7. KIEFER BOUNDS FROM CENSORED SAMPLES

7.1 INTRODUCTION

In the non-regular family of distributions the lower bounds on the variance such as Cramer- Rao bound, Bhattacharayya bounds cannot be used as the support depends upon the parameter and the regularity conditions are violated. The bounds provided by the inequalities due to Chapman and Robbins(1951), Fraser and Guttman(1952), Hammersley(1950), Kiefer (1952), Vincze(1976) can be applied in non-regular situations. Amongst these bounds only the bound due to Kiefer is known to attain by the variance of UMVU Estimators of parameter $\theta$ in a few situations. Then also Kiefer bound is not very familiar though it is regarded as the final word in bounds. Our efforts are to find its more and more applications. Blischke et.al. (1965-69), Polfeldt (1970), Akahira and Ohyauchi (2007) extended Kiefer’s results for asymptotic situations. Bartlett (1982) extended them for parameters of a few more probability distributions. Jadhav and Prasad (1986-87) extended those for some parametric functions in a family of distributions and gave a necessary and sufficient condition for the attainment of Kiefer bound. Jadhav and Shanubhogue (2014) provided attainable Kiefer bounds for left and right truncated distributions and the distributions when both the ends of the support of the distribution depend on the parameter. All these studies are based on either a small complete sample or a large sample. But in the situations such as life testing experiments these results can not be suitable. In such situations censored samples are chosen using various censoring schemes. Now, we obtain Kiefer bounds for the type II left, right and doubly censored samples from left and right truncated distributions as well as distributions with both the ends of the support depending on parameter. In Section 7.2 we obtain Kiefer bound on the variance of unbiased estimator of parametric function involved in left truncated probability distributions from type II left censored sample and prove that it is attained by the variance of UMVUE. Section 7.3 deals with providing a UMVUKBE of parametric function in the pdf of right truncated family of distributions. Sections 7.4 and 7.5 deal with doubly censored samples from left and right truncated families of distributions respectively. UMVUKBE of parametric functions in densities are obtained. The results are compared with those from uncensored samples and are illustrated with examples. In Section 7.6 we deal with
doubly censored sample from a distribution both ends of whose support depend on the parameter. The range has UMVUKBE. The results are compared with those from uncensored sample and are illustrated by examples.

7.2 KIEFER BOUND IN TYPE II LEFT CENSORED SAMPLE IN LEFT TRUNCATED FAMILY

We consider a sample in which first (r-1) failures are not observed or are not available. Therefore, observations from r\textsuperscript{th} order statistics onwards only are available. Thus, we have, \(X_{(r)}, X_{(r+1)}, ..., X_{(n)}\). Let the parent population be left truncated r.v. with p.d.f.;

\[
f(x; \theta) = \frac{q(x)}{Q(b) - Q(\theta)} ; -\infty < a < \theta < x < b < \infty \tag{7.2.1}
\]

Note that \(q(.)\) is a positive real valued function with integration \(Q(.)\) so that (7.2.1) is quite scientific and natural. It can be seen that the \(r\textsuperscript{th}\) order statistics \(X_{(r)}\) is complete sufficient for \(\theta\). Then we have;

**Theorem 7.2.1**

The variance of UMVU estimator of \(Q(\theta)\) based on minimum observation in left censored sample from left truncated distribution (7.2.1) attains Kiefer bound. That is, the estimator is minimum variance unbiased Kiefer bound estimator (UMVUKBE).

**Proof.**

For \(X\) with p.d.f. (2.1), \(F(x) = \frac{Q(x) - Q(\theta)}{Q(b) - Q(\theta)}\) and \(1 - F(x) = \frac{Q(b) - Q(x)}{Q(b) - Q(\theta)}\), the p.d.f. of \(X_{(r)}\) is given by

\[
g(x; \theta) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x), \quad 0 < x < b \tag{7.2.2}
\]

\[
= \frac{n!}{(r-1)! (n-r)!} \frac{[Q(x) - Q(\theta)]^{r-1} [Q(b) - Q(x)]^{n-r}}{[Q(b) - Q(\theta)]^n} q(x), Q(\theta) < Q(x) < Q(b) \tag{7.2.3}
\]

Let \(Q(\theta) = \varphi\). Therefore, the pdf of \(X_{(r)}\) becomes

\[
g(x; \theta) = \frac{n! [Q(x) - \varphi]^{r-1} [Q(b) - Q(x)]^{n-r} q(x)}{(r-1)! (n-r)! [Q(b) - \varphi]^n}, \quad \varphi < Q(x) < Q(b) \tag{7.2.4}
\]
For each fixed $\varphi \in \Phi = \{\varphi; \ g_{r:n}(x_{(r)}; \varphi) > 0 \} = (0, Q(b))$.

