CHAPTER V
IDENTIFICATION OF MODELS BY INCOME CHARACTERISTICS

5.1 Introduction

The present study so far focussed attention on the approach to modeling income data by using the quantile functions as an alternative to the distribution function traditionally employed in most of the situations. A particular quantile function proposed in Chapter III capable of generating the flexible family of distributions named as the modified lambda family, was seen to represent a potential income model in the sense of rendering a good fit to income data. A serious limitation to this approach was the lack of a stochastic mechanism that account for the distribution. Alternatively in the absence of stochastic arguments, one may also think of inherent characteristics of income data that may give rise to a unique distribution so that the data generating mechanism can be spelt out through the concerned characteristic. In other words the model appropriate to a given population of incomes can be based on a characterization satisfying the particular nature of an income characteristic suitable to that population. The objective of this chapter is to build up a theoretical frame work for this purpose. This needs well accepted choice of income characteristics that can distinguish various distributions and amenable to analytic treatment. We have selected the concept of income gap ratio and the truncated form of Gini index for characterizing income distributions. Results are obtained for both distribution functions and quantile functions, by starting with the former and then making deductions to the latter case. The rest of the chapter consists of five more sections. In Section 2 we introduce the definitions of the Income Gap Ratio and Truncated Gini Index. We show in Section 3 that \( \alpha(t) \) uniquely determines the distribution \( F(x) \) and that the power distribution is the only continuous distribution for which \( \alpha(t) \) and \( G(t) \) are independent of the truncation point \( t \). Further it is proved that \( (1+G(t))\mu(t) \) can characterize the income distribution. Similar results
concerning the affluent are proved in Section 4. The monotonic behaviour of the income gap ratio can be used as a criterion to classify income distributions that help the choice of the distribution as model of income. In Section 5 some results in this connection are presented. In the last Section almost all the results in the above sections are converted into the context of quantile functions and the income gap ratio and truncated gini index of the MLF have been evaluated.

5.2 Income gap ratio and truncated Gini index

Most of the indexes of poverty or affluence associated with income data are generally based on the proportion of people belonging to that category along with their income distribution through the income gap ratio and some measure of income inequality like the Gini index truncated at the appropriate level of income. Sen (1976), Takayama (1979) and Sen (1986, 1988) deal with such indexes and their properties. Since the income gap ratio and truncated Gini coefficient have a vital role in the definition of an index, it is important to investigate their relationships with the basic income distribution. In situations where these quantities are estimated from the observations without knowing the form of the distribution of incomes, (e.g. non-parametric estimation of income gap ratio and Gini coefficient) one basic question is whether the values of these functions at different levels of income enable the determination of the income distribution of the population. Theoretically the problem looks at the derivation of the distribution function of incomes based on the functional form of the income gap ratio and the truncated Gini coefficient. The present chapter focuses attention on this problem and some related issues like classification of income distributions on the basis of the behaviour of income gap ratio. Analogous results for quantile function of incomes have also been discussed in this chapter. We first present the basic definitions of the income gap ratio and truncated Gini index using the distribution function approach.

Let $X$ be a non-negative random variable representing the income of a community of individuals with absolutely continuous distribution function
\(F(x)\), survival function \(\bar{F}(x) = 1 - F(x)\) and density function \(f(x)\).

Assuming the poverty line \(X = t\), the proportion of poor people is \(F(t)\) and their income distribution becomes that of the random variable \((X \mid X \leq t)\) viz.

\[
\begin{align*}
F(x) &= \frac{F(x)}{F(t)} \quad x \leq t \\
&= 1 \quad x > t.
\end{align*}
\]  

(5.1)

The income gap ratio of the poor people is defined as

\[
\alpha(t) = 1 - E\left(\frac{X}{t} \mid X \leq t\right) = 1 - \frac{\int_y yf(y)dy}{tF(t)}
\]

(5.2)

Using the standard definition of the Gini coefficient in (2.33) the truncated version relating to those below the poverty line is

\[
G(t) = 1 - 2 \left[ \mu(t) \right]^{-1} \int_0^t yF(y)F(t)dy
\]

\[
= 1 - 2 \left[ \mu(t) \right]^{-1} \int_0^t y \left(1 - \frac{F(y)}{F(t)}\right) \frac{f(y)}{F(t)}dy
\]

(5.3)

where

\[
\mu(t) = \frac{1}{F(t)} \int_0^t yf(y)dy = E(X \mid X \leq t)
\]

(5.4)

is the average income below the poverty line.

