1 INTRODUCTION AND SUMMARY

1.1 INTRODUCTION

Suppose that $X$ is a random variable or a random vector taking values in a set $\mathcal{X}$, whose probability distribution $P_\theta$ depends on an unknown parameter $\theta \in \Theta$, where $\Theta$ is a subset of the real line. The problem is to draw definite and sufficiently reliable conclusions about $\theta$ or a function $m(\theta)$ of $\theta$ using the observed data, say, $(X_1, X_2, \ldots, X_n) = \bar{X}$ on $X$ for fixed $n$. We shall denote $\bar{X}$ by $X$ for convenience. Some function of observations is used to do so. Any function $T(X)$ of observations may, in principle, be called an estimator of $m(\theta)$ and its particular value $T(x)$ an estimate. To select one among the several estimators, it is necessary to formulate criteria for their performance. The traditional requirements are those of unbiasedness and minimum variance.

Unbiasedness is a natural requisite for an estimator to be impartial. It is often introduced for convenience, when weaker conditions would be sufficient. The results of unbiased estimators can be extended for biased estimators with some minor modifications. Similarly, only small modifications are necessary to extend the results to mean square error instead of the variance.

Minimum variance is required to be introduced to ensure that the estimate is close to the true value of the parameter. A large class of estimators has asymptotic normality. In this situation, the variance is the best measure of dispersion when asymptotic normality occurs. In other cases, especially if an asymptotic distribution is well defined, it is still an important measure of how good an estimator is. When $\theta$ is a location parameter, there is a further justification for using the variance that in this case the best invariant estimator is unbiased when the squared error loss function is used. In view of this we restrict our attention to minimum variance unbiased estimation.

There are three approaches to obtain uniformly minimum variance unbiased estimator (UMVUE). The first one is to identify the class of unbiased estimators of zero and then to find an estimator which has zero covariance with every unbiased estimator of zero. The second approach is through Lehmann-Scheffe-Rao-Blackwell theorem. If there is a complete sufficient statistic, then a function of complete sufficient statistic is UMVUE of its expectation. Third approach is to obtain a bound
on the variance of any unbiased estimator of the parametric function, say \( m(\theta) \), to be estimated, and to search for the estimator whose variance achieves this bound.

The approach through the variance bound, to obtain UMVUE, leads to two important problems. The first, which we call, ‘the problem of construction’, is to construct a lower bound for the variance of all the unbiased estimators in a certain situation. The second, which we call, ‘the problem of attainment’, is to investigate whether the variance bound is attainable, and to find, if possible, an explicit expression for an estimator with minimum variance, in a specified situation. Different lower bounds on the variance are provided by different inequalities. Therefore, the lower bounds on the variance of the same estimator provided by different inequalities can be different. Therefore, a natural and third problem which we call, ‘the problem of comparison’, comes up.

Once the form of a bound on the variance, in a situation, is provided, it is used not only in unbiased estimation but also for biased estimators. The lower bound on the variance has two applications—(i) to prove if a given estimator is locally or uniformly minimum risk estimator among all estimators with the same expectation, and (ii) to prove if a given estimator is admissible.

1.2 BRIEF HISTORY

The history of lower bounds on the variance of estimators is long and has many contributors. The widely known bound and the basis of this theory is the Cramér-Rao bound (Cramér(1946), Rao (1945)). It is equal to the inverted value of Fisher’s information quantity (Fisher (1922),(1925)). The earliest expression involving ‘Fisher information’ is given by Pearson and Filon (1898) in a different context. Doob(1936) and Dugué (1937) also used Fisher information in their expressions. There intension is not to obtain UMVU estimator or to provide a bound on the variance of the estimator. In fact, what Doob has done is as follows- he considered a class \( \{T_n, n \geq 1\} \) of maximum likelihood estimators of \( \theta \) which follows asymptotic normal distribution and proved that the asymptotic variance of \( \sqrt{n} (T_n - \theta) \) is the inverted value of Fisher information quantity. He has also obtained least upper bound on the Fisher information quantity. The actual problem of obtaining minimum variance unbiased estimator is considered by Aitken and Silverstone (1942). Under
certain restrictions they proved that there exists a minimum variance unbiased estimator $T(X)$ of $\theta$, if the derivative of the $\log f(x, \theta)$ can be written as

$$\frac{d \log f(x; \theta)}{d\theta} = \frac{[T(x) - \theta]}{\lambda(\theta)} \quad (1.1)$$