Let $\Phi_\varphi = \{h; (\varphi + h) \in \Phi\} = (-\varphi, Q(b) - \varphi)$ on subset $(0, Q(b) - \varphi)$ of $(-\varphi, Q(b) - \varphi)$. Let us define prior distributions-

$$dG_1(h) = \frac{[n+1][Q(b) - \varphi - h]^{n+1}dh}{[Q(b) - \varphi]^{n+1}}, \ 0 < h < Q(b) - \varphi \quad \text{and} \quad G_2(h) = (7.2.5)$$

$$\Delta_1(Q(b) - \varphi) = E_1(h)$$

$$= \int_0^{Q(b) - \varphi} h dG_1(h)$$

$$= \frac{(n+1)}{[Q(b) - \varphi]^{n+1}} \int_0^{Q(b) - \varphi} h(Q(b) - \varphi - h)^n dh$$

$$= \frac{-n(n + 1)}{[Q(b) - \varphi]^{n+1}} \left\{ \left[ \frac{(Q(b) - \varphi - h)^{n+1}Q(b) - \varphi}{-(n+2)} \right]_0^{Q(b) - \varphi} - (Q(b) - \varphi) \left[ \frac{(Q(b) - \varphi - h)^{n+2}}{-(n+1)} \right]_0^{Q(b) - \varphi} \right\}$$

$$= \frac{-n(n + 1)(Q(b) - \varphi)^{n+2}}{[Q(b) - Q(\theta)]^{n+1}} \left\{ \frac{1}{n+2} - \frac{1}{n+1} \right\}$$

$$= \frac{-n(n + 1)(Q(b) - \varphi)^{n+2}}{[Q(b) - Q(\theta)]^{n+1}} \left\{ \frac{-1}{(n+1)(n+2)} \right\}$$

$$= \frac{Q(b) - \varphi}{n + 2} \quad (7.2.6)$$

If $\varphi$ is incremented to $\varphi + h$, we have,

$$g(x; \varphi + h) = \frac{n! [Q(x) - \varphi - h]^{r-1} [Q(b) - Q(x)]^{n-r} q(x)}{(r-1)! (n-r)! [Q(b) - \varphi - h]^n} \quad (7.2.7)$$
\[ \varphi + h < Q(x) < Q(b) \] which implies that \( 0 < h < Q(x) - \varphi \).

Then we have,
\[
\int_{\varphi}^{g(x; \varphi + h)} dG_1(h) = \frac{n!(Q(b) - Q(x))^{n-r}q(x)[n+1]}{(r-1)!(n-r)!} \int_0^{Q(x) - h} [Q(x) - h]^{r-1} dh
\]
\[
= \frac{n!(Q(b) - Q(x))^{n-r}q(x)[n+1]}{(r-1)!(n-r)!} \left[ \frac{[Q(x) - Q(b)]^r}{r} \right]_0^{Q(x) - h}
\]
\[
= \frac{n!(Q(b) - Q(x))^{n-r}q(x)[n+1]}{r!(n-r)!} [Q(x) - \varphi]^r
\]
\[
\therefore \Delta_1 g(x; \varphi) = \frac{n!(Q(x) - \varphi)[Q(x) - Q(b)]^{n-r}q(x)[n+1]}{(r-1)!(n-r)!} \left[ \frac{[Q(x) - Q(b)]^r}{r} \right]_0^{Q(x) - h} - 1
\]

Using this and (7.2.6), we get,
\[
\frac{\Delta_1 g(x; \varphi)}{g(x; \varphi) \Delta_1 \varphi} = \frac{(n+2)}{[Q(b) - \varphi]} \left[ \frac{(n+1)[Q(x) - \varphi] - r[Q(b) - \varphi]}{r[Q(b) - \varphi]} \right]
\]
\[
= \frac{(n+2)}{[Q(b) - \varphi]} \left[ \frac{(n+1)Q(x) - rQ(b) - (n-r+1)\varphi}{rQ(b) - \varphi} \right]
\]
\[
= \frac{(n-r+1)(n+2)}{r(Q(b) - \varphi)^2} \left[ \frac{(n+1)Q(x)}{(n-r+1)} - \frac{rQ(b)}{(n-r+1)} - \varphi \right]
\]

The equation (2.8) is the ideal estimation equation which implies that,
\[
T(x) = \frac{(n+1)Q(x)}{(n-r+1)} - \frac{rQ(b)}{(n-r+1)}
\]
is UMVUKBE of \( \varphi = Q(\theta) \) with its variance \( Var[T(x)] = \frac{r(Q(b) - \varphi)^2}{(n-r+1)(n+2)} \):

Kiefer bound on variance of estimator of \( \varphi \).

**Remark 7.2.1**

If the whole sample is observed, \( x_{(r)} = x_{(1)} \), Kiefer bounds from complete and censored samples coincide to \( \frac{r(Q(b) - \varphi)^2}{n(n+2)} \) and is attained by variance of UMVUE.