Analogously with reference to an affluence line \(X = t^*\) the incomes for the affluent \(\{X \mid X > t^*\}\) has distribution specified by

\[
F_{t^*}(x) = 0 \quad x \leq t^*
\]
\[
F(x) - F(t^*) = \frac{1}{1 - F(t^*)} \quad x > t^* \tag{5.5}
\]

The income gap ratio of the affluent is then

\[
\alpha^*(t^*) = 1 - \frac{t^*}{E(X|X \geq t^*)} = 1 - \frac{t^*F(t^*)}{\int_{t^*}^{\infty} yf(y)dy} \tag{5.6}
\]

and the corresponding truncated Gini coefficient becomes,

\[
G^*(t^*) = 1 - 2 \left[ \mu^*(t^*) \right]^{-1} \int_{t^*}^{\infty} \frac{\bar{F}(y)}{F(t^*)} \frac{f(y)}{F^*(t^*)} dy \tag{5.7}
\]

where

\[
\mu^*(t^*) = \frac{1}{\bar{F}(t^*)} \int_{t^*}^{\infty} yf(y)dy = E(X|X > t^*). \tag{5.8}
\]

These definitions will be employed in the next section to develop characterization of \( F \) in terms of \( \alpha(t), \alpha^*(t^*), G(t) \) and \( G^*(t^*) \).

### 5.3 Characterization of income distributions

First we establish a one-to-one correspondence between income gap ratio and the base line income distribution.

**Theorem 5.1:**

If \( X \) has an absolutely continuous distribution function over \((0, \infty)\) with finite mean and income gap ratio \( \alpha(t) \) which is differentiable, then

\[
F(t) = \exp \left[ - \int_{t}^{\infty} \frac{1 - \alpha(y) - y\alpha'(y)}{y\alpha(y)} dy \right], t > 0. \tag{5.9}
\]

**Proof:** From the definition (5.2),
\[(1 - \alpha(t))tF(t) = \int_0^t yf(y)dy\]

Differentiating with respect to \(t\) and re-arranging terms

\[\frac{f(t)}{F(t)} = \frac{1 - \alpha(t)}{t\alpha(t)} - \frac{\alpha'(t)}{\alpha(t)}\]

Integrating from \(t\) to \(\infty\),

\[\left[ \ln F(t) \right]_t^\infty = \int_t^\infty \frac{1 - \alpha(y) - y\alpha'(y)}{y\alpha(y)} dy\]

which leads to (5.9).

This theorem shows that using the functional form of \(\alpha(t)\) one can determine the income distribution. Usually income gap ratios computed at several points of income are available directly from the income data without making assumptions about the income distribution. Empirically it is possible to draw conclusion about the form of the income gap ratio by plotting \(\alpha(t)\) against \(t\). We now establish some sample functional forms of \(\alpha(t)\) that characterize income distributions.

**Theorem 5.2:**

The only continuous distribution for which the income gap ratio is a constant is the power distribution

\[F(x) = \left(\frac{x}{c}\right)^\alpha , \quad 0 < x < c, \alpha > 0.\]  (5.10)

**Proof:** Suppose \(X\) follows the power distribution (5.10). Then from definition (5.2),

\[\alpha(t) = \frac{1}{\alpha + 1}, \text{ which is a constant.}\]

Conversely, let \(\alpha(t) = k\), a constant less than unity. Then from (5.9),

\[F(x) = \left(\frac{x}{c}\right)^{1-k}\]
which is the power distribution with
\[ \alpha = \frac{1-k}{k} > 0. \]

**Remark:**

The above result can be used to ascertain the changes in the income distribution (e.g. number of individuals whose income has to be raised to the next level) inorder to have a designated reduction in the income gap ratio.

An analogous result for the power distribution exists regarding the truncated Gini coefficient relating to the poor.

**Theorem 5.3:**

If \( X \) has absolutely continuous distribution function \( F(x) \) satisfying \( E(XF(X)) < \infty \), then the truncated Gini coefficient \( G(t) \) is independent of \( t \) if and only if \( X \) has power distribution (5.10).

Proof: From (5.3) and (5.4),
\[
G(t) = 1 - 2 \left[ \mu(t) \right]^{-1} \left[ \int_0^t y f(y) dy - \int_0^t \frac{y^2}{F(t)} \right] \\
= 1 - 2 + 2 \left[ \mu(t) \right]^{-1} \int_0^t \frac{y^2}{F(t)} dy
\]
which gives
\[
\frac{1}{2} \mu(t) [1 + G(t)] = \int_0^t \frac{yF(y)f(y)}{F^2(t)} dy 
\]
(5.11)

When \( X \) follows power distribution (5.10), from (5.11)
\[
\frac{1}{2} \mu(t) [1 + G(t)] = \frac{1}{(t/c)^{2\alpha}} \int_0^t \left( \frac{y}{c} \right)^\alpha y^\alpha dy \\
= \frac{\alpha t}{2\alpha + 1}
\]
and further
\[ \mu(t) = \frac{\alpha}{(\alpha + 1)t} \]
so that \( G(t) = \frac{1}{2\alpha + 1} \), a constant. This proves the if part.

