CHAPTER VI
OPTIMUM UTILISATION OF RESOURCES FOR A PRODUCTION PLANNING PROBLEM, UNDER DIFFERENT COST STRUCTURES AND MARK-UP OF PRICES

6.1 INTRODUCTION:

A business firm always decides about the amount of inventory that it should keep in order to run its business more smoothly and efficiently. Due to this, it is of considerable importance as to determine the optimum order quantity that the firm should keep from time to time. In view of this, certain standard inventory models, like EOQ & ELS model may be considered. The decision variables contain those resources which are to be produced, planned, organised, co-ordinated and harmoniously related and controlled with an objective of achieving desirable results to increase the profitability of the business firm.

This in turn requires to fix up the production schedules with an objective or fulfilling business man's requirements.

The purpose of this chapter is to formulate a production planning problem considering an appropriate inventory model with an objective of maximising the net return of the producer and to determine the optimum levels of
output, optimum values of labour employed and capital invested. The costs due to the advertisement expenditure and obsolescence are also to be considered.

The sensitivity analysis is carried out to illustrate the application of this model.

6.2 ASSUMPTIONS :-

Following are the assumptions for this model.

(1) Lots of size q are replenished for each replenishment.

(2) Demand rate is R units/year which is a function of the unit selling price p &. Here we have the case of a variable demand function.

(3) Shortages are not allowed to occur.

(4) Lead time is zero.

(5) Replenishing size is constant with the lot size of q units per replenishment.

(6) Unit inventory holding cost is $C_1 \ (\&/\text{unit/year}).$

(7) The unit replenishment cost is $C_3 \ &. per order which is fixed.

(8) Advertisement cost is a fraction of the total revenue and it is given by $\alpha_1 p R (\&/\text{year})$ ($\alpha_1$ is a given constant, $0 < \alpha_1 < 1$).
(9) Cost due to the damaged goods is given by \( \alpha_2 C(q)R \), where \( \alpha_2 \) is lot fraction defective and \( C(q) \) represents the cost of \( q \) units replenished, thus allowing for the quantity discount case.

(10) The net revenue (profit) is a real and continuous function of the lot size \( q \).

(11) The production firm produces only a single commodity having its output \( Q \) by employing the linear production function model which consists of only two variable input factors of production—viz capital investment \( K \) & labour employed \( L \).

6.3 FORMULATION OF THE MODEL:

A manufacturer is implementing the frequent advertisement efforts to promote the sales of the produced units. The units which are stored in the warehouse can get damaged. The production function of the manufacturing firm is assumed to have linear form which can be written as

\[ Q = \alpha K + \beta L \]

where \( \alpha \) & \( \beta \) are the marginal physical products of capital and labour respectively.
The problem for the manufacturer is to determine the optimum lot size to be replenished under the given market conditions, with a view to maximise his net return.

6.4 DERIVATION OF THE MODEL:

Using the above assumptions the model is derived as under.

Here we use the production function in its linear form as given by

$$Q = \alpha K + \beta L \quad (\alpha > 0, \beta > 0) \quad ...(6.1)$$

Then we consider the ratio of the capital K and the labour L as fixed. Suppose it is given by

$$\frac{K}{L} = m \quad \text{or} \quad K = Lm \quad ... (6.2)$$

This ratio determines the share of input factors of production, where both L and K are in monetary units.

Thus:

$$Q = \alpha m L + \beta L$$

$$= (\alpha m + \beta)L$$

also let \( q = a + b Q \) where \( q < Q \)

(which means that a fraction of the lot produced is considered for replenishment).

$$q = a + A L \quad ...(6.3)$$

Where \( A = b(\alpha m + \beta) \) & \( A > 0 \)
Production cost = C Q

= C(q - a)/b \hspace{1cm} (6.4)

= C A L/b \hspace{1cm} (6.5)

Advertising cost = \alpha_1 p R \hspace{1cm} (6.6)

Let P = \theta C(q) \hspace{1cm} (\theta > 1) \hspace{1cm} (6.7)

denote the unit price related with the unit cost function by means of mark-up parameter \theta.