**Remark 7.2.2**

\[ Var[T(x)] = \frac{r(Q(b) - \varphi)^2}{(n-r+1)(n+2)} = \{K(\varphi), r \geq 1\} \]
\( K(\varphi) \) is increasing function of \( r \). That is, though the variance of the estimator continues to attain its Kiefer bound it increases with increase in the number of censored observations.
Remark 7.2.3

The estimators based upon left censored samples from left truncated distributions have larger variances $\frac{r}{(n-r+1)(n+2)} [Q(b) - \varphi]^2$ as compared to those based on complete sample $\frac{[Q(b) - \varphi]^2}{n(n+2)}$.

Example 7.2.1

Let

$$f(x; \theta) = e^{-(x-\theta)}, \theta < x < \infty$$

Here, $q(x) = e^{-x}, Q(x) = -e^{-x}, Q(b = \infty) = 0, Q(\theta) = -e^{-\theta}$.

$F(x) = 1 - e^{-\theta}$. Here, Kiefer bound on the variance of u-estimator of $e^{-\theta}$ based on

(i) complete sample is $\frac{e^{-2\theta}}{n(n+2)}$ and

(ii) censored sample is $\frac{re^{-2\theta}}{(n-r+1)(n+2)}, 1 < r < n$

Example 7.2.2

Let

$$f(x; \theta) = (b - \theta)^{-1}, \theta < x < b.$$ 

Here, $(x) = 1, Q(x) = x$.

Here, Kiefer bound on the variance of u-estimator of $\theta$ based on

(i) complete sample is $\frac{(b-\theta)^2}{n(n+2)}$ and (ii) censored sample is $\frac{r(b-\theta)^2}{(n-r+1)(n+2)}; 1 \leq r \leq n$.

Example 7.2.3

Let

$$f(x; \theta) = \frac{e^{-x}}{e^{-\theta} - e^{-1}}, \theta < x < b$$

Here, $q(x) = e^{-x}, Q(x) = -e^{-x}$

Here, attainable Kiefer bound on the variance of u-estimator of $e^{-\theta}$ based on

(i) Complete sample $= \frac{(e^{-\theta} - e^{-1})^2}{n(n+2)}$. (ii) Censored sample $= \frac{r(e^{-\theta} - e^{-1})^2}{(n-r+1)(n+2)}$.

7.3 KIEFER BOUND IN TYPE II RIGHT CENSORED SAMPLE IN RIGHT TRUNCATED FAMILY

In this section we extend results due to Kiefer(1952), Jadhav and Prasad(1986-87) and Jadhav and Shanubhogue (2014) to a situation in which the experiment is terminated after $r$ failures are observed. Thus, we have observations $X_(1), X_(2), ..., X_(r)$ from right truncated probability distribution having p.d.f.
\[
f(x; \theta) = \frac{q(x)}{Q(\theta) - Q(a)}, \quad -\infty < a < x < \theta < b < \infty \tag{7.3.1}
\]

Then we have the following-

**Theorem 7.3.1**

The unbiased estimator of \(Q(\theta)\) based on maximum observation from right censored sample from right truncated distribution (7.3.1) is UMVUKBE.

**Proof**

The pdf of \(r\)th order statistic \(X_{r:n} = x\) is given by,

\[
f_{r:n}(x; \theta) = \frac{n!F(x)^{r-1}(1-F(x))^{n-r}f(x; \theta)}{(r-1)!(n-r)!} = \frac{n!}{(r-1)!(n-r)!} \frac{[Q(x) - Q(a)]^{r-1} [Q(\theta) - Q(x)]^{n-r} q(x)}{[Q(\theta) - Q(a)]^{n-1}}.
\]

Let \(y = \frac{Q(x) - Q(a)}{Q(\theta) - Q(a)} \Rightarrow t = [Q(x) - Q(a)] = y[Q(\theta) - Q(a)].\)

Clearly, \(\frac{dy}{dx} = \frac{q(x)}{Q(\theta) - Q(a)}.\)

Therefore, the pdf of \(y\) is given by,

\[
g_{r:n}(y; \theta) = \frac{n!y^{r-1}(1-y)^{n-r}}{(r-1)!(n-r)!}, \quad 0 < y < 1 \tag{7.3.2}
\]

Thus, \(y\) is a beta variable of first kind with parameters \(r\) and \((n-r+1)\). Hence,

\[
E[Q(X) - Q(a)] = \frac{r[Q(\theta) - Q(a)]}{(n+1)}.\]

Therefore, \(\frac{(n+1)[Q(X) - Q(a)]}{r}\) is UMVUKB estimator of \([Q(\theta) - Q(a)]\) with its variance

\[
\frac{(n-r+1)[Q(\theta) - Q(a)]^2}{r(n+2)}\]

which equals Kiefer bound.