Where, $E(T(X)) = \theta$ and $\text{Var}[T(X)] = \lambda(\theta)$. They also showed that $E\left[\frac{d \log f(x; \theta)}{d\theta}\right] = 0$ (a result already proved by Pearson (1898)). If $T(X)$ satisfies (1.1), they showed that

$$\text{Var}(T(X)) = \frac{1}{E\{-\frac{d^2 \log f(X; \theta)}{d\theta^2}\}} = \frac{1}{\left[[E\left\{\frac{d \log f(X; \theta)}{d\theta}\right]\right]^2]}$$

Thus, it is clear that Aitken and Silverstone have calculated Cramér-Rao bound as the variance of the estimator $T(X)$ when (1.1) holds. But, they as well as others mentioned above, did not obtain the bounds for the variances.

It seems that Fréchet (1942) has given the inequality which is now known as the Cramér-Rao inequality in the statistical literature, after its explicit and independent publication by Cramér (1946) and Rao (1945). Bhattacharyya (1946) generalized Rao’s results, under some additional conditions, to give a sequence of sharper bounds. Darmois (1945) extended Fréchet’s inequality to n-dimension.

Cramér-Rao inequality and Bhattacharyya inequality hold under certain regularity conditions. Therefore, many other authors tried to provide the lower bounds on the variance of estimators, by dropping the regularity assumptions or by giving sharper bounds.

Barankin (1949), with a complementary remark in the (1951) paper, starts with the goal to obtain (locally) attainable variance bounds. His results are very general, but difficult to apply. He demonstrates that the Cramér-Rao and Bhattacharyya bounds are special cases of bound obtained by him. The variance bounds “without regularity conditions” appeared around 1950. Hammersley (1950) and Chapman and Robbins (1951) use the same idea to give bounds without regularity conditions. Fraser and Guttman (1952) used the idea applied by Chapman and Robbins to give Bhattacharyya bounds without regularity assumptions. In (1952),
Kiefer gave a modified form of Barankin’s bound, and showed at the same time that it is a generalization of Hammersley-Chapman and Robbins bound. Polfeldt (1970) writes: The Kiefer bound has found little application, except for the examples given by Kiefer himself, of already well-known results derived afresh, and the recent results of Weibull(1969)( which we could not trace as they were not published). This is due to the difficulties involved in finding the maximizing priors involved in the inequality. On the other hand, it has been regarded as the last word in variance bounds, since it has been held to be attainable, on the ground of Barankin’s results. Blischke (1969), Polfeldt (1967),(1970) and many others including Mitra(1954), Vincze(1979), Khatri (1980), Chatterji(1982) etc., have provided some bounds.

Fréchet (1943) as well as many others mentioned that Cramér-Rao lower bound is attained only if the family of distributions of $X$ is one parameter exponential. In this way, together with providing a lower bound on the variance of unbiased estimators, Fréchet has studied its attainment. Fend (1959) gives proof of the above result but his proof is not rigorous. He provided the situation where Bhattacharyya bound is attained by considering a larger family of distributions. His result for the larger family seems to be vacuous. Revealing this we modified Fend’s results on attainment of Bhattacharyya bound. Our results are similar to those by Zacks (1971) and are presented in my M. Phil. dissertation. A rigorous account on attainment of Cramér-Rao lower bound is provided by Wijsman (1973) and Joshi (1976). Sen and Ghosh (1976) provided results about attainment of Chapman-Robbins bound. The attainment of Kiefer bound is investigated by Bartlett (1982), Jadhav and Prasad (1986-87).

