Chapter 5

Conclusion

Fluid mechanics is the source of many of the ideas and concepts that are central to modern mathematics. Mathematicians have abstracted and vastly generalised many of the fluid mechanical concepts and have a deep and powerful body of knowledge. But many of them are unfortunately now unknown to fluid mechanicians, while mathematicians themselves have lost all but a passing knowledge of physical origins of many of their basic notions. It will be surprising to a student of classical fluid mechanics to see that early mathematicians like C.F Gauss had thought of applying topological ideas in electricity and magnetism and had inspired Lord Kelvin to develop a theory of matter based on vortex knots and links. The past two decades have witnessed a revival of interest in these studies and has resulted in the origin of a new branch of study called topological fluid mechanics. The present thesis is the outcome of our investigation of some of these topological aspects of hydrodynamics and magnetohydrodynamics.

In this thesis we are studying possible invariants in hydrodynamics and hydromagnetics. The concept of flux preservation and line preservation of vector fields, especially vorticity vector fields, have been studied from the very beginning of the study of fluid mechanics by Helmholtz and others. In ideal magnetohydrodynamic flows the magnetic fields satisfy the same conservation laws as that of vorticity field in ideal hydrodynamic flows. Apart from these there are many other fields also in ideal hydrodynamic and
magnetohydrodynamic flows which preserves flux across a surface or whose vector lines are preserved.

A general study using this analogy had not been made for a long time. Moreover there are other physical quantities which are also invariant under the flow, such as Ertel invariant. Using the calculus of differential forms Tur and Yanovsky classified the possible invariants in hydrodynamics. This mathematical abstraction of physical quantities to topological objects is needed for an elegant and complete analysis of invariants.

Many authors used a four dimensional space-time manifold for analysing fluid flows. We have also used such a space-time manifold in obtaining invariants in the usual three dimensional flows.

In chapter one we have discussed the invariants related to vorticity field using vorticity field two form \( w^2 \) in \( E^4 \). Corresponding to the invariance of four form \( w^2 \land w^2 \) we have got the invariance of the quantity \( E \cdot w \). We have shown that in an isentropic flow this quantity is an invariant over an arbitrary volume.

In chapter three we have extended this method to any divergence-free frozen-in field. In a four dimensional space-time manifold we have defined a closed differential two form and its potential one from corresponding to such a frozen-in field. Using this potential one form \( \omega^1 \), it is possible to define the forms \( d\omega^1 \), \( \omega^1 \land d\omega^1 \) and \( d\omega^1 \land d\omega^1 \). Corresponding to the invariance of the four form we have got an additional invariant in the usual hydrodynamic flows, which can not be obtained by considering three dimensional space.

In chapter four we have classified the possible integral invariants associated with the physical quantities which can be expressed using one form or two form in a three dimensional flow. After deriving some general results which hold for an arbitrary dimensional manifold we have illustrated them in the context of flows in three dimensional Euclidean space \( \mathbb{R}^3 \). If the Lie derivative of a differential \( p \)-form \( \omega \) is not vanishing,
then the surface integral of $\omega$ over all $p$-surfaces need not be constant of flow. Even then there exist some special $p$-surfaces over which the integral is a constant of motion, if the Lie derivative of $\omega$ satisfies certain conditions. Such surfaces can be utilised for investigating the qualitative properties of a flow in the absence of invariance over all $p$-surfaces. We have also discussed the conditions for line preservation and surface preservation of vector fields. We see that the surface preservation need not imply the line preservation. We have given some examples which illustrate the above results.

The study given in this thesis is a continuation of that started by Vedan et.al. As mentioned earlier, they have used a four dimensional space-time manifold to obtain invariants of flow from variational formulation and application of Noether's theorem. This was from the point of view of hydrodynamic stability studies using Arnold's method.

The use of a four dimensional manifold has great significance in the study of knots and links. In the context of hydrodynamics, helicity is a measure of knottedness of vortex lines. We are interested in the use of differential forms in $E^4$ in the study of vortex knots and links. The knowledge of surface invariants given in chapter 4 may also be utilised for the analysis of vortex and magnetic reconnections.