Cost of damaged goods = \alpha_2 C(q) R \hspace{1cm} (6.8)

Total inventory cost = T I C

= \left[ C_1 q/2 \right] + \left[ C_3 R/q \right] \hspace{1cm} (6.9)

Hence total cost

T C = (Production cost) + (Advertising cost) +

(cost of damaged goods) + (T I C) \hspace{1cm} (6.10)

So that

T C = \left( C(q - a)/b \right) + \alpha_1 p R + \alpha_2 C(q) R + \left( C_1 q/2 \right) + \left( C_3 R/q \right) \hspace{1cm} (6.11)

Total Revenue = TR = R \left[ p - C(q) \right] \hspace{1cm} (6.12)

= \Psi(q) \left[ p - C(q) \right] \hspace{1cm} (6.13)

Hence Net Revenue (PROFIT) P = T R - T C \hspace{1cm} (6.14)

\Rightarrow P = \Psi(q) \left[ p - C(q) \right] - \left[ \left( C(q - a)/b \right) + \alpha_1 p C(q) + \alpha_2 C(q) \Psi(q) + \left( C_1 q/2 \right) + \left( C_3 \Psi(q)/q \right) \right] \hspace{1cm} (6.15)
Where \( R = \psi(q) = f(\theta C(q)) \)

In the above profit function, let us consider here a specific demand function given by Kotler [10], as under

\[ R = s \frac{f}{p^n} \quad \cdots (6.16) \]

Where \( s = \text{Constant } (s > 0) \)

\( f = \text{Frequency of advertisement} \)

\( \eta = \text{Elasticity of demand } (\eta > 0) \)

So that \( R = \psi(q) = s \frac{f}{\theta^{\eta} (C(q))} \)

We further assume that \( \eta = 1 \)

then \( R = \psi(q) = s \frac{f}{\theta C(q)} \) \quad \cdots (6.17)

Substituting equation (6.15) into equation (6.15) we get the profit function as

\[ P = \psi(q)(p - C(q)) - \left[ \frac{(C(q-a) + \alpha_1 s f + (\alpha_2 s f/\theta)}{2} + \frac{(C_1 q/2) + [C_3 s f/\theta C(q) q]} \right] \quad \cdots (6.18) \]

An attempt is made to investigate the optimum value of the input factors \( L^* \) and \( K^* \), optimum amount of production \( Q^* \) and optimum order quantity \( q^* \), so as to maximize the profit \( P \) of the system as a whole.

**Particular Cases**

To illustrate the above model, let us consider here the specific forms of quantity discounts as under.
6.4.1. Case I : Linear Quantity Discount :

The unit cost function \( C(q) \) assumes the linear quantity discount case as given by

\[ C(q) = g - h q \quad (g > 0, h > 0, g > h) \]

Hence from (6.18) and (6.3)

\[ \text{Profit } P = \left( \frac{\theta - 1}{\theta} \right) s f + \left( C_{\text{A}} L/b \right) + \{ \left( C_{\text{A}} (a + AL)/2 \right) + C_{\text{S}} f/\theta \} [g(a + AL) - h(a+AL)^2] \]

\[ \ldots (6.19) \]

The necessary conditions for maximization of profit is

\[ \frac{d P}{d L} = 0 \quad \text{which gives} \]

\[ h^2 L^4 - 2Ah(g - 2ah)L^3 + \left[ A^2 (g - 2ah^2) - 2h(ga - \frac{ha^2}{2}) \right] L^2 \]

\[ \left[ 4bhAC_{\text{S}} f/(2C + bC_{\text{I}}) \theta + 2A(ga - ha^2) (g - 2ah) \right] L + \]

\[ [(ga - ha^2)^2 - 2bC_{\text{S}} f(g - 2ah)/(2C + bC_{\text{I}}) \theta] = 0 \]

\[ \ldots (6.21) \]

and the sufficiency condition \( \frac{d^2 P}{dL^2} < 0 \implies \)

\[ -(2A^2 C_{\text{S}} f/\theta) \left[ g^2 - 3h(a + AL)(g - h(a + AL))/(a + AL)^3 (g - h(a + AL))^3 \right] < 0 \]

Solving (6.19), we get the optimum labour input \( L^* \) from which \( K^* \) and \( P^* \) can be obtained.

6.4.1.1. Application :

To illustrate the above model, let us consider here a numerical example as under.
Constants: \( g = 50, h = 0.1, a = 1, b = 0.1, \alpha = 0.3, \alpha_1 = 0.1, \alpha_2 = 0.05, m = 2, \beta = 0.4, s = 50, f = 40. \)

Parameters: \( c = 0.5, c_3 = 10, c_1 = 1. \)

Mark-up Parameter: \( \theta = 2, 3, 4. \)

With the above hypothetical values, the polynomial equation (6.21) is solved to determine the optimum value of \( L \) such that \( P \) is maximum. The optimum solution is as given in table 6.1.