The lower bounds on the variance of an estimator provided by different inequalities may not be the same. This raises a problem of comparison of the various bounds. Though various authors like Bhattacharyya, Chapman and Robbins compared the bounds obtained by them with Cramér-Rao lower bound, up to some extent detailed study of relative magnitudes of bounds is done for the first time by Sen and Ghosh(1976). They compared bounds due to Fréchet-Cramér-Rao, Bhattacharyya, and Chapman and Robbins. Similar results comparing Kiefer bound with the other bounds are given by Jadhav and Prasad (1986-87). We restrict ourselves to the field of finite and censored samples.
1.3 CHAPTER-WISE SUMMARY

Chapter 1 consists of introduction, brief history, chapter wise summary, notation and conventions.

In Chapter 2, some results due to Blyth and Roberts (1972) to provide a general inequality of Cramér-Rao type are given. This is used to obtain lower bounds on the variance of an unbiased estimator. This approach unifies the method of deriving lower bounds on the variance. Some of the standard inequalities giving lower bounds for the variance are derived as particular cases. The various solutions to the problem of construction of lower bound on variance such as bounds due to Cramér and Rao, Bhattacharayya, Chapman and Robbins, Barankin, Vincze and Kiefer are discussed. Section 2.1 is an introduction. In Section 2.2 we discuss the best Swarz’s inequality which would be sharp as well as translation invariant as it has to provide lower bound on the variance. In Section 2.3, we discuss the inequalities of Cramér-Rao type. These are the translation invariant inequalities providing lower bounds on the variance of unbiased estimator of the parameter or its function which has unbiased estimator. Section 2.3 deals with the general results on Cramér-Rao type bounds, their existence and the functions giving bounds. The bound giving functions (bgfs) are introduced and their properties are discussed. This section includes interesting results like: A function $V(X, \theta)$ of both $X$ and $\theta$ can provide a lower bound on the variance of an estimator $T(X)$ if $V(X, \theta)$ depends on $T(X)$ only through $(T(X))$.

i. A necessary condition for $V = V(X, \theta)$ to give an inequality of Cramer Rao type is that $V$ depends on $X$ only through the minimal sufficient statistic.

ii. The above condition is also sufficient, when the minimal sufficient statistic is complete.

The bound giving function(bgf) $V = V(X, \theta)$,with $0 < Var(V) < \infty$, gives an inequality of Cramér-Rao type if and only if, $V$ has 0 covariance with every finite variance unbiased estimator of zero.

A necessary and sufficient condition for the existence of an achievable Cramér-Rao type bound for the variance of an estimator $T$ having a specified expectation $m(\theta)$ is that $m(\theta)$ possess UMVU estimator with positive variance. In Section 2.4, we give the forms of bgf's $V$ which yield inequalities obtained by Cramér(1946) and Rao(1945), Bhattacharyya(1946), Barankin(1949), Chapman and Robbins(1951), Kiefer(1952) and Vincze(1979).
In Chapter 3, Section 3.1 deals with introduction and summary. In Section 3.2, stating Kiefer’s inequality and bound in brief we provide necessary and sufficient condition for its attainment. The ideal estimation equation and generalized difference are discussed in Sections 3.3 and 3.4 respectively. In 3.4 the necessary and sufficient condition for the attainment of Kiefer bound by Jadhav and Prasad and Bartlett’s criterion through ideal estimation equation and generalized difference are revealed to be equivalent. In Section 3.5 we state Kiefer’s inequality for variance of unbiased estimator of any parametric function \( m(\theta) \). This Section makes it clear that the (bgf) \( V(X, \theta) \) and the prior probability distributions \( G_1, G_2 \) change with the parametric function to be estimated. Thus, the bgf and prior distributions applicable to provide Kiefer bound when \( \theta \) is to be estimated are shown to be not applicable when some function \( m(\theta) \) is to be estimated. The need to reparametrize the distributions is revealed to obtain Kiefer bound. This brings out the complications in finding Kiefer bound and the reasons of its limited applications.