To see that the only if part holds we assume \( G(t) = k \) and write (5.11) as
\[
\frac{1}{2} F(t)(1 + G(t)) \int_0^t yf(y)dy = \int_0^t yF(y)f(y)dy
\]
Differentiating the last equation twice
\[
(1 + k)t f(t) + \frac{1}{2}(1 + k)F(t) = tf(t) + F(t)
\]
or
\[
\frac{f(t)}{F(t)} = \frac{(1 - k)}{2kt}
\]
Integrating from \( t \) to \( c \),
\[
\left[ \ln F(t) \right]_t^c = \int_t^c \frac{1 - k}{2ky} dy
\]
\[ = \ln \left( \frac{c}{t} \right)^{\frac{1 - k}{2k}}
\]
or
\[
F(t) = \left( \frac{t}{c} \right)^{\frac{1 - k}{2k}}
\]
which is the power distribution (5.10) with \( \alpha = \frac{(1 - k)}{2k} \).

The problem of obtaining a general inversion formula for finding \( F(t) \) in terms of \( G(t) \) was found to be difficult in view of the presence of \( \mu(t) \) in (5.11). However, if one sets \( g(t) = \mu(t) \left[ 1 + G(t) \right] \), we can express \( F(t) \) in terms of \( g(t) \).
From (5.11)
\[
\frac{1}{2} \left[ F(t) \right]^2 g(t) = \int_0^t y F(y) f(y) dy
\]
Differentiating with respect to \( t \), we obtain
\[
f(t) g(t) + \frac{1}{2} F(t) g'(t) = tf(t)
\]
or
\[
f(t) = \frac{g'(t)}{F(t)} = \frac{g'(t)}{2[t - g(t)]}
\]
Integrating from \( t \) to \( \infty \),
\[
\ln F(t) = \int_t^\infty \frac{g'(y)}{\left[ y - g(y) \right]} \, dy
\]
or
\[
F(t) = \exp \left\{ -\frac{1}{2} \int_t^\infty \frac{g'(y)}{y - g(y)} \, dy \right\}, \quad t > 0.
\]

The practical utility of (5.12) is the identification of \( F(x) \) through the functional form of \( g(x) \). Expressions of \( g(x) \) for some income distributions are evaluated below, to indicate its usefulness in a practical situation.

(i) Pareto distribution:
\[
f(x) = \alpha \sigma^\alpha x^{-\alpha - 1}, \quad x > \sigma > 0, \quad \alpha > 0
\]
\[
g(x) = \left[ 1 + G(x) \right] \mu(x)
\]
\[
= 2 \int_\sigma^x y F(y) f(y) \, dy
\]
\[
= \frac{2 \int_\sigma^x y F(y) f(y) \, dy}{F^2(x)}
\]
\[
= \frac{2}{\left[ 1 - \left( \frac{x}{\sigma} \right)^{-\alpha} \right]^2} \int_\sigma^x \left[ 1 - \left( \frac{y}{\sigma} \right)^{-\alpha} \right] \alpha \sigma^\alpha y^{-\alpha - 1} \, dy
\]

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\[ F(x) = \frac{2\alpha x}{1+(\frac{x}{\alpha} - \frac{1}{\alpha})^2} \left\{ \frac{1 - \left(\frac{x}{\alpha} - \frac{1}{2\alpha}\right)}{\alpha - 1} - \frac{1 - \left(\frac{x}{\alpha} - \frac{1}{2\alpha}\right)^2}{2\alpha - 1} \right\}. \]

(ii) Exponential:

\[ F(x) = 1 - e^{-\lambda x}, \ x > 0, \ \lambda > 0 \]

\[ g(x) = 2 \int_{0}^{\infty} y \left[ 1 - e^{-\lambda y} \right] \frac{\lambda e^{-\lambda y}}{1 - e^{-\lambda x}} dy \]

\[ = \frac{2}{[1 - e^{-\lambda x}]^2} \left[ \frac{e^{-2\lambda x}}{2} \left( x + \frac{1}{2\lambda} \right) - e^{-\lambda x} \left( x + \frac{1}{\lambda} \right) + \frac{3}{4\lambda} \right]. \]

(iii) Power:

\[ F(x) = \left( \frac{x}{c} \right)^{\alpha} \]

\[ g(x) = \frac{2}{[\left( \frac{x}{c} \right)^{\alpha}]} \int_{0}^{\infty} y \left[ \frac{y}{c} \right]^{\alpha - 1} c^{\alpha} \alpha y^{\alpha - 1} dy \]

\[ = \frac{2\alpha x}{(2\alpha + 1)} \]

(iv) Dagum:

\[ F(x) = \left[ 1 + \left( \frac{b}{x} \right)^{a} \right]^{p}, \ x > 0 \]

\[ g(x) = \frac{2}{[1 + \left( \frac{b}{x} \right)^{a} - 2p]} \int_{0}^{\infty} y \left[ 1 + \left( \frac{b}{y} \right)^{a} \right]^{-p} \left[ 1 + \left( \frac{b}{y} \right)^{a} \right]^{-p-1} \ dy \]

\[ = 2ap \left[ 1 + \left( \frac{b}{x} \right)^{a} \right]^{-2p} \int_{0}^{\left( \frac{y}{b} \right)} \left[ 1 + \left( \frac{y}{b} \right)^{a} \right]^{-a} -2p-1 \ dy \]

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\[ B_x(p,q) = \int_0^\infty t^{p-1}(1+t)^{-(p+q)} \, dt. \]

where \( B_x(p,q) = \int_0^\infty t^{p-1}(1+t)^{-(p+q)} \, dt. \)

### 5.4 Measures of affluence

The income gap ratio and Gini coefficient for the affluent hold analogous properties as for the poor.