Let \(\varphi = [Q(\theta) - Q(a)], t = [Q(X) - Q(a)], a < x < \theta < b.\)

Therefore, \(0 < t < \varphi\) and the pdf of \(t\) is given by,

\[
f_{r:n}(y; \varphi) = \frac{y^{r-1}(\varphi-y)^{n-r}}{\varphi^n \beta(r,n-r+1)}, \quad 0 < y < 1.
\]

For each \(\varphi \in \Phi\), let \(\Phi_{\varphi} = \{h; (\varphi + h) \in \Phi\} = (-\varphi, \infty).\) On the subset \((-\varphi, 0)\) of \(\Phi_{\varphi}\), let us define the probability distributions:

\[
dG_1(h) = \frac{[n+1](\varphi + h)^n dh}{\varphi^{n+1}}, \quad -\varphi < h < 0 \quad \text{and} \quad G_2(h) = l_0(h).
\]

Then,

\[
E_1(h) = \frac{-\varphi}{(n+2)}
\]

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The ideal estimation equation is given by
\[ \Delta_{f,r,n} \frac{f_{r,n}(t,\varphi)}{f_{r,n}(t,\varphi) \Delta_{r,n} \varphi} = \frac{r(n+2)}{\varphi^2(n+r+1)} \left( \frac{n+1}{r} t - \varphi \right) \] (7.3.3)

Therefore, variance \[ \frac{(n+1)}{r} Q(x) - \frac{n-r+1}{r} Q(a) \] of \( Q(\theta) \) attains its Kiefer bound \[ K(Q(\theta)) = \frac{\varphi^2(n+r+1)}{r(n+2)}. \]

**Remark 7.3.1**

To compare the results based on complete and censored samples, consider,
\[
\frac{\text{Var}[T(X_{(n)})]}{\text{Var}[T(X_{(r)})]} = \frac{(Q(\theta) - Q(a))^2}{n(n+2)} \frac{r(n+2)}{(n-r+1)(Q(\theta) - Q(a))^2} = \frac{r}{n(n-r+1)} < 1.
\]

That is, variance of estimator based on complete sample is quite less than that based on left censored sample. In other words, UMVUE based on complete sample is statistically more efficient than that based on censored type II sample. But from the time, money, efforts and cost point of view UMVUE based on censored type II sample can be more efficient. ■

**Remark 7.3.1**

If \( r = n \) both the schemes coincide.

**Example 7.3.1**

In the density given by Tate (1959),
\[ f(x; \theta) = \frac{a\theta}{(\theta-a)x^2}, \quad 0 < a < x < \theta; \]
\[ q(x) = \frac{1}{x^2}, \quad Q(x) = -\frac{1}{x}. \] In complete sample, \( x_{(n)} \) is complete sufficient statistic.
\[
\left( \frac{n+1}{n} \right) \left[ \frac{x_{(n)} - a}{ax_{(n)}} \right] \text{ is UMVUKBE of } \left[ \frac{\theta-a}{a\theta} \right] \text{ with variance } \frac{1}{n(n+2)} \left[ \frac{\theta-a}{a\theta} \right]^2 = \text{Kiefer bound.}
\]

In right censored sample, \( \left( \frac{n+1}{r} \right) \left[ \frac{x_{(r)} - a}{ax_{(r)}} \right] \) is UMVUKBE of \( \left[ \frac{\theta-a}{a\theta} \right] \) with variance
\[
\left( \frac{n-r+1}{r(n+2)} \right) \left[ \frac{\theta-a}{a\theta} \right]^2 \text{ which equals Kiefer bound.}
\]
In life testing experiments some initial and last failures are not observed. Thus, we have ordered observations, say, from $x_{(r)}$ to $x_{(s)}$. This is doubly censored sample and the inferences are based upon that.

**Theorem 7.4.1.**

Doubly censored sample on variable having left truncated probability density function (pdf) provides UMVU estimator of function of truncation parameter involved in pdf with its variance attaining Kiefer bound.

**Proof.**

Let the pdf of the variable be from left truncated family as in (7.2.1). Consider a doubly censored sample in which only the observations from $r$th order statistic $X_{(r)}$ to $s$th order statistic $X_{(s)}$ are observed. Then the likelihood function of these observations is as below.

$$L(x|\theta) = \frac{n!}{(r-1)!(n-s)!}[F(x_{(r)})]^{r-1}[1 - F(x_{(s)})]^{n-s} \prod_{i=r}^{s} f(x_{(i)})$$

$$= \frac{n!}{(r-1)!(n-s)!} \left[\frac{Q(x_{(r)}) - Q(\theta)}{Q(\theta) - Q(\theta)}\right]^{r-1} \left[\frac{Q(b) - Q(x_{(s)})}{Q(x_{(s)}) - Q(\theta)}\right]^{n-s} \prod_{i=r}^{s} q(x_{(i)})$$

$$= g(x_{(r)}, \theta), h(x)$$

Therefore, by factorization theorem, $x_{(r)}$ is sufficient statistic for $\theta$. The pdf of $x_{(r)}$ is given by,

$$f_{r:n}(x_{(r)}; \theta) = \frac{n!}{(r-1)!(n-r)!}[F(x_{(r)})]^{r-1}[1 - F(x_{(r)})]^{n-r} f(x_{(r)}), \quad \theta < x_{(r)} < b$$