**Table 6.1**

<table>
<thead>
<tr>
<th>Mark-up Parameter (( \theta ))</th>
<th>Labour (( L^* ))</th>
<th>Capital (( K^* ))</th>
<th>Output (( Q^* ))</th>
<th>Lot Size (( q^* ))</th>
<th>Profit (( P^* ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>70.26</td>
<td>140.51</td>
<td>70.26</td>
<td>8.026</td>
<td>33.33</td>
</tr>
<tr>
<td>3</td>
<td>68.32</td>
<td>136.64</td>
<td>68.32</td>
<td>7.832</td>
<td>70.20</td>
</tr>
<tr>
<td>4</td>
<td>67.12</td>
<td>134.24</td>
<td>67.12</td>
<td>7.712</td>
<td>88.77</td>
</tr>
</tbody>
</table>

From table 6.1, it may be concluded that with the increase in the mark-up parameter \( \theta \), the optimum profit \( P^* \) also increases and the optimum value of the lot size \( q^* \), Labour \( L^* \) and Capital \( K^* \) are decreasing. This is also visualised from the graph G/6.1.
6.4.1.2 Sensitivity Analysis:

The model can also be tested for its sensitivity by changing the values of all the parameters by 5% and 10% simultaneously or varying only one of the parameters at a time while all other remaining parameters are kept fixed.

**TABLE 6.2**

TOTAL SENSITIVITY

<table>
<thead>
<tr>
<th>Mark-up</th>
<th>Parameter</th>
<th>Value of P*</th>
</tr>
</thead>
<tbody>
<tr>
<td>All parameters increasing:All parameters decreasing simultaneously</td>
<td>θ by 5% : by 10% : by 5% : by 10%</td>
<td></td>
</tr>
<tr>
<td>2 : 31.14 : 29.16 : 35.44 : 37.43</td>
<td>(-6.57) : (-12.51) : (6.33) : (12.30)</td>
<td></td>
</tr>
<tr>
<td>3 : 68.11 : 66.21 : 72.18 : 74.12</td>
<td>(-2.98) : (-5.68) : (3.82) : (5.58)</td>
<td></td>
</tr>
<tr>
<td>4 : 86.75 : 84.89 : 90.70 : 92.59</td>
<td>(-2.28) : (-4.37) : (2.17) : (4.30)</td>
<td></td>
</tr>
</tbody>
</table>

(Note: The figures shown in parenthesis indicate percentage change in the expected net profit as compared to the corresponding solution given in table - 6.1).

In table 6.2, the values of the optimum profit are given when the mark-up parameter θ takes some particular values and all the other parameters observe simultaneous increase by 5% and 10% respectively. It is observed that as θ increases, the
optimum profit $P^*$ also increases when all the parameters increase their values simultaneously. Similar fact is also observed when for increasing the value of $\theta$, all the parameters observe simultaneous reduction by 5% or 10% respectively. Thus general tendency is to find an increase in $P^*$ when all the parameters increase or decrease simultaneously.

### Table 6.3

PARTIAL SENSITIVITY

<table>
<thead>
<tr>
<th align="left">Chang:Marking</th>
<th align="left">Value of $P^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td align="left">para-meter:</td>
<td align="left">Value of one parameter increasing</td>
</tr>
<tr>
<td align="left">$\theta$</td>
<td align="left">by 5%</td>
</tr>
<tr>
<td align="left">2</td>
<td align="left">31.69</td>
</tr>
<tr>
<td align="left">C</td>
<td align="left">3</td>
</tr>
<tr>
<td align="left">4</td>
<td align="left">87.18</td>
</tr>
<tr>
<td align="left">$\beta$</td>
<td align="left">3</td>
</tr>
<tr>
<td align="left">3</td>
<td align="left">69.99</td>
</tr>
<tr>
<td align="left">$\beta_0$</td>
<td align="left">4</td>
</tr>
<tr>
<td align="left">2</td>
<td align="left">33.24</td>
</tr>
<tr>
<td align="left">$\beta_3$</td>
<td align="left">3</td>
</tr>
<tr>
<td align="left">4</td>
<td align="left">88.58</td>
</tr>
</tbody>
</table>
(Note: The figures shown in parenthesis indicate percentage change in the expected net profit $\theta$ compared to the corresponding solution given in table 6.1).