**Theorem 5.4:**

If \( X \) has absolutely continuous distribution function over \((0,\infty)\) with finite mean and income gap ratio \( \alpha^*(t^*) \), then

\[
F(t^*) = \exp \left[ - \int_0^{t^*} \frac{1 - \alpha^*(y) + y \alpha''(y)}{\alpha^*(y)(1 - \alpha^*(y))} \, dy \right] \quad (5.13)
\]

**Proof:** From (5.6),

\[
\left[ 1 - \alpha^*(t^*) \right] \int_{t^*}^{\infty} yf(y) \, dy = t^*F(t^*)
\]

Differentiating with respect to \( t^* \) and simplifying

\[
\frac{f(t^*)}{F(t^*)} = \frac{1 - \alpha^*(t^*) + t^* \alpha''(t^*)}{\alpha^*(t^*) (1 - \alpha^*(t^*)) t^*}
\]

Integrating from 0 to \( t^* \), we get (5.13).

**Remarks:**

1. The only continuous distribution over the set of positive reals for which \( \alpha^*(t^*) \) is a constant is the Pareto distribution. This is easily
verified by noting that, \( \alpha^*(t^*) = \frac{1}{\alpha} \) (in the form given above) and substituting in (5.13) gives the desired form for \( F(x) \).

2. The generalized Pareto family with

\[
\bar{F}(x) = \left( \frac{b}{ax + b} \right)^{a+1}, x > 0, b > 0, a > -1
\]  

(5.14)

is characterized by an income gap ratio in the bilinear form

\[
\alpha^*(t^*) = \frac{at^* + b}{(a+1)t^* + b}
\]

Note that (5.14) contains the exponential distribution as \( a \to 0 \), the Pareto II distribution of the form

\[
\bar{F}(x) = \alpha^c (x + \alpha)^{-c}, x, \alpha > 0
\]

with \( a = (c-1)^{-1}, \ b = \alpha (c-1)^{-1} \) when \( 0 < a < 1 \) in (5.14) and the beta

\[
\bar{F}(x) = \left( 1 - \frac{x}{R} \right)^d, 0 < x < R, d > 0,
\]

\[
a = -(d+1)^{-1}, \ b = R(d+1)^{-1} \] when \( -1 < a < 0 \) in (5.14).

3. Ord et. al (1983) have used the gamma entropy measure

\[
e^\gamma(t) = \int_0^\infty \frac{f(x)}{F(t)} \left[ 1 - \frac{f^\gamma(x)}{F^\gamma(t)} \right] dx / \gamma
\]  

(5.15)

as a measure of inequality and shows that (5.15) is truncation invariant (independent of \( t \)) if and only if \( X \) has exponential distribution. We now discuss the relationship of the measure \( e^\gamma(t) \) with the income gap ratio.

Defining the random variable \( Y_\gamma = \left( \frac{f^\gamma(X)}{t^\gamma \bar{F}^\gamma(t^*)} \right) \mid X > t^* \) it is easy to see that

\[
E(Y_\gamma) = \frac{1}{t^\gamma} \left( 1 - \gamma e^\gamma(t^*) \right)
\]
Now the geometric mean of $Y_{t^*}$ is $G(t^*)$, where

$$\log G(t^*) = E \log Y_{t^*} = \frac{\gamma}{F(t^*)} \int (\log f(x) - \log \bar{F}(t^*) - \log t^*) f(x) dx$$

$$= \frac{\gamma}{F(t^*)} \int f(x) \log f(x) dx - \gamma \log \bar{F}(t^*) - \gamma \log t^* \quad (5.16)$$

Again defining $Z_{t^*} = \left( \frac{f(X)}{t^*F(x)} \right)|X > t^*$, $Z_{t^*}$ has geometric mean $p(t^*)$ with

$$\log p(t^*) = E \left( \log \frac{f(X)}{t^*F(x)} \mid X > t^* \right)$$

$$= \frac{1}{F(t^*)} \int (f(x) \log f(x) - f(x) \log \bar{F}(x) - f(x) \log t^*) dx$$

$$= \frac{1}{F(t^*)} \int f(x) \log f(x) dx - \log \bar{F}(t^*) - \log t^* + 1 \quad (5.17)$$

Comparing with (5.16)

$$\log p(t^*) = 1 + \frac{1}{\gamma} \log G(t^*)$$

or

$$p(t^*) = e^{G(t^*)} \quad (5.18)$$

Finally $Z_{t^*}$ has harmonic mean $H(t^*)$ where

$$H^{-1}(t^*) = E \left( \frac{t^*F(X)}{f(X)} \mid X > t^* \right)$$

$$= \frac{t^*}{F(t^*)} \int \bar{F}(x) dx$$

$$= t^* \left[ \frac{1}{F(t^*)} \int_{t^*}^{\infty} yf(y) dy - t^* \right]$$
\begin{align*}
= t^* \left[ \frac{t^*}{1 - \alpha^* \left( t^* \right)} - t^* \right] \\
\text{or} \\
H(t^*) = \frac{1 - \alpha^* \left( t^* \right)}{t^* \alpha^* \left( t^* \right)}
\end{align*}