Then, the results follow from (7.4.3),(3.2.5) and (7.2.8). □

**Remark 7.4.1**

The results from type II left censored and doubly censored sample in left truncated family of probability distributions are the same.
7.5 KIEFER BOUND FROM DOUBLY CENSORED SAMPLE FROM RIGHT TRUNCATED FAMILY

In double-censored sample some observations on failures in the beginning of the life testing experiment and some last failures are not available. Thus, we have a sample of ordered observations, say, from \( x_{(r)}, x_{(r+1)}, \ldots, x_{(r+s)} \). We have to draw the inferences using this sample. In this case, we have the following-

**Theorem 7.5.1**

Doubly censored sample on variable having right truncated probability density function (pdf) provides UMVUKBE of function of truncation parameter involved in pdf.

**Proof**

Let the pdf of the variable be from right truncated family as in (7.3.1). Consider a doubly censored sample in which only the observations from \( x_{(r)}, x_{(r+1)}, \ldots, x_{(r+s)} \) are observed. Then the likelihood function of these observations is as below.

\[
L(x|\theta) = \frac{n!}{(r-1)! (n-r-s)!} \left[ F(x_{(r)}) \right]^{r-1} \left[ 1 - F(x_{(r+s)}) \right]^{n-r-s} \prod_{i=r}^{r+s} f(x_{(i)}) \]

\[
= \frac{n!}{(r-1)! (n-r-s)!} \left[ \frac{Q(x_{(r)}) - Q(a)}{Q(\theta) - Q(a)} \right]^{r-1} \left[ \frac{Q(\theta) - Q(x_{(r+s)})}{Q(\theta) - Q(a)} \right]^{n-r-s} \prod_{i=r}^{r+s} \frac{q(x_{(i)})}{Q(\theta) - Q(a)} \]

\[
= \frac{n!}{(r-1)! (n-r-s)!} \left[ \frac{Q(x_{(r)}) - Q(a)}{Q(\theta) - Q(a)} \right]^{r-1} \left[ \frac{Q(\theta) - Q(x_{(r+s)})}{Q(\theta) - Q(a)} \right]^{n-r-s} \prod_{i=r}^{r+s} \frac{q(x_{(i)})}{Q(\theta) - Q(a)} \]

\[
= f(x_{(r+s)}, \theta)q(x_{(i)}) \quad \cdotp \quad (7.5.2)
\]

Therefore, \( x_{(r+s)} = x_{(s)} \), say, is sufficient statistic. Its pdf is given by,

\[
f_{s:n}(x_{(s)}; \theta) = \frac{n!}{(s-1)! (n-s)!} \left[ F(x_{(s)}) \right]^{s-1} \left[ 1 - F(x_{(s)}) \right]^{n-s} f(x_{(s)}) \quad \text{if} \quad a < x_{(s)} < \theta. \]

\[
= \frac{n!}{(s-1)! (n-s)!} \left[ \frac{Q(x_{(s)}) - Q(a)}{Q(\theta) - Q(a)} \right]^{s-1} \left[ \frac{Q(\theta) - Q(x_{(s)})}{Q(\theta) - Q(a)} \right]^{n-s} q(x_{(s)}) \quad (7.5.3)
\]

Let \( y = \frac{Q(x_{(s)}) - Q(a)}{Q(\theta) - Q(a)} \).
Therefore, \( \frac{dy}{dx} = \frac{q(x)}{Q(\theta) - q(a)} \), \( 0 < y < 1 \). Then pdf of \( Y \) becomes

\[
g_{s;n}(y; \theta) = \frac{n!}{(s-1)!(n-s)!} y^{r-1} (1 - y)^{n-s}, \quad 0 < y < 1 \tag{7.5.4}
\]

Then the results follow from (7.3.2), (7.3.3) and (7.5.4). ■

**Remark 7.5.1**

The results from type II right censored and doubly censored samples from right truncated families of probability distributions are the same.

### 7.6 KIEFER BOUND FROM DOUBLY CENSORED SAMPLE FROM DISTRIBUTIONS ON \( (C_1(\theta), C_2(\theta)) \)

In this sort of probability distributions both the ends of the support of the distribution depend on the parameter. Therefore, it is suitable, natural and rational to consider doubly censored samples for effective inference when dealing with life testing experiments. This gives us the following-

**Theorem 7.6.1**

Doubly censored sample on variable having probability density function (pdf)

\[
f(x, \theta) = \frac{1}{[C_2(\theta) - C_1(\theta)]}, \quad C_1(\theta) < x < C_2(\theta) \tag{7.6.1}
\]

provides UMVUKBE of function \( \varphi = [C_2(\theta) - C_1(\theta)] \).

