{ and } i_{J(t)} i_J \Omega = \omega. \]  

\[ (4.6) \]

So

\[ \omega^{(1)} = h \omega \Leftrightarrow i_{J(t)} i_J \Omega = f \omega, \text{ for the scalar function } f = h - \nabla \cdot u \]

\[ \Leftrightarrow i_{J(t)} i_J \Omega - f i_{J(1)} i_J \Omega = 0 \]

\[ \Leftrightarrow i_{J(t)} - f i_{J(1)} i_J \Omega = 0, \text{ since } i_J = f i_X \]

\[ \Leftrightarrow J^{(2)} - f J^{(1)} = gJ, \text{ for some scalar function } g \]

(since \( i_X i_Y \Omega = 0 \Leftrightarrow X = h' Y \) for some scalar function \( h' \))

\[ \Leftrightarrow J^{(2)} = fJ + gJ^{(1)} \text{ where } f \text{ and } g \text{ are scalar functions.} \]

So \( \omega^{(1)} = h \omega \Leftrightarrow J^{(2)} = fJ + gJ^{(1)} \) for some scalar functions \( h, f \) and \( g \). Thus if any of these equivalent conditions is satisfied and if initially there exists a family of integral surfaces for \( \omega \), then on this family of integral surfaces, \( \omega \) is annihilated initially. So from the previous section it follows that the integration of \( \omega \) over an arbitrary material curve on this family of surfaces vanishes initially and hence identically. So this family of integral surfaces for \( \omega \) will remain as a family of integral surfaces for \( \omega \). Then it
follows that this family of integral surfaces which is a family of orthogonal surfaces for $J \times J^{(1)}$ will remain as a family of orthogonal surfaces for $J \times J^{(1)}$. Hence the vector fields $J$ and $J^{(1)}$ which span this family of material surfaces initially will remain spanning this family of material surfaces (also see the above remark). A vector field is said to preserve a material surface if initially this material surface is generated by the field lines of this vector field and this material surface remains generated by the field lines of the same vector field throughout the flow. That is, if the field lines initially lie on this comoving surface, then they will remain lying on this surface. Hence if there exist an integral surface for $J$ and $J^{(1)}$ initially, then (4.4) is the condition for the surface preservation of the vector field $J$. Here a material line initially which is a $J$-line lying on this invariant surface need not be a $J$-line as the field evolves. When the field $J^{(1)}$ vanishes identically, then as given in case I all the vector lines are material lines. Then clearly all the vector surfaces of the vector field $J$ are preserved, that is they are material surfaces. If $J^{(2)} = 0$, then from (4.6) $\omega^{(1)} = (\nabla \cdot u)\omega$. Then also the integral surfaces of $J$ and $J^{(1)}$ are preserved by the flow, if they exist initially.

4.6 Invariant surfaces related to two forms

Let $\omega$ be a two form associated with a physical quantity in a continuous media. Consider the derived space of two forms as given in section 3. We will consider three cases given below.

Case I: Here we will consider the case where $\omega^{(1)} = 0$ for a flow. Then the integral of $\omega$ over any 2-surface will be a constant of motion. In terms of associated vector field $H$ it says that $\int_C H \cdot dS$ is a constant for all 2-surfaces.

Case II: In some cases it may happen that $\omega^{(2)} = \lambda \omega^{(1)}$, then for all $m \geq 2$, $\omega^{(m)} = \lambda_m \omega^{(1)}$ for some scalar functions $\lambda_m$. Then consider a surface over which $\omega^{(1)}$ vanishes initially. Any 2-surface in this surface will be an integral invariant surface for $\omega$. That is, $\int_C \omega$ is a constant. Let $H$ be the associated vector field of the two form $\omega^{(1)}$. Here
initially consider the vector sheets of $H$. Then clearly $w^{(1)}$ is annihilated on these vector sheets of $H$. So all 2-surfaces lying on these vector sheets of $H$ constitute a class of surfaces with constant surface integral of the two form $w$.