Since \( p(t^*) \geq H(t^*) \), from (5.18)

\[
G(t^*) \geq \frac{\left( 1 - \alpha^* \left( t^* \right) \right)^\gamma}{e^\gamma t^{2\gamma} \left( \alpha^* \left( t^* \right) \right)^\gamma}
\]  (5.19)

Further \( E(Y^*_*) \geq G(t^*) \) gives

\[
\frac{1}{t^* \gamma} \left( 1 - \gamma e_{y_*} (t^*) \right) \geq \frac{\left( 1 - \alpha^* \left( t^* \right) \right)^\gamma}{e^\gamma t^{2\gamma} \alpha^* \left( t^* \right)^\gamma}
\]

which on simplification provides the following lower bound to the income gap ratio in terms of the entropy measure

\[
\alpha^* \left( t^* \right) \geq \left[ 1 + et^* \left( 1 - \gamma e_{y_*} (t^*) \right)^\frac{1}{\gamma} \right]^{-1}
\]  (5.20)

The Gini coefficient for the affluent is defined in equations (5.7) and (5.8). Ord et. al (1983) have shown that this coefficient is constant among the class of absolutely continuous distributions with positive density almost everywhere in \((k, \infty)\), if and only if the distribution is Pareto. This result corresponds to Theorem 5.3. Writing

\[
g^* \left( x \right) = \left( 1 - G^* \left( x \right) \right) \mu^* \left( x \right)
\]

one could see that

\[
\frac{1}{2} \bar{F}^2 \left( t^* \right) \left( 1 - G^* \left( t^* \right) \right) \mu^* \left( t^* \right) = \int_{t^*}^{\infty} y\bar{F} \left( y \right) f \left( y \right) dy
\]

Proceeding as in the earlier theorems we have the following result.
Theorem 5.5:

If \( X \) has absolutely continuous distribution function with 
\( E\left(X\bar{F}(X)\right) < \infty \) with Gini coefficient \( G^*(t^*) \) and average income \( \mu^*(t^*) \) above the affluence line \( X = t^* \), then

\[
\bar{F}(x) = \exp\left[ -\frac{1}{2} \int_0^x \frac{g^*(y)}{g^*(y)-y} \, dy \right]
\]

(5.21)

where \( g^*(y) = (1 - G^*(y)) \mu^*(y) \).

Remarks:

1. The generalized Pareto family (5.14) is characterized by

\[
\mu^*(t^*)(1 - G^*(t^*)) = \frac{2at^* + 2t^* + b}{a + 2} = At^* + B
\]

(5.22)

which is a unification of the result in Sathar, Rajesh and Nair (2003) separately proved for the exponential, Pareto II and beta distributions. They have also proved that the income gap ratio is in constant proportion with the \( G^*(t^*) \) for the above three distributions, the proportionality being \( \frac{1}{2} \) for exponential, \( > \frac{1}{2} \) for Pareto II and \( < \frac{1}{2} \) for the beta. From (5.22) the property \( G^*(t^*) = \frac{1 + a}{2 + a} \alpha^*(t^*) \) characterizes the generalized Pareto family.

2. \( g^*(y) = cy \), where \( c \) is a constant greater than unity characterizes the Pareto distribution.

5.5 Classes of Income Distributions

Based on the monotonic behaviour of the income gap ratio \( \alpha^*(t^*) \) it is possible to classify income distributions. These results are helpful in
modeling incomes where the empirical evaluation of the income gap ratios at different values of $r^*$ will give us an idea about the class of distributions from which the appropriate model should be chosen.

**Definition:** A distribution function $F(x)$ is increasing in income gap ratio for the rich IIR($r$) (decreasing in income gap ratio – DIR($r$)) if $\alpha^*(x^*)$ is non-decreasing (non-increasing) in $x^*$.

Belzunce et al. (1998) defines the class of decreasing mean left proportional residual income (DMLPRI) if

$$E\left(\frac{X}{t} | X > t\right) = \frac{1}{tF(t)} \int_t^\infty xf(x)dx$$

is decreasing in $t$. Since this criterion is equivalent to DIR($r$) all results proved there are true for DIR($r$) also, and accordingly we establish some new implications of the DIR($r$) class which can supplement the existing results on DMLPRI.

1. The classes of income distributions based on monotonicity of $\alpha^*(t^*)$ are well defined, as the exponential model is DIR($r$), the beta discussed in Section 5.4 is IIR($r$) while the Pareto distribution is both DIR($r$) and IIR($r$) with $\alpha^*(t^*)$, a constant.

2. A sufficient condition for $F(x)$ to be DIR($r$) is that either of the following conditions hold.

   (a) $f(x)$ is log-concave (b) $F(x)$ has increasing failure rate.