**Proof**

Let the pdf of the variable be as in (7.6.1). In this case,

\[
F(x) = \frac{x - C_1(\theta)}{\varphi}, \quad 1 - F(x) = \frac{[C_2(\theta) - x]}{\varphi}
\]

For convenience, let us write, \( X_r = z \) and \( X_s = y \), \( r < s \)

\[
f_{rs}(z, y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \left[ \frac{z-C_1(\theta)}{\varphi} \right]^{r-1} \left[ \frac{y-z}{\varphi} \right]^{s-r-1} \left\{ \frac{C_2(\theta) - y}{\varphi} \right\}^{n-s} \frac{1}{\varphi^n}
\]

\[
f_{rs}(z, W_{rs}) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \left[ \frac{z-C_1(\theta)}{\varphi} \right]^{r-1} \left[ \frac{W_{rs} - z}{\varphi} \right]^{s-r-1} \left\{ \frac{C_2(\theta) - W_{rs} - z}{\varphi} \right\}^{n-s} \frac{1}{\varphi^n}
\]

\[
f(W_{rs}; \varphi) = \frac{n! \left[ \frac{W_{rs} - z}{\varphi - W_{rs}} \right]^{s-r-1} \left\{ \frac{C_2(\theta) - W_{rs} - z}{\varphi - W_{rs}} \right\}^{n-s} \left\{ \frac{z-C_1(\theta)}{\varphi - W_{rs}} \right\}^{r-1} \left[ \frac{C_2(\theta) - z}{\varphi - W_{rs}} \right]^{n-s} dz}{(s-r-1)!(n-s+r+1)! \varphi^n \beta(r,n-s+1) C_1(0) C_2(0) [\varphi - W_{rs}]^{n-s+r}}
\]
But \( \frac{1}{\beta(r,n-s+1)} \int_{C_1(0)}^{C_2(0)} \frac{(z-C_1(0))^{r-1}[C_2(0)-Wrs-z]^{n-s}dz}{[\varphi-Wrs]^{n-s+r}} = 1 \) being the \( \beta \) density in the form
\[
g(y;p,q) = \frac{1}{\beta(p,q)} \frac{(y-a)^{p-1}(b-y)^{q-1}}{(b-a)^{p+q-1}}, \quad a \leq y \leq b.
\]

Therefore, the pdf of \( w_{rs} \) is given by
\[
f(w_{rs};\varphi) = \frac{n! [w_{rs}]^{s-r-1}[\varphi - w_{rs}]^{n-s+r}}{(s-r-1)! (n-s+r+1)! \varphi^n \beta(r,n-s+1)} (7.6.2)
\]

\( \Phi = \{ \varphi; f(w_{rs};\varphi) > 0 \} = (0, \infty) \). For each fixed \( \varphi \in \Phi \), let \( \Phi_\varphi = \{ h; (\varphi + h) \in \Phi \} = (-\varphi, \infty) \). On \( (-\varphi, 0) \) let us define the prior probability distribution
\[
G_1(h) = \frac{(n+1)(\varphi + h)^n I((-\varphi,0))(h)}{\varphi^{n+1}}
\]

Note that, \( E_1(h) = \frac{-\varphi}{n+2} = \Delta_1 \varphi \).

If \( \varphi \) is increased to \( \varphi + h, 0 < w_{rs} < \varphi + h \Rightarrow w_{rs} - \varphi < h \).

\[
\Delta f(w_{rs};\varphi) = \int_{w_{rs} - \varphi}^{0} f(w_{rs};\varphi + h) dG_1(h) - f(w_{rs};\varphi)
\]

\[
= \int_{w_{rs} - \varphi}^{0} \frac{[w_{rs}]^{s-r-1}[\varphi + h - Wrs]^{n-s+r}}{\beta(s-r,n-s+r+1)(\varphi + h)^n} (n+1)(\varphi + h)^n I((-\varphi,0))(h) dh
\]

\[
= \int_{w_{rs} - \varphi}^{0} \frac{[w_{rs}]^{s-r-1}(n+1)}{\beta(s-r,n-s+r+1)\varphi^{n+1}} \int_{w_{rs} - \varphi}^{0} [\varphi + h - Wrs]^{n-s+r} dh
\]

\[
= \frac{[w_{rs}]^{s-r-1}(n+1)\beta(s-r,n-s+r+1)\varphi^{n+1}}{\beta(s-r,n-s+r+1)\varphi^{n}}
\]

\[
= \frac{\varphi^{n}}{(n+1)(\varphi - w_{rs})} \left[ \frac{\varphi^{n}(n-s+r+1)}{(n+1)(\varphi - w_{rs})} - 1 \right]
\]

\[
= \frac{(n+1)(\varphi - w_{rs})}{\varphi^{n}(n-s+r+1)} - 1
\]

\[
= \frac{(n+1)(\varphi - w_{rs}) - \varphi^{n}(n-s+r+1)}{\varphi^{n}(n-s+r+1)(\varphi - w_{rs})}
\]

\[
= \frac{-n+2}{\varphi^{n}(n-s+r+1)}
\]

\[
= \frac{(n+1)w_{rs} - (s-r)\varphi}{\varphi^{n}(n-s+r+1)}
\]

\[
= \frac{(n+1)w_{rs} - (s-r)\varphi}{\varphi^{n}(n-s+r+1)}
\]