Let $w$ be a two form for which $w^{(2)} = f w^{(1)}$ and $J$ is an invariant vector field (that is $J^{(1)} = 0$). If $\alpha = i_J w$, then $\alpha^{(2)} = h \alpha^{(1)}$, from case 2 of theorem 5. Let $H$ be the associated vector field of $w^{(1)}$. Then $H \times J$ will be the associated vector field of the one form $\alpha^{(1)}$ (since, $\alpha^{(1)} = i_J i_H \Omega$). Then on the vector lines of $H$ and $J$, $\alpha^{(1)}$ is annihilated and these vector lines are integral invariant families of curves for $\alpha$. If the Frobenius integrability condition is satisfied initially for the vector fields $H$ and $J$, then any curve on these integral surfaces form an integral invariant 1-surface for $\alpha$. A similar discussion is possible when $w^{(1)} = 0$ and $J^{(2)} = f J^{(1)}$, using case 1 of theorem 5. The third case of theorem 5 can also be discussed similarly.

If $w^{(1)} = \lambda \omega$, then the constant flux surfaces will include vector sheets of the field $K$, where $\omega = i_K \Omega$. Here the value of the constant flux across the surface is zero. Here we can also see that the associated vector field $K$ is line preserving. We have

$$w^{(1)} = \partial_t (i_K \Omega) + L_u (i_K \Omega)$$

$$= (i_{\partial_t K} + i_K \partial_t) \Omega + (i_{[u,K]} + i_K) L_u \Omega, \text{ since } \partial_t i_Y - i_Y \partial_t = i_{\partial_Y}$$

and $L_u i_X - i_X L_u = i_{[u,X]}$

$$= i_{\partial_t K + [u,K]} \Omega + i_K L_u \Omega, \text{ since } \partial_t \Omega = 0$$

$$= i_{K^{(1)}} \Omega + (\nabla \cdot u) i_K \Omega$$

$$= i_{K^{(1)} + (\nabla \cdot u) K} \Omega. \quad (4.7)$$

So $w^{(1)} = \lambda \omega$ implies that $i_{K^{(1)} + (\nabla \cdot u) K} \Omega = \lambda i_K \Omega = i_{\lambda K} \Omega$.

Hence,

$$i_{K^{(1)} + (\nabla \cdot u) K - \lambda K} \Omega = 0 \Rightarrow K^{(1)} + (\nabla \cdot u) K - \lambda K = 0 \Rightarrow K^{(1)} = \beta K$$
where $\beta = \lambda - \nabla \cdot u$. Hence it follows that the field lines of $K$ are preserved.

Similarly, if $\omega^{(2)} = \lambda \omega^{(1)}$, then in addition to the flux preservation of the two form $\omega$ over all the 2-surfaces over which $\omega^{(1)}$ is annihilated, the field lines of the vector field $H$ are preserved, where $\omega^{(1)} = i_H \Omega$.

Case III: Here we consider the case where

$$\omega^{(3)} = f \omega^{(1)} + g \omega^{(2)}$$

(4.8)

for some scalar functions $f$ and $g$. If there exists a surface $C$ which annihilates both $\omega^{(1)}$ and $\omega^{(2)}$ initially, then $\int_C \omega$ is a constant. Let $J$ and $K$ be two vector fields satisfying $\omega^{(1)} = i_J \Omega$ and $\omega^{(2)} = i_K \Omega$. If $[J, K]$ is a linear combination of $J$ and $K$ initially, that is

$$[J, K] = \lambda J + \beta K,$$

(4.9)

then on the integral surfaces of $J$ and $K$, $\omega^{(1)}$ and $\omega^{(2)}$ are vanishing initially, so that the surface integral of $\omega$ over any of the 2-surface lying on this family of integral surfaces are constant of the motion.

Now we are going to show that the condition (4.8) is equivalent to the condition that the vector field $J^{(2)}$ is a linear combination of the vector fields $J$ and $J^{(1)}$.

Clearly

$$K = J^{(1)} + (\nabla \cdot u)J$$

(4.10)

since $\omega^{(2)} = \partial_t \omega^{(1)} + \mathcal{L}_u \omega^{(1)}$ and proceeding as in (4.7).