**Proof:** If $g(x)$ is a monotonic (increasing or decreasing) function on $(a,b)$ with either $g(a) = 0$ or $g(b) = 0$, then if $g'(x)$ is log-concave then $g(x)$ is also log-concave on $(a,b)$. We use this result repeatedly for different functions in the proof.
To prove (a) Assume that \( f(x) \) is log-concave. Then by definition

of log-concavity \( \frac{f''(x)}{f(x)} \) is decreasing, and since \( \frac{f'(x)}{f(x)} = \frac{\bar{F}''(x)}{\bar{F}'(x)} \), \( \frac{\bar{F}''(x)}{\bar{F}'(x)} \) and hence \( \frac{F'(x)}{F(x)} \) are decreasing. Defining \( H(x) = \int_{x}^{\infty} \bar{F}(t) \, dt \),

\[ H'(x) = -\bar{F}(x) \quad \text{and} \quad H''(x) = -\bar{F}'(x). \]

Thus \( \frac{H''(x)}{H'(x)} \) and hence \( \frac{H'(x)}{H(x)} \) are decreasing functions. Thus we find that

\[ \frac{1 - \alpha^*(t^*)}{\alpha^*(t^*)} = \frac{t^*H'(t^*)}{H(t^*)} \]

is increasing and this implies \( \alpha^*(t^*) \) is decreasing or \( F(x) \) is DIR(r).

To prove (b) we note that whenever the failure rate

\[ h(x) = \frac{f(x)}{F(x)} = -\frac{\bar{F}'(x)}{\bar{F}(x)} \]

is increasing \( \frac{\bar{F}'(x)}{\bar{F}(x)} \) is decreasing. The rest of the proof follows from that of part (a).

Remarks:

1. Part (a) gives a simple criterion to distinguish income distributions with decreasing income gap ratio. For IIIR(\( r \)) models the words increasing and log-concave in (a) and (b) are to be replaced by decreasing and log-convex. A classification of some distributions used to model incomes according to the above criteria is given below.

Note: log-concavity properties are preserved under linear transformations so that scale and location parameters can be introduced without affecting their classifications. Also, the classifications hold for truncated versions of the above distributions.

2. Belzunce et. al (1998) defines the class of increasing proportional failure rate (IPFR) distributions in which \( xh(x) \) is an increasing function and shows that IPFR \( \Rightarrow DMLPRI \). When the class has increasing
failure rate $xh(x)$ is also increasing so that $IFR \Rightarrow IPFR$, but the converse need not be as seen from the case of the distribution

$$f(x) = \frac{1}{2} x^{-\frac{3}{2}} e^{-x^{\frac{1}{2}}}, x > 0.$$  

Therefore resulting from (b) above we can write the implications

$$IFR \Rightarrow IPFR \Rightarrow DMLPRI \Leftrightarrow DIR(r).$$

3. A necessary and sufficient condition that $F(x)$ is $DIR(r)$ ($IIR(r)$) is that

$$\alpha^*(t^*) < (>) \frac{1}{t^*h(t^*)}.$$  

**Proof:** From the definition in equation (5.6)

$$\left(1 - \alpha^*(t^*)\right) \int_{t^*}^{\infty} yf(y)dy = t^* \bar{F}(t^*)$$

Differentiating with respect to $t^*$ and simplifying,

$$t^* \alpha^*(t^*)f(t^*) - \frac{t^* \bar{F}(t^*)}{1 - \alpha^*(t^*)} \alpha'(t^*) = \bar{F}(t^*)$$

or

$$\alpha^*(t^*) = \frac{1 - \alpha^*(t^*)}{t^*} (\alpha^*(t^*)h(t^*)t^* - 1)$$

For $IIR(r)$ distribution, $\alpha'' > 0$ and hence

$$\alpha^*(t^*) > \frac{1}{t^*h(t^*)}.$$  

An analogous discussion holds in the case of income gap ratio for the poor.

**Definition:** A distribution function $F(x)$ is increasing in income gap ratio for the poor-$IIR(p)$ (decreasing-$DIR(p)$) if $\alpha(x)$ is non-decreasing (non-increasing) in $x$. This class is the same as that discussed by Belzunce et al.
(1998) in the name of decreasing mean right proportional residual income (DMRPRI). They provide an exhaustive discussion of the properties of this class. We observe further that

(i) a necessary and sufficient condition for $F(x)$ to be $IIR(p)$ is that

$$\alpha < (1 + t\lambda(t))^{-1},$$

where $\lambda(t) = \frac{f(t)}{F(t)}$, the reversed failure rate of $X$.

(ii) $\alpha(t)$ is increasing or $F(x)$ is $IIR(p)$ if and only if $\int_0^t F(t)dt$ is log-concave.

5.6 Quantile Forms of Income Gap Ratio and Truncated Gini Coefficient

Since the major theme in the present work is the modeling of income data using the lambda distribution, the transformation of the expressions of inequality measures discussed in the previous sections of this chapter into quantile forms is relevant.