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\[
\frac{\Delta_1 f (w_{rs}; \varphi)}{f (w_{rs}; \varphi) \Delta_1 \varphi} = \frac{(s - r)(n + 2)}{\varphi^2 (n - s + r + 1) [s - r]^{w_{rs} - \varphi}}
\]  
(7.6.3)

Equation (7.6.3) is the ideal estimation equation. Therefore, \( \frac{(n+1)^2}{(s-r)} \) is UMVUKBE of \( \varphi \) with Kiefer bound \( K(\varphi) = \frac{\varphi^2 (n-s+r+1)}{(s-r)(n+2)} \).

**Remark 7.6.1**

In this situation, for complete sample \( \hat{\varphi} = \left( \frac{n+1}{n-1} \right) U \) is UMVUBE with

\[
K(\varphi) = \frac{2\varphi^2}{(n-1)(n+2)} \text{ while for doubly censored sample, } \hat{\varphi} = \left( \frac{n+1}{s-r} \right) w_{rs} \text{ is UMVUBE with}
\]

\[
K(\varphi) = \frac{\varphi^2 (n-s+r+1)}{(s-r)(n+2)} .
\]

**Remark 7.6.2.**

To compare the results based on complete and censored samples, consider,

\[
\text{Variance of UMVUE based on complete sample} = \frac{2\varphi^2}{(n-1)(n+2)} \text{ and }
\]

\[
\text{Variance of UMVUE based on censored sample} = \frac{2\varphi^2}{(n-1)(n+2)} \varphi^2 (n-s+r+1) = \frac{2(s-r)}{(n-1)(n-s+r+1)} < 1.
\]

**Remark 7.6.3.**

If \( s=n \) and \( r=1 \), the results of complete and doubly censored samples coincide.

**Example 7.6.1.**

Consider a doubly censored sample from variable having pdf

\[
f(x; \theta) = \frac{1}{2\theta}, \quad -\theta < x < \theta
\]  
(7.6.4)

\( \varphi = 2\theta \). \( F(x) = \frac{x+\theta}{2\theta} \). \( 1 - F(x) = \frac{\theta-x}{2\theta} \).

Suppose that we have the ordered observations \( X_{(r)}, X_{(r+1)}, ..., X_{(s)} \). The joint pdf of \( X_{(r)} = z \) and \( X_{(s)} = y \) is given by

\[
f_{rs}(z, y, \theta) = \frac{n!(z+\theta)^{r-1}(y-z)^{s-r-1}(\theta-y)^{n-s}}{(r-1)!(s-r-1)!(n-s)!2^n}.
\]

Therefore,

\[
f_{rs}(z, w_{rs}, \varphi) = \frac{n!(z+\varphi/2)^{r-1}(w_{rs})^{s-r-1}[(\varphi/2-w_{rs})^2-z)^{n-s}}{(r-1)!(s-r-1)!(n-s)!2^n}.
\]

\[
f_{rs}(z, w_{rs}, \varphi) = \frac{n!(w_{rs})^{s-r-1}[(\varphi-w_{rs})^{s-r}z)^{n-s+r}}{(s-r-1)!(n-s+r+1)\varphi^n} \frac{(n-s+r+1)(\varphi/2-w_{rs})^{n-s}(z+\varphi/2)^{r-1}}{(r-1)!(n-s)!(\varphi-w_{rs})^{n-s+r}}
\]

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But \( \int_0^\infty (n-s+r+1)\phi(2-w_{rs}, -z)^{-n-s} (z+\phi/2)^{r-1}dz \) 
\( (r-1)(n-s)[\phi-w_{rs}]^{n-s+r} \) 
\( \frac{1}{\beta(r,n-s+1)} \int_0^\infty (\phi-w_{rs})^{-n-s} (z+\phi)^{r-1}dz \)
\( = 1. \)

Therefore, the pdf of \( w_{rs} \) is given by

\[
f_{rs}(w_{rs}, \varphi) = \int_0^\infty f_{rs}(z, w_{rs}, \theta)dz
\]

\[
f_{rs}(w_{rs}, \varphi) = \frac{(w_{rs})^{s-r-1}[\varphi-w_{rs}]^{n-s+r}}{\beta(s-r,n-s+r+1)\varphi^{n+1}}< w_{rs} < \varphi. \tag{7.6.5}
\]