Then

$$\omega^{(3)} = f \omega^{(1)} + g \omega^{(2)} = fi_J \Omega + gi_K \Omega = i_{fJ} \Omega + i_{gK} \Omega = i_{fJ + gK} \Omega$$

(4.11)

So vector field corresponding to $\omega^{(3)}$ is $fJ + gK$. But, from the definition of $\omega^{(3)}$ and proceeding as in (4.7), we have

$$\omega^{(3)} = i_{J^{(1)} + (\nabla \cdot u)K} \Omega.$$
So from (4.11) and (4.12)

\[ K^{(1)} + (\nabla \cdot u)K = fJ + gK. \]  

(4.13)

Hence

\[ K^{(1)} = fJ + hK \]  

(4.14)

where \( h = g - \nabla \cdot u \).

Also from equation (4.10)

\[ K^{(1)} = (\partial_t + Lu)[J^{(1)} + (\nabla \cdot u)J] = J^{(2)} + (\nabla \cdot u)J^{(1)} + kJ, \]  

(4.15)

for some scalar function \( k \).

Also substituting the value of \( K \) from (4.10) in (4.14) we have

\[ K^{(1)} = tJ + hJ^{(1)} \]  

(4.16)

where \( t = f + \nabla \cdot u \). Now from (4.15) and (4.16) we get

\[ J^{(2)} = f_1J + f_2J^{(1)} \]  

(4.17)

for some scalar functions \( f_1 \) and \( f_2 \). Hence \( J^{(2)} \) is a linear combination of the vector fields \( J \) and \( J^{(1)} \).

If initially

\[ [J, J^{(1)}] = g_1J + g_2J^{(1)} \]  

(4.18)

for some scalar functions \( g_1 \) and \( g_2 \), then the integral surfaces for the fields \( J \) and \( J^{(1)} \) exist initially. Then from (4.17) it follows that these integral surfaces of \( J \)-lines (and \( J^{(1)} \)-lines) are preserved during the flow as discussed in previous section. Here note that
(4.9) and (4.18) are also equivalent (since, $[J,K] = \lambda J + \beta K \leftrightarrow [J,J^{(1)} + (\nabla \cdot u)J] = \\
\lambda J + \beta K \leftrightarrow [J,J^{(1)} + J, (\nabla \cdot u)J] = \lambda J + \beta K \quad \{\text{since} \quad [J,\gamma J] = (J \cdot \nabla \gamma)J, \quad \text{where} \quad \gamma = \nabla \cdot u \quad \text{and} \quad \lambda' = \lambda - J \cdot \nabla \gamma\} = g_1J + g_2J^{(1)} \quad \text{where} \quad g_1 = \lambda' + \beta(\nabla \cdot u) \quad \text{and} \quad g_2 = \beta, \quad \text{from} \quad (4.10))$

### 4.7 Some illustrative examples

In this section we will discuss some examples in the Euclidean space $\mathbb{R}^3$ which illustrate some of the developments given in the previous sections. For convenience differential forms are represented in vector notations.

**Example I**

Consider the equation of motion of a homogeneous incompressible fluid in a potential field of external forces, taking into account viscous friction in Rayleigh's form [26, 44, 45].

\[ \partial_t u + (\nabla \times u) \times u = -\nabla f - ku \quad (4.19) \]

where $k$ is the coefficient of viscous friction constant. This equation can be written as

\[ \partial_t \Theta + L_u \Theta = d(u^2 - f) - k\Theta, \quad \text{where} \quad \Theta = u_1 dx^1 + u_2 dx^2 + u_3 dx^3 \quad (4.20) \]

This equation can be again written as

\[ \partial_t \omega + L_u \omega = -k\omega \quad (4.21) \]

under some gauge transformation $\omega = \Theta + d\phi$, where the potential $\phi$ is to be so chosen that it satisfies $u^2 - f + d\phi/dt + k\phi = 0$. Here the one form $\omega$ satisfies $\omega^{(1)} = -k\omega$. So from section 4.4, for any comoving curve $C$ for which $\omega$ vanishes $\int_C \omega$ is a constant. If $\omega \wedge d\omega = 0$ initially, then there exists a family of integral surfaces for the one form
\( \omega \) and all curves on this family of surfaces will be integral invariant curves. Here the one form \( \omega \) corresponds to the velocity vector under some gauge potential. Similar potentials have been used in [31, 80].