Let $p$ be the proportion of the poor people of a population. Then by the transformation $u = F(x), 0 < u < 1$ or $x = Q(u)$ where $Q(u) = F^{-1}(u)$ in the equation (5.2) we get the income gap ratio of poor in terms of quantile functions and is given by

$$\alpha(p) = 1 - \frac{\int_0^p Q(u)du}{pQ(p)} \quad (5.23)$$

Similarly, the income gap ratio of the rich is given by

$$\alpha^*(p^*) = 1 - \frac{(1 - p^*)Q(p^*)}{\int_{p^*}^1 Q(u)du} \quad (5.24)$$

where $(1 - p^*)$ is the proportion of rich people of the population.
Now the truncated Gini coefficient for poor and rich are given respectively by

\[
G(p) = \frac{2 \left[ \mu(p) \right]^{-1}}{p^3} \int_0^p Q(u) \, du - 1
\]  
(5.25)

where
\[
\mu(p) = \frac{1}{p} \int_0^p Q(u) \, du
\]

and

\[
G^*(p^*) = 1 - \frac{2 \left[ \mu^*(p^*) \right]^{-1}}{\left( 1 - p^* \right)^2} \int_{p^*}^1 (1-u)Q(u) \, du
\]  
(5.26)

where
\[
\mu^*(p^*) = \frac{1}{1-p^*} \int_{p^*}^1 Q(u) \, du.
\]

**Theorem 5.6:**

The quantile function \( Q(p) \) can be uniquely determined by the income gap ratio of the poor, \( \alpha(p) \) as

\[
Q(p) = \frac{\mu}{p \left[ 1 - \alpha(p) \right]} \exp \left\{ - \int_{p}^{1} \frac{1}{u \left[ 1 - \alpha(u) \right]} \, du \right\}
\]  
(5.27)

**Proof:** From definition (5.23)

\[
\left[ 1 - \alpha(p) \right] p Q(p) = \int_0^p Q(u) \, du
\]

\[
\frac{Q(p)}{\int_0^p Q(u) \, du} = \frac{1}{p \left[ 1 - \alpha(p) \right]}
\]

Integrating from \( p \) to 1,

\[
\left[ \ln \int_0^p Q(u) \, du \right]_{p}^{1} = \int_{p}^{1} \frac{1}{u \left[ 1 - \alpha(u) \right]} \, du
\]
\[ \ln \mu - \ln \int_0^p Q(u) \, du = \int_0^1 \frac{1}{u[1-\alpha(u)]} \, du \]

\[ \ln \frac{\int_0^p Q(u) \, du}{\mu} = -\int_0^1 \frac{1}{u[1-\alpha(u)]} \, du \]

\[ \int_0^p Q(u) \, du = \mu \exp \left\{ -\int_0^1 \frac{1}{u[1-\alpha(u)]} \, du \right\} \]

Now differentiating with respect to \( p \), we get 5.27.

**Theorem 5.7:**

The quantile function can be uniquely determined by the income gap ratio of the rich, \( \alpha^*(p^*) \) as

\[ Q^*(p^*) = \mu \left[ \frac{1-\alpha^*(p^*)}{1-p^*} \right] \exp \left\{ -\int_0^{p^*} \frac{1-\alpha^*(u)}{1-u} \, du \right\} \]  (5.28)

**Proof:** From definition 5.24,

\[ \left[ 1 - \alpha^*(p^*) \right] \int_{p^*}^1 Q(u) \, du = (1-p^*) Q(p^*) \]

\[ \frac{-Q(p^*)}{\int_{p^*}^1 Q(u) \, du} = \frac{\alpha^*(p^*)-1}{1-p^*} \]

Integrating from 0 to \( p^* \),

\[ \left[ \ln \int_{p^*}^1 Q(u) \, du \right]_0^{p^*} = \int_0^{p^*} \frac{\alpha^*(u)-1}{1-u} \, du \]

\[ \ln \frac{\int_{p^*}^1 Q(u) \, du}{\mu} = \int_0^{p^*} \frac{\alpha^*(u)-1}{1-u} \, du \]
\[ \int_{\rho^*-2}^{\rho^*} Q(u) \, du = \mu \exp \left\{ - \int_{0}^{\rho^*} \frac{1 - \alpha^*(u)}{1 - u} \, du \right\} \]

Now differentiating with respect to \( p^* \), we get 5.28.

Now (5.25) can be written as

\[ \left[ 1 + G(p) \right] \mu(p) = \frac{2}{p^2} \int_{0}^{p} uQ(u) \, du \] (5.29)

Let

\[ g(p) = \left[ 1 + G(p) \right] \mu(p) \]

Thus from (5.29),

\[ p^2 g(p) = 2 \int_{0}^{p} uQ(u) \, du \]

Differentiating with respect to \( p \),

\[ 2pg(p) + p^2g'(p) = 2pQ(p) \]

Dividing by \( 2p \),

\[ Q(p) = g(p) + \frac{pg'(p)}{2} \] (5.30)

Thus it is possible to determine the quantile function from \( g(p) \).