In Chapter 4, Section 4.1 is introduction and summary. In Section 4.2 the probability density function of left truncated random variable is considered in its natural form, namely,

\[
f(x; \theta) = \frac{q(x)}{Q(b) - Q(\theta)}, -\infty < a < \theta < x < b < \infty
\]

(1.3.1)

It is proved that \([Q(b) - Q(\theta)]^r, r > n/2\), has uniformly minimum variance unbiased estimator whose variance attains Kiefer bound (UMVUKBE). It also proved that the survival function, Kiefer bound (i.e., the variance of UMVUE), etc. have UMVUKBEs. In Section 4.3, the estimation of any function \( m(\theta) \) is considered. Tate (1959) has given the expression for unbiased estimator of this function \( m(\theta) \). We modified it for the left truncated density in the natural form and tried to find its probability distribution. We apply a novel idea for this. The idea is successful when the normalizing constant is independent of the parameter. Though, we intended to get lower bound on the variance of the estimator, instead of the bound, we could provide UMVU estimator for the variance of UMVU estimator of \( m(\theta) \). The results are illustrated by several examples.

In Chapter 5, Section 5.1 is introduction and summary. In Section 5.2 the pdf of right truncated r.v. is written in its natural form as
Using this density, it is proved that the parametric functions such as \([Q(\theta) - Q(a)r > -n/2];\) variance of this estimator, the survival function etc. have UMVUKBE. Section 5.3 considers the estimation of any parametric function \(m(\theta).\) The expression of unbiased estimator for \(m(\theta)\) for the density (5.3.2) is obtained. We obtain UMVUE of variance of UMVUE of \(m(\theta).\) It is tried to obtain probability distribution of UMVUE of \(m(\theta)\) using a novel idea in Section 5.4. The idea is successful if the normalizing constant is independent of the parameter. The results are illustrated by several examples. The origin of most of these ideas is uniform probability distribution on \((0, \theta).\) Therefore, in Section 5.5, we consider various parametric functions such as raw as well as central moments, distribution function, survival function, Kiefer bounds, various probabilities of uniform r.v. on \((0, \theta).\) It is shown that most of the parametric functions in use occurring in uniform probability distribution have UMVUKBE. This shows that the applications of Kiefer bound can still be enhanced on large scale.

Chapter 6 deals with the probability distributions both the ends of whose support depend on the parameter. Here, we consider the density given by

\[
f(x; \theta) = \frac{q(x)}{Q(\theta) - Q(a)}, \quad -\infty < a < x < \theta < b < \infty \tag{1.3.2}
\]

Section 6.1 consists of introduction and summary. In this case the problem to provide Kiefer bound on the variance of unbiased estimators of parameter or functions of the parameter was not dealt earlier. We deal with this problem in this chapter. Let the support be \((C_1(\theta), C_2(\theta)).\) The probability distributions on this support may or may not admit a single sufficient statistic. In Section 6.2 we consider the case when there is no single sufficient statistic. Here, \(Z = X_{(1)}\) and \(Y = X_{(n)}\) are jointly sufficient for \(\theta.\) Based on \(Z, Y\) we deal with the estimation of \(\varphi = C_2(\theta) - C_1(\theta)\) and provide attainable Kiefer bound on variance of its estimator. In the following sections we consider the situations admitting single sufficient statistic and prove that this statistic provides UMVUKBE of \(\varphi.\) In Section 6.3 we consider \(C_1(\theta)\) to be monotone.
decreasing and \( C_2(\theta) \) to be monotone increasing. Here the single sufficient statistic provides UMVUKBE of \( \varphi \). In Section 6.4 we consider \( C_1(\theta) \) to be monotone increasing and \( C_2(\theta) \) to be monotone decreasing. Here also the single sufficient statistic provides UMVUKBE of \( \varphi \). These results are illustrated by suitable examples.