Let \( \mathbf{A} \) be the vector field corresponding to the one form \( \omega \). and let \( \mathbf{B} \) be an arbitrary vector field not parallel to the field \( \mathbf{A} \). Then on the vector lines of the field \( \mathbf{A} \times \mathbf{B} \) the one form \( \omega \) is annihilated, so these vector lines forms family of integral invariant curves for \( \omega \).

Let there exist a family of integral surface for the one form \( \omega \) initially. Also let \( \beta \) be a one form associated with some physical quantity. Then, from case 2 of theorem 3, the two form \( \omega \wedge \beta \) will have this family of integral surfaces for \( \omega \) as integral invariant surfaces.

**Example 2:**

Here we discuss the integral invariant curves of a one form corresponding to the magnetic potential of a magnetic field using particle drift velocity. Let a uniform electric field \( \mathbf{E} \) drive a current through an infinitely long straight wire. The magnetic field \( \mathbf{B} \) consists of a uniform external field parallel to the wire, in addition to the field produced by the current in the wire [64]. Then the fields out side the wire may be written in cartesian coordinates as follows:

\[
\mathbf{B} = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 1 \right) \quad \text{and} \quad \mathbf{E} = (0, 0, 1).
\]

Then the particle drift velocity is given by

\[
\mathbf{v} = \left( \frac{-x}{1 + x^2 + y^2}, \frac{-y^2}{1 + x^2 + y^2}, 0 \right).
\]
Magnetic potential is taken to be

\[ \mathbf{A} = \left( -\frac{y}{2}, \frac{x}{2}, -\log \sqrt{x^2 + y^2} \right). \]

Let \( \omega_\mathbf{A} \) be the one form associated with the magnetic potential \( \mathbf{A} \). In vector notation the first three derived fields of this one form are given below.

\[ \omega^{(1)}_\mathbf{A} \leftrightarrow \frac{1}{1 + x^2 + y^2} (y, -x, 1), \]
\[ \omega^{(2)}_\mathbf{A} \leftrightarrow \frac{2}{(1 + x^2 + y^2)^3} (-y, x, x^2 + y^2) \quad \text{and} \]
\[ \omega^{(3)}_\mathbf{A} \leftrightarrow \frac{4(-1 + 2x^2 + 2y^2)}{(1 + x^2 + y^2)^5} (-y, x, x^2 + y^2) \]

Clearly \( \omega^{(3)}_\mathbf{A} = f \omega^{(2)}_\mathbf{A} \), where \( f = 2(-1 + 2x^2 + 2y^2)/(1 + x^2 + y^2)^2 \). So for all \( m \geq 3 \), \( \omega^{(m)}_\mathbf{A} = f_m \omega^{(2)}_\mathbf{A} \) for some scalar function \( f_m \). If initially \( \omega^{(1)}_\mathbf{A} \) and \( \omega^{(2)}_\mathbf{A} \) are annihilated on some curve \( C \), then \( \int_C \omega_\mathbf{A} \) is an invariant. This follows from section 4.4. Clearly vector lines of the vector field \( \mathbf{A}_1 \times \mathbf{A}_2 \) are such curves, where \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) are the associated vector fields of the one forms \( \omega^{(1)}_\mathbf{A} \) and \( \omega^{(2)}_\mathbf{A} \) respectively. Also we may consider the two form \( \alpha = \omega^{(1)}_\mathbf{A} \wedge \omega^{(2)}_\mathbf{A} \). Then \( \alpha^{(1)} = g \alpha \), where \( g \) is some scalar function. Here the one form \( \omega^{(2)}_\mathbf{A} \) is a surface forming one form, since \( \omega^{(2)}_\mathbf{A} \wedge d\omega^{(2)}_\mathbf{A} = 0 \). So initially on this family of surfaces \( \alpha \) is annihilated. Hence over these 2-surfaces the integrals of \( \alpha \) are vanishing invariants.