Table 3 gives optimum profit when the mark-up parameter $\theta$ takes some particular value and only one of the other parameters increases or decreases by 5% or 10% respectively.

It is observed that when particular parameter increases by 5% or 10%, the profit $P^*$ also increases for the changing value of the parameter $\theta$. Similar fact is also observed when that particular parameter alone decreases by 5% or 10% respectively for the increasing value of $\theta$. Such a variation in $P^*$ is found to be more or less similar as compared to the original solution given in table - 6.1. Thus it may be concluded that with respect to sensitivity, the three parameters $C$, $C_1$ & $C_3$ are almost alike in nature.

6.4.2. Case : Hyperbolic Quantity Discount:

Let us consider here the hyperbolic quantity discount case given by $C(q) = d + e/q$ $(d > 0, e > 0)$ ... (6.23)

Hence $P = \theta (d + e/q)$

So that the profit function $P$ given in (6.18) reduces to

$P = \{(\theta - 1)/\theta\}s/f - [(C_1a/2) + ((2C + bC_1)/2b)AL + s(f\alpha + \alpha/\theta) + C_3s f/\theta(d(a + AL) + e)]$ ... (6.24)
The necessary condition \( \frac{dP}{dL} = 0 \) ...(6.25)
gives
\[
A^2 d^2 L^2 + (2aAd^2 + 2eAd)L + (a^2 d^2 + 2aed + e^2 - \{2bC_3 sfd/\theta(2C+bC_1)\}) = 0
\] ...(6.26)
That is
\[
M_1 L^2 + M_2 L + M_3 = 0
\] ...(6.27)
Where
\[
M_1 = A^2 d^2
\]
\[
M_2 = (2aAd^2 + 2eAd)
\]
\[
M_3 = a^2 d^2 + 2aed + e^2 - \{2bC_3 sfd/\theta(2C+bC_1)\}
\]
Hence solving equation (6.26) we get
\[
L = \frac{-M_2 \pm \sqrt{M_2^2 - 4M_1M_3}}{2M_1}
\] ...(6.28)
and the sufficiency condition \( \frac{d^2P}{dL^2} < 0 \) ensures to take
\[
L^* = \frac{-M_2 + \sqrt{M_2^2 - 4M_1M_3}}{2M_1}
\] ...(6.29)
Hence \( K^* \) and \( P^* \) can be determined from (6.2) and (6.24) respectively.

6.4.2.1 Application:

To illustrate the above model, let us consider here a numerical example as under.

Constants : \( \lambda = 0.1, \beta = 0.5, m = 5, a = 1, b = 0.1, d = 1, e = 2, \lambda_1 = 0.1, \lambda_2 = 0.2, s = 100, f = 20. \)

Parameters : \( C = 2, C_1 = 1, C_3 = 1. \)

Mark-up parameter : \( \theta = 2, 3, 4, 5. \)
With the above hypothetical values, the quadratic equation (6.25) is solved to determine the optimum value of $L$ so that $P$ is maximum. The optimum solution for the above numerical example is as given in table - 6.4.

**Table 6.4**

**Optimum Solution**

<table>
<thead>
<tr>
<th>Mark-up</th>
<th>Optimum</th>
<th>Optimum</th>
<th>Optimum</th>
<th>Optimum</th>
<th>Optimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>parameter</td>
<td>Labour</td>
<td>Capital</td>
<td>Output</td>
<td>Lot size</td>
<td>Profit</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$L^*$</td>
<td>$K^*$</td>
<td>$Q^*$</td>
<td>$q^*$</td>
<td>$P^*$</td>
</tr>
<tr>
<td>2</td>
<td>39.85</td>
<td>199.25</td>
<td>39.85</td>
<td>4.985</td>
<td>374.64</td>
</tr>
<tr>
<td>3</td>
<td>27.028</td>
<td>135.14</td>
<td>27.028</td>
<td>3.7028</td>
<td>827.19</td>
</tr>
<tr>
<td>4</td>
<td>19.387</td>
<td>96.935</td>
<td>19.387</td>
<td>2.9387</td>
<td>1058.52</td>
</tr>
<tr>
<td>5</td>
<td>14.173</td>
<td>70.865</td>
<td>14.173</td>
<td>2.4173</td>
<td>1177.25</td>
</tr>
</tbody>
</table>

From table 6.4 it may be concluded that with the increases in the mark-up parameter $\theta$, the optimum profit $P^*$ also increases and the optimum value of the lot size $q^*$, Labour $L^*$ and capital $K^*$ are decreased, which is also apparent from the graph G/6.2.

