Chapter 3

ANALYTIC SOLUTIONS OF
DIFFERENTIAL-DIFFERENCE
EQUATIONS OF ORDER \((2, 1)\)

\(^0\)A part of this chapter is presented in the form of the paper entitled “Laplace decomposition methods for solving certain class of differential-difference equations”, *The Journal of the Indian Mathematical Society*, (Accepted for publication).
### 3.1 Introduction

In continuation with the study of chapter 2, the analytic solutions of differential-difference equations of order $(2, 1)$ are studied in the present chapter. These problems are related to the problem of solving a singularly perturbed second order differential-difference equation where singular perturbation parameter and the delay parameter are selected as small as possible. Such problems play an important role in variety of physical problems ([41] and references quoted in it) such as microscale heat transfer, diffusion in polymers, control of chaotic systems and so on. In the present chapter, we study the following differential-difference equation of order $(2, 1)$:

$$
\epsilon u''(t) = u'(t) - \left[ f(t) + F(u(t - \omega)) \right], \quad t > \omega.
$$

Further, we set $\epsilon = 1$ and work with one initial interval condition:

$$
u(t) = a + bt, \quad t \in [0, \omega].$$

Keeping these facts in mind, the present chapter is devoted to the study of two analytical methods, namely, Laplace decomposition method and modified Laplace decomposition method [5, 11, 13, 25, 30, 38], [43]-[45], [57, 75, 83, 88, 90] which solve any linear or nonlinear second order differential equations with the main idea to make them applicable to solve linear or nonlinear differential-difference equations of order $(2, 1)$ with appropriate initial interval condition.

The following simple types of differential-difference equations are considered for the study:
1. Linear differential-difference equation of order (2, 1):

\[ u''(t) + c_1 u'(t) + c_2 u(t - \omega) = f(t), \quad t > \omega, \]  
\[ u(t) = a + bt, \quad 0 \leq t \leq \omega, \]  

where \( c_2 \neq 0, c_1, a \) and \( b \) are real constants, \( f(t) \) is a given function of exponential order and \( \omega \) is a positive difference parameter.

2. Nonlinear differential-difference equation of order (2, 1):

\[ u''(t) = f(u(t - \omega)), \quad t > \omega, \]  
\[ u(t) = a + bt, \quad 0 \leq t \leq \omega, \]  

where \( a \) and \( b \) are real constants, \( f(u) \) is a given nonlinear function of exponential order and \( \omega \) is a positive difference parameter.

In the next section, the Laplace decomposition method and modified Laplace decomposition method are described for a more general problem which includes both (3.1.1) and (3.1.2). In the ensuing section a set of three test problems are worked out. The last section includes concluding remarks as well as scope for further work to be done in the coming chapters.
3.2 Description of the methods to solve certain Differential-Difference Equations of order \((2, 1)\)

Let us consider the linear or nonlinear differential-difference equation with second order differential and first order difference of the following simple type:

\[
\begin{align*}
  u''(t) &= cu'(t) + f(u(t - \omega)), \quad t > \omega, \\
  u(t) &= a + bt, \quad 0 \leq t \leq \omega,
\end{align*}
\]

where \(a, b, c\) are real constants, \(f(u)\) is a given linear or nonlinear function of exponential order and \(\omega\) is a positive difference parameter. First we note that,

\[
\int_{\omega}^{\infty} u'(t) e^{-st} \, dt = L\{u'(t)\} - \int_{0}^{\omega} b e^{-st} \, dt = L\{u'(t)\} + \frac{b}{s} (e^{-\omega s} - 1)
\]

and

\[
\int_{\omega}^{\infty} u''(t) e^{-st} \, dt = 0 \Rightarrow \int_{\omega}^{\infty} u''(t) e^{-st} \, dt = L\{u''(t)\}.
\]

Hence we shall describe the Laplace decomposition method and modified Laplace decomposition method for the nonlinear differential-difference equation (3.2.1).

Laplace decomposition method

Now let us multiply both sides of (3.2.1) by \(e^{-st}\), \(s > 1\) and integrate between \(\omega\) and \(\infty\), to obtain

\[
\int_{\omega}^{\infty} u''(t) e^{-st} \, dt = c \int_{\omega}^{\infty} u'(t) e^{-st} \, dt + \int_{\omega}^{\infty} f(u(t - \omega)) e^{-st} \, dt.
\]

Let us apply suitable shifting of variables and use initial interval condition to obtain

\[
L\{u''(t)\} = c L\{u'(t)\} + \frac{bc}{s} (e^{-\omega s} - 1) + e^{-\omega s} L\{f(u(t))\}.
\]
After applying formula of Laplace transform for second order derivative, finally, we arrive at

\[ L\{u(t)\} = \frac{a}{s} + \frac{b}{s^2} - \frac{c a}{s^3} + \frac{b c}{s^3} e^{-\omega s} + \frac{c}{s} L\{u(t)\} + \frac{e^{-\omega s}}{s^2} L\{f(u(t))\}. \] (3.2.2)

Now we seek the following type of decomposition for \(L\{u(t)\}\):

\[ L\{u(t)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\}. \] (3.2.3)

which may be regarded as Laplace decomposition.