Also form \( \omega^{(3)}_\mathbf{A} = f \omega^{(2)}_\mathbf{A} \), it follows that for all curves \( C \) on which \( \omega^{(2)}_\mathbf{A} \) is annihilated, \( \int_C \omega^{(1)}_\mathbf{A} \) is a constant of motion. Since \( \omega^{(2)}_\mathbf{A} \) is a surface forming one form, it is annihilated on all curves lying on these surfaces and hence all such curves will be integral invariant curves for \( \int_C \omega^{(1)}_\mathbf{A} \).
Example 3:

Consider the magnetic field

$$\mathbf{B} = (\cos z, \sin z, 0)$$

produced by the current

$$\mathbf{J} = -\frac{1}{4\pi} (\cos z, \sin z, 0)$$

and the electric field

$$\mathbf{E} = (1, 0, 0).$$

Then the particle drift velocity is given by

$$\mathbf{v} = (0, 0, \sin z)$$

and the magnetic potential is

$$\mathbf{A} = (-\cos z, -\sin z, 0).$$

Let $\omega_{\mathbf{A}}$ be the one form associated to the magnetic potential $\mathbf{A}$. Then the first two derived fields of $\omega_{\mathbf{A}}$ are given below:

$$\omega_{\mathbf{A}}^{(1)} \longleftrightarrow (\sin^2 z, -\sin z \cos z, 0) \quad \text{and}$$

$$\omega_{\mathbf{A}}^{(2)} \longleftrightarrow (\sin z \sin 2z, -\sin z \cos 2z, 0).$$

Then $\omega_{\mathbf{A}}^{(2)} = f \omega_{\mathbf{A}} + g \omega_{\mathbf{A}}^{(1)}$, where $f = -\sin^2 z$ and $g = \cos z$. So $\omega_{\mathbf{A}}^{(m)} = f_m \omega_{\mathbf{A}} + g_m \omega_{\mathbf{A}}^{(1)}$, for all $m \geq 2$. Hence, if $\omega_{\mathbf{A}}$ and $\omega_{\mathbf{A}}^{(1)}$ are annihilated initially on some curve $C$, then $\int_C \omega_{\mathbf{A}}$ is a vanishing invariant of the flow. Clearly $\omega_{\mathbf{A}}$ and $\omega_{\mathbf{A}}^{(1)}$ are annihilated on the lines parallel to the $z$-axis. So they are curves with constant line integral of $\omega_{\mathbf{A}}$. Here note that $\mathbf{B}$ is the magnetic field of a circularly polarized electromagnetic wave in free space.
Example 4:

Again consider the equation of motion of a fluid in Rayleigh’s form given by (4.19). Taking the curl of this equation we get the vorticity equation

\[ \partial_t \omega + \nabla \times (\omega \times \mathbf{u}) = -k\omega \]  

where \( \omega = \nabla \times \mathbf{u} \) is the vorticity. Then consider a two form \( \omega \) defined by \( \omega = i_\omega \Omega \). Then the above equation is equivalent to

\[ \partial_t \omega + L_\mathbf{u} \omega = -k\omega. \]

That is

\[ \omega^{(1)} = -k\omega. \]

So if \( \omega \) is annihilated on some 2-surface initially, then such 2-surfaces will be integral invariant surfaces for \( \omega \). Clearly such surfaces include vector sheets of \( \omega \).

Here the equation 4.22 can also be expressed as

\[ \partial_t \omega + L_\mathbf{u} \omega = l\omega \]

where \( l = -k - \nabla \cdot \mathbf{u} \). This is equivalent to the equation \( \omega^{(1)} = l\omega \). So from case 2 of section 4.5 it follows that the vector lines of \( \omega \) are preserved by the flow.