**6.4.2.2 Sensitivity Analysis**

The model can also be tested for its sensitivity by changing the values of all the parameters by 5% and 10% simultaneously or varying only one of the parameter at a time while all other remaining parameters are kept fixed.
TABLE 6.5
TOTAL SENSITIVITY

<table>
<thead>
<tr>
<th>Value of $p^*$</th>
<th>(\theta) by 5%</th>
<th>by 10%</th>
<th>by 5%</th>
<th>by 10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mark-up: All parameters increasing: All parameters decreasing simultaneously</td>
<td>((-3.01))</td>
<td>((-6.01))</td>
<td>((3.01))</td>
<td>((6.02))</td>
</tr>
<tr>
<td>2</td>
<td>363.38</td>
<td>352.11</td>
<td>385.91</td>
<td>397.18</td>
</tr>
<tr>
<td></td>
<td>((-3.01))</td>
<td>((-6.01))</td>
<td>((3.01))</td>
<td>((6.02))</td>
</tr>
<tr>
<td>3</td>
<td>818.55</td>
<td>809.91</td>
<td>835.83</td>
<td>844.47</td>
</tr>
<tr>
<td></td>
<td>((-1.04))</td>
<td>((-2.09))</td>
<td>((1.04))</td>
<td>((2.09))</td>
</tr>
<tr>
<td>4</td>
<td>1051.44</td>
<td>1044.37</td>
<td>1056.59</td>
<td>1072.66</td>
</tr>
<tr>
<td></td>
<td>((-0.67))</td>
<td>((-1.34))</td>
<td>((0.67))</td>
<td>((1.34))</td>
</tr>
</tbody>
</table>

(Note: The figures shown in the parenthesis are percentage change in the expected net profit as compared to the corresponding solution given in table 6.4).

The optimum profit increases with decrease (5% or 10%) in the values of all the parameters simultaneously for some change in the mark-up parameter \(\theta\), and exactly reverse phenomena is observed for corresponding increase in the parameter value as compared to the original solution.
### Table 6.6

**PARTIAL SENSITIVITY**

<table>
<thead>
<tr>
<th>Chang:Mark-:</th>
<th>Value of $p^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter:</td>
<td>Increasing</td>
</tr>
<tr>
<td>$\theta$</td>
<td>by 5% by 10%</td>
</tr>
<tr>
<td>: 2</td>
<td>370.74 367.00</td>
</tr>
<tr>
<td>: (-1.041)</td>
<td>(-2.039) (1.089)</td>
</tr>
<tr>
<td>C</td>
<td>824.56 822.05</td>
</tr>
<tr>
<td>: (-0.318)</td>
<td>(-0.621) (0.335)</td>
</tr>
<tr>
<td>: 4</td>
<td>1056.64 1054.87</td>
</tr>
<tr>
<td>: (-0.178)</td>
<td>(-0.345) (0.189)</td>
</tr>
<tr>
<td>$C_1$</td>
<td>827.10 827.01</td>
</tr>
<tr>
<td>: (-0.011)</td>
<td>(-0.022) (0.011)</td>
</tr>
<tr>
<td>: 4</td>
<td>1058.44 1058.37</td>
</tr>
<tr>
<td>: (-0.0076)</td>
<td>(-0.0142) (0.0066)</td>
</tr>
<tr>
<td>$C_3$</td>
<td>821.42 815.78</td>
</tr>
<tr>
<td>: (-0.698)</td>
<td>(-1.379) (0.716)</td>
</tr>
<tr>
<td>: 4</td>
<td>1053.52 1048.63</td>
</tr>
<tr>
<td>: (-0.472)</td>
<td>(-0.934) (0.484)</td>
</tr>
</tbody>
</table>

(Note: The figures shown in the parenthesis are percentage change in the expected net profit as compared to the corresponding solution given in table 6.4).
The optimum profit is increased for increase in the value of the parameter \( \theta \). When any one of the parameter decreases while all other parameters remain as fixed as compared to the original solution and exactly reverse phenomena is observed when any one of its parameter increased its value. When other parameters remain as fixed as compared to the original solution. From table 6.6 we find that the parameter \( C_3 \) is the most sensitive parameter as compared to all other parameters. The next sensitive parameters is \( C \). This means that the replenishment cost and the production cost play a significant role in changing the net profit of the inventory system.