As an illustration we have found below the expression of \( g(p) \) that characterizes \( Q(p) \) for some important distributions.

(i) Pareto:

\[ Q(p) = \sigma (1 - p)^{-1} \]

\[ g(p) = \frac{2}{p^2} \int_{0}^{p} uQ(u) \, du \]

\[ = \frac{2}{p^2} \int_{0}^{p} u \sigma (1 - u)^{-1/a} \, du \]
\[
= \frac{2\sigma \alpha}{p^2 (\alpha - 1)(2\alpha - 1)} \left\{ \alpha - (1 - p)^{\frac{1}{\sigma + 1}} \right\} - \frac{2\sigma \alpha (1 - p)^{\frac{1}{\sigma + 1}}}{p(\alpha - 1)}
\]

(ii) Exponential:
\[
Q(p) = -\frac{1}{\lambda} \ln(1 - p)
\]
\[
g(p) = \frac{2}{\lambda^2} \int_0^p u \left( -\frac{1}{\lambda} \ln(1 - u) \right) du
\]
\[
= \frac{1}{\lambda} \left[ \frac{1}{p} + \frac{1}{2} \ln(1 - p) + \frac{\ln(p - 1)}{p^2} \right]
\]

(iii) Power:
\[
Q(p) = cp^\alpha
\]
\[
g(p) = \frac{2}{p^2} \int_0^p u \frac{1}{\alpha} du
\]
\[
= \frac{2c \alpha p^\alpha}{2\alpha + 1}
\]

(iv) Dagum:
\[
Q(p) = b \left[ p^{\frac{1}{\beta}} - 1 \right]^{-\frac{1}{\alpha}}
\]
\[
g(p) = \frac{2}{\lambda^2} \int_0^p ub \left[ u^{\frac{1}{\beta}} - 1 \right] du
\]
\[
= \frac{2b \beta}{p^2} \int_0^{u^{\frac{1}{\beta}}} v^{\frac{1}{\alpha + 2\beta - 1}} (1 - v)^{\frac{1}{\alpha}} dv, \quad \text{when} \quad v = u^{\frac{1}{\beta}}
\]
\[
= \frac{2b \beta}{p^2} B \frac{1}{\beta} \left( \frac{1}{\alpha} + 2\beta, 1 - \frac{1}{\alpha} \right).
\]

Similarly \( Q(p^*) \) can be uniquely determined by the function
\[
g^*(p^*) = \left[ 1 - G^*(p^*) \right] \mu^*(p^*)
\]
which can be proved as follows.
From (5.26)
\[ g^*(p^*) = \frac{2}{(1-p^*)^2} \int_0^1 (1-u)Q(u) \, du \]

Differentiating with respect to \( p^* \), we obtain
\[ Q(p^*) = g^*(p^*) \frac{(1-p^*)g''(p^*)}{2} \] (5.31)

The characterization results obtained in the previous sections can also be proved using quantile function approach.

Now for MLF,
\[ \alpha(p) = \frac{1}{\lambda_2} \left[ \frac{p^{\lambda_2+1}}{\lambda_3+1} + \frac{1-(1-p)^{\lambda_4}}{\lambda_4(\lambda_4+1)} \right] \] (5.32)

\[ \alpha^*(p^*) = \frac{1}{\lambda_2} \left[ \frac{p^{\lambda_2}}{\lambda_3+1} + \frac{(1-p^{\lambda_2})^{\lambda_4+1}}{\lambda_4+1} - \frac{p^{\lambda_2}}{\lambda_3} + \frac{1}{\lambda_3(\lambda_3+1)} \right] \] (5.33)

\[ G(p) = \frac{1}{\lambda_2} \left[ \frac{p^{\lambda_3}}{(\lambda_3+2)(\lambda_3+1)} + \frac{1+(1-p)^{\lambda_4+1}}{p\lambda_4(\lambda_4+1)} + \frac{2[(1-p)^{\lambda_4+2} - 1]}{p^2\lambda_4(\lambda_4+1)(\lambda_4+2) - \frac{1}{\lambda_4}} \right] \] (5.34)
In conclusion, we have shown that the modified lambda family has the potential to be used as a model of income, because of its flexibility to assume different distributional shapes. In view of the quantile functions involved in the distribution, it is easier to generate random numbers than many of the competing parametric models. Since analysis of income data usually involves a large number of observations, the asymptotic properties seem to apply with a good amount of accuracy. Further there is closed form expressions for many of the measures of income inequality, making them easier to compute. We have also presented a few theoretical results that help the identification of the distribution of income given the income gap ratio or the truncated Gini coefficient at different values of the poverty or affluence limit. The poverty and affluence measures being directly expressed in terms of the $\alpha$'s and $G$'s, the results established here have relevance in that context also.

In the present study MLF is fitted to one income data remarkably well. It is essential to verify the goodness of fit of the family to the income datas of various countries in different time periods. This model can also be used to project the income distributions of future period. These problems are expected to be presented in a subsequent work.