Example 5:

Consider the motion of the magnetic lines of force under drift velocity, as given in example 2. Let \( \omega_\mathbf{B} \) is the two form corresponding to this magnetic field

\[ \mathbf{B} = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 1 \right). \]
Then the corresponding derived fields of $\omega_B$ are given by

\[
\begin{align*}
\omega_B^{(1)} & \longleftrightarrow \frac{2}{(1 + x^2 + y^2)^2} (-y, x, -1) \\
\omega_B^{(2)} & \longleftrightarrow \frac{4(-1 + 2x^2 + 2y^2)}{(1 + x^2 + y^2)^4} (-y, x, -1).
\end{align*}
\]

So $\omega_B^{(2)} = f \omega_B^{(1)}$, where $f = 2(-1 + 2x^2 + 2y^2)/(1 + x^2 + y^2)^2$. Hence for all $m \geq 2$, $\omega_B^{(m)} = f^m \omega_B^{(1)}$. So any surface $C$ on which $\omega_B^{(1)}$ is initially annihilated will be an integral invariant 2-surface for the two form $\omega_B$. Let $B_1$ be the vector field associated with the two form $\omega_B^{(1)}$. Then clearly on the vector sheets of the vector field $B_1$ the two form $\omega_B^{(1)}$ is annihilated initially and hence any 2-surface in this family of vector sheets will be integral invariant surfaces for $\omega_B$. So these 2-surfaces are constant flux surfaces of the associated magnetic field $B$.

**Example 6:**

Here we will consider a possible incompressible viscous flow under no external body forces and with vanishing pressure gradient. An exact solution to the corresponding Navier-Stokes equation is given by

\[
u = (e^{-(y+at)}, -(a + v), b)
\]

where $a$ and $b$ are constants and $\nu$ is the coefficient of kinematic viscosity. Then the vorticity field $w$ is given by

\[
w = (0, 0, e^{-(at+y)})
\]

Let $w_\nu$ is the two form associated to this vorticity vector field. Then

\[
\begin{align*}
\omega_w^{(1)} & \longleftrightarrow (0, 0, \nu e^{-(at+y)}) \\
\omega_w^{(2)} & \longleftrightarrow (0, 0, \nu^2 e^{-(at+y)})
\end{align*}
\]
So we have $\omega^{(1)}_w = \nu \omega_w$ and hence $\omega^{(m)}_w = \nu^m \omega_w$ for all $m \geq 1$. Hence for all 2-surfaces $C$ on which the two form $\omega_w$ vanishes initially, we have $\int_C \omega_w = 0$ throughout the flow. The family of planes initially parallel to $z$-axis will constitute such a class of invariant surfaces.

Example 7:

Here we are giving an example which illustrates the surface preservation of vector fields. Consider the fields as given in example 3. Here $\mathbf{B}$ is a vector field in the three dimensional Euclidean space $\mathbb{R}^3$. Then consider the derived fields of the vector field $\mathbf{B}$.

\[
\mathbf{B}^{(1)} = (-\sin^2 z, \cos z \sin z, 0) \\
\mathbf{B}^{(2)} = (-2 \cos z \sin^2 z, \cos 2z \sin z, 0) \\
\mathbf{B}^{(3)} = 
\left( (-1 + 3 \cos 2z) \sin^2 z, -\frac{1}{2} (\cos z - 3 \cos 3z) \sin z, 0 \right)
\]

Then we have $\mathbf{B}^{(2)} = f \mathbf{B} + g \mathbf{B}^{(1)}$, where $f = -\sin^2 z$ and $g = \cos z$. Here clearly $[\mathbf{B}, \mathbf{B}^{(1)}] = 0$ and hence there exist a family of integral surfaces, which are planes perpendicular to the $z$-axis. Initially consider such a plane material surface. Then from case 3 of section 4.5, it follows that this material surface will remain as the integral surface for $\mathbf{B}$ and $\mathbf{B}^{(1)}$.