On \( (-\varphi, 0) \) let us define a prior probability distribution

\[
G_1(h) = \frac{(n+1)(\varphi+h)^n1_{[(-\varphi,0)]}(h)}{\varphi^{n+1}}.
\]

\[
\Delta_1 f(w_{rs}; \varphi) = \int_{w_{rs} = \varphi}^0 f(w_{rs}; \varphi + h) dG_1(h) - f(w_{rs}; \varphi)
\]

\[
\Delta_1 f(w_{rs}; \varphi) = \int_{w_{rs} = \varphi}^0 \frac{(w_{rs})^{s-r-1}[\varphi+h-w_{rs}]^{n-s+r}}{\beta(s-r,n-s+r+1)\varphi^{n+1}} (n+1) - f(w_{rs}; \varphi)
\]

\[
= \frac{(w_{rs})^{s-r-1}(n+1)}{\beta(s-r,n-s+r+1)\varphi^{n+1}} \frac{[\varphi-w_{rs}]^{n-s+r+1}}{n-s+r+1} - f(w_{rs}; \varphi)
\]

\[
= \frac{(w_{rs})^{s-r-1}[\varphi-w_{rs}]^{n-s+r}}{\beta(s-r,n-s+r+1)\varphi^{n+1}} \frac{(n+1)[\varphi-w_{rs}]}{n-s+r+1} - 1
\]

\[
\Delta_1 f(w_{rs}; \varphi) = \left[ \frac{(n+1)[\varphi-w_{rs}]}{\varphi[n-s+r+1]} - 1 \right] \frac{(n+2)}{(-\varphi)}
\]

\[
\Delta_1 f(w_{rs}; \varphi) = \left[ \frac{[n+2]}{\varphi^2(n-s+r+1)} \right] \frac{(n+1)}{(s-r)} w_{rs} - \varphi \tag{7.6.6}
\]

Thus, \( \frac{(n+1)}{s-r} w_{rs} \) is UMVUKBE of \( \varphi \) with Kiefer bound \( K(\theta) = \frac{\varphi^2(n-s+r+1)}{[n+2]s-r} \).

If \( s=n, r=1 \), the results reduce to the results of complete sample. That is, if \( s=n, r=1 \),

\[
K(\theta) = \frac{\varphi^2(n-s+r+1)}{[n+2]s-r}
\]

\[
= \frac{2\varphi^2}{(n-1)(n+2)}.
\]

For comparison, let us consider,

\[
\frac{\text{Kiefer bound from doubly censored sample}}{\text{Kiefer bound from complete sample}} = \frac{\varphi^2(n-s+r+1)(n-1)(n+2)}{[n+2](s-r)2\varphi^2} = \frac{(n-1)(n-s+r+1)}{2(s-r)} \tag{7.6.7}
\]

**Example 7.6.2.**
Let, 

\[ f(x; \varphi) = \frac{1}{\varphi}, \theta < x < \frac{1}{\theta}, \quad \varphi = \frac{1 - \theta^2}{\theta} \]  

(7.6.8)

\[ F(x) = \frac{x - \theta}{\varphi}. 1 - F(x) = \frac{1 - x}{\varphi}. F(y) - F(z) = \frac{y - z}{\varphi} \]

Let, \( X_{(r)} = z \) and \( X_{(s)} = y \) be the order statistics of order \( r \) and \( s \) respectively. But these are minimum and maximum in the observed censored sample. Their joint pdf is given by

\[ f_{rs}(z, w_{rs}; \theta) = \frac{n!w_{rs}^{s-r-1}(z-\theta)^{r-1}(1-w_{rs}-z)^{n-s}}{(r-1)!s!(n-s)!\left(1-\frac{\theta^2}{\theta}\right)^n}, \theta < z < \frac{1}{\theta}, 0 < w_{rs} < 1 \]

Therefore,

\[ f_{rs}(z, w_{rs}; \varphi) = \frac{1}{\theta}f_{rs}(z, w_{rs}; \theta)dz \]

\[ f(w_{rs}; \varphi) = \frac{(w_{rs})^{s-r-1}[\varphi-w_{rs}]^{n-s+r}}{\beta(s-r,n-s+r+1)\varphi^{n}}, \quad 0 < w_{rs} < \varphi \]  

(7.6.9)

From (7.6.5), (7.6.6) and (7.6.9), it is clear that

\[ E(w_{rs}) = \left(\frac{s-r}{n+1}\right)\varphi \quad \text{and} \quad Var(w_{rs}) = \left(\frac{s-r}{n+1}\right)\frac{\varphi^2}{(n+2)(n+1)^2}. \]

\( \left(\frac{n+1}{s-r}\right)w_{rs} \) is UMVUKBE of \( \varphi \) with Kiefer bound \( K(\varphi) = \frac{\varphi^2(n-s+r+1)}{[n+2](s-r)}. \]

\[ \blacksquare \]