Chapter 7 deals with censored samples from truncated distributions and the distributions both ends of whose support depend on the parameter. This is the first time when Kiefer bound is thought of in the context of censored samples. The Section 7.1 is introduction and summary. In Section 7.2, it is proved that the variance of unbiased estimator of \( Q(\theta) \) based on type II left censored sample from left truncated family of distributions attains Kiefer bound. This bound is compared with its value computed from a complete (uncensored) sample. The Kiefer bound comes out to be an increasing function of number of censored observations \( r \). In this case first \((r-1)\) observations are not available. If \( r = 1 \), the results from censored and complete sample coincide. The variance of the estimator continues to attain the concerned Kiefer bound. Section 7.3 is devoted to Kiefer bound from right censored sample from right truncated distribution. It is proved that the unbiased estimator based on maximum observation from this censored sample is UMVUKBE. Kiefer bound giving the variance is an increasing function of number of censored observations. The variance of UMVUKBE based on censored sample is more than or equal to that based on complete sample. In Section 7.4, doubly censored sample from left truncated family is considered. These results match with the results obtained from left censored sample from left truncated family of distributions discussed in Section 7.2. Section 7.5 deals with doubly censored sample from right truncated distribution. It is shown that these results match with those from type II right censored sample from right truncated distributions discussed in Section 7.3. In Section 7.6, we consider the density (1.3.3) and find Kiefer bound on the variance of unbiased estimator of \( \varphi = C_2(\theta) - C_1(\theta) \) based on doubly censored sample. It is proved that the estimator based on the sample range is UMVUKBE. The results are compared with those obtained from the complete sample. The results are illustrated with several examples.

Chapter 8 is devoted to the comparison of magnitudes of various lower bounds on variance of an unbiased estimator. Section 8.1 is introduction and short summary. In Section 8.2 we provide the results such as: If Cramér-Rao bound exists, it is not
greater than Chapman - Robbins bound. Kiefer bound is not less than Chapman-Robbins bound. Thus, Cramér-Rao bound is smaller than or equal to Kiefer bound. The results are illustrated by examples. Section 8.3 is devoted to compare Chapman-Robbins bound and Kiefer bound. Kiefer bound is proved to be tighter than Chapman-Robbins bound in non-regular families of distributions. Even Sen and Ghosh (1976) did not talk about Chapman-Robbins bound when both the ends of the support of the distribution depend on the parameter. We compute Chapman-Robbins bound and Kiefer bound and compare them. We prove that Kiefer bound is attainable and Chapman-Robbins bound is smaller than Kiefer bound which is not attainable. The results are illustrated by examples in Section 8.3 itself.

Chapter 8 is followed by appendix. It is to provide a brief introduction of life and work of Professor Jack Karl Kiefer who introduced the variance bound which is known by Kiefer bound.

1.4 NOTATION AND CONVENTIONS

\( X \) A random variable or a random vector
\( \mathcal{X} \) The set of values of \( X \).
\( \Theta \) The set of possible values of the unknown parameter \( \theta \)
\( f(x; \theta) \) The probability density or probability mass function of \( X \) when \( \theta \) is the true value of the parameter.
\( \mathcal{X}_\theta = \{ x; f(x; \theta) > 0 \} \). Obviously, \( \mathcal{X}_\theta \subset \mathcal{X} \).
\( \Theta_\theta = \{ \gamma; \theta + \gamma \in \Theta \} \).
\( m(\theta) \) The parametric function to be estimated.
\( \Phi_\theta = \{ \varphi; \varphi \in \Theta, \varphi \neq \theta, m(\theta) \neq m(\varphi), \mathcal{X}_\varphi \subset \mathcal{X}_\theta \} \).
\( A(\theta) \) Cramér-Rao bound on the variance of unbiased estimator of \( \theta \).
\( B_k(\theta), k \geq 1 \) \( k \)th Bhattacharayya bound on the variance of unbiased estimator of \( \theta \).
\( C(\theta) \) Chapman-Robbins bound on the variance of unbiased estimator of \( \theta \)
\( B(\theta) \) Barankin bound on the variance of unbiased estimator of \( \theta \)
\( K(\theta) \) Kiefer bound on the variance of unbiased estimator of \( \theta \).
\( V(\theta) \) Vincze bound on the variance of unbiased estimator of \( \theta \).
\( (\mathcal{X}, \mathcal{F}, \mu) \) The measure space on \( \mathcal{X} \) with \( \mu \ \sigma \)-finite
\( (\Theta, \mathcal{B}, \nu) \) The measure space on \( \Theta \) and unless otherwise specified \( \nu \) will be Lebesgue
Unless otherwise specified, the quantities like expected value, variance, covariance, probability etc. are calculated when \( \theta \) is assumed to be fixed at the true value of the parameter. Therefore, while writing the above quantities suffix \( \theta \) will be suppressed. The result (a.b.c) is the \( c^{th} \) result in the section b of chapter a.