Here the one form $\omega = i_{\mathbf{B}^{(1)}} \mathbf{i} \Omega$ is given by

\[
\omega \longleftrightarrow (0, 0, \sin z)
\]

Also $\omega^{(1)}$ is given by

\[
\omega^{(1)} \longleftrightarrow (0, 0, \sin 2z))
\]
So $\omega^{(1)} = f \omega$, where $f = 2 \cos z$. Here $\omega \wedge d\omega = 0$ and the integral surfaces of $\omega$ are clearly that of $\mathbf{B}$ and $\mathbf{B}^{(1)}$, namely planes perpendicular to $z$-axis. For any curve on this family of surfaces, $\omega$ is a vanishing integral invariant. So this family of surfaces annuls $\omega$ initially and hence identically.

The material surface which is an integral surface of $\mathbf{B}$ and $\mathbf{B}^{(1)}$ initially, remain as an integral surface of $\mathbf{B}$ and $\mathbf{B}^{(1)}$. We conclude that the material planes perpendicular to the $z$-axis remains perpendicular to $z$-axis. Hence the material surface spanned by $\mathbf{B}$ and $\mathbf{B}^{(1)}$ remain spanned by $\mathbf{B}$ and $\mathbf{B}^{(1)}$. So the surface preservation of $\mathbf{B}$ (and $\mathbf{B}^{(1)}$) is obtained. Here note that the vector lines of $\mathbf{B}$ are not material lines. That is, a material line initially parallel to a $\mathbf{B}$-line need not be parallel to the $\mathbf{B}$-line during the flow.

### 4.8 Discussion

In this chapter we have obtained some sufficient conditions for the integral invariance of one forms and two forms under a flow field in $\mathbb{R}^3$. Also we have obtained some sufficient conditions for surface preservation of vector fields.

We have proved some general results which holds for any $n$-dimensional manifold. we have given sufficient conditions for the invariance of integral of a $p$-form $\omega$ over a $p$-surface $C^p$. We explained these results in the context of three dimensional flows in $\mathbb{R}^3$.

Let $\omega$ be a one form in the Euclidean space $\mathbb{R}^3$. Also let $\omega^{(2)} = f \omega^{(1)}$. Then it is possible to find out some particular curves for which the line integral of $\omega$ is a constant of motion. If $\mathbf{A}$ is the associated vector field of $\omega^{(1)}$ and $\mathbf{B}$ is any arbitrary smooth vector field not collinear with $\mathbf{A}$, then the vector lines of $\mathbf{A} \times \mathbf{B}$ will constitute a family of curves over which the line integral of $\omega$ is a constant. Moreover, if Frobenius integrability condition is satisfied for $\omega^{(1)}$, then there exist a family of orthogonal surfaces for the vector field $\mathbf{A}$. So any curve on this surfaces will be integral invariant.
We have shown that equation (4.4) is the sufficient condition for the surface preservation of vector fields provided that there exist an integral surfaces for $J$ and $J^{(1)}$. Also we have given conditions for the invariance of surface integrals of a two form in different cases.

At the end of the chapter we have given some examples which illustrate the concepts developed in the chapter. Now let us consider a particular example of viscous ABC flow, which is used in [40, 41] in the study of vortex reconnection. Here the velocity field and vorticity field given by

$$\mathbf{u} = \omega = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x),$$

where $A = A_0 e^{-\nu t}, B = B_0 e^{-\nu t}, C = C_0 e^{-\nu t}$ and $A_0, B_0, C_0$ are constants, is an exact solution of the Navier-Stokes equation. Here topology of vortex lines is invariant but helicity changes because the magnitude of vorticity changes. Also note that when there is a reconnection of vortex lines, then the topology of vortex lines changes. Let $\omega$ be the two form associated with this vorticity field. Then we have $\omega^{(1)} = -\nu \omega$. So $\omega^{(m)} = (-\nu)^m \omega$, for all $m \geq 1$. Then from section 4.6 it is clear that the vortex sheets of $\omega$ are integral invariant 2-surfaces for this two form. So a material surface which is initially a vortex sheet will remain as a vortex sheet for this flow.