The Laplace decomposition for \(L\{f(u(t))\}\) is given by,

\[
L\{f(u(t))\} = L\{f(u_0(t))\} \\
+ e^{-\omega s} L\left\{ \left[ \frac{d}{du} f(u(t)) \right]_{u=u_0} \right\} u_1(t) \\
+ e^{-2\omega s} L\left\{ \left[ \frac{d}{du} f(u(t)) \right]_{u=u_0} \right\} u_2(t) + \left[ \frac{d^2}{du^2} f(u(t)) \right]_{u=u_0} \frac{1}{2!} u_1^2(t) \\
+ e^{-3\omega s} L\left\{ \left[ \frac{d}{du} f(u(t)) \right]_{u=u_0} \right\} u_3(t) \\
+ \left[ \frac{d^2}{du^2} f(u(t)) \right]_{u=u_0} \frac{1}{2!} 2u_1(t)u_2(t) + \left[ \frac{d^3}{du^3} f(u(t)) \right]_{u=u_0} \frac{1}{3!} u_1^3(t) \\
+ \ldots \\
= \sum_{n=0}^{\infty} e^{-n\omega s} L\{A_n(t)\}. \] (3.2.4)

In (3.2.4), we have

\[ A_0(t) = f(u_0(t)). \]
\[ A_1(t) = \left[ \frac{d}{du} f(u(t)) \right]_{u=u_0} \frac{1}{1!} u_1(t) \]
and in general, for \( n \geq 2 \), \( A_n \) is \( n^{th} \) degree Adomian Polynomial [28] of \( f(u(t)) \) in the powers of \( u_1(t), u_2(t), \ldots, u_n(t) \) given by,

\[
A_n(t) = \left[ \frac{d}{du} f(u(t)) \bigg|_{u=u_0} \right] u_n(t) + \left[ \sum_{k=2}^{n} \frac{d^k}{du^k} f(u(t)) \bigg|_{u=u_0} \right] \sum_{i_1+i_2+\ldots+i_k=n} u_{i_1}(t)u_{i_2}(t)\ldots u_{i_k}(t).
\]

Now the main idea of Laplace decomposition is to set an iteration as follows:

\[
(1 - \frac{c}{s}) \sum_{n=0}^{\infty} e^{-nw_0} L\{u_n(t)\} = \left( \frac{a}{s} + \frac{b}{s^2} - \frac{ca}{s^2} - \frac{bc}{s^3} \right) + \frac{bc}{s^3}e^{-ws}
\]

\[
+ \frac{1}{s^2} \sum_{n=1}^{\infty} e^{-nw_0} L\{A_{n-1}(t)\}
\]

\[
= \left( 1 - \frac{c}{s} \right) \left( \frac{a}{s} + \frac{b}{s^2} \right) + \left( \frac{bc}{s^3} + \frac{1}{s^2} L\{A_0(t)\} \right) e^{-ws}
\]

\[
+ \frac{1}{s^2} \sum_{n=2}^{\infty} e^{-nw_0} L\{A_{n-1}(t)\}.
\]

(3.2.5)

One may compute \( L\{u_n(t)\} \) iteratively as follows :

\[
L\{u_0(t)\} = \frac{a}{s} + \frac{b}{s^2}.
\]

\[
L\{u_1(t)\} = \left( 1 - \frac{c}{s} \right)^{-1} \left( \frac{bc}{s^3} + \frac{1}{s^2} L\{A_0(t)\} \right),
\]

\[
L\{u_n(t)\} = \left( 1 - \frac{c}{s} \right)^{-1} \left( \frac{1}{s^2} L\{A_{n-1}(t)\} \right), \quad n = 2, 3, 4, \ldots.
\]

By applying inverse Laplace transform for the Laplace decomposition series, we get

\[
u(t) = \sum_{n=0}^{\infty} u_n(t - n\omega)e(t - n\omega),
\]

(3.2.6)

where \( e(t - n\omega) \) is a unit step function given by,

\[
e(t - n\omega) = \begin{cases} 
0, & t < n\omega, \\
1, & t > n\omega.
\end{cases}
\]
Hence the approximate solution for each interval is given by,

\[ u(t) = \sum_{n=0}^{N} u_n(t - n\omega), \quad N\omega \leq t \leq (N + 1)\omega, \quad (3.2.7) \]

\[ N = 0, 1, 2, \ldots . \]

**Modified Laplace decomposition method**

In the Laplace decomposition method, \( L\{u_0(t)\} = \frac{a}{s} + \frac{b}{s^2} \). The term \( \frac{b}{s^2} \) will lead to unnecessary simplifications in each iteration. So in order to avoid this, we modify the Laplace decomposition method only at the first and second term. Let us apply the following modified Laplace decomposition

\[ L\{u(t)\} = L\{\tilde{u}(t)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{\tilde{u}_n(t)\}. \quad (3.2.8) \]

Using (3.2.5), we obtain

\[
\left(1 - \frac{c}{s}\right) \left[L\{\tilde{u}_0(t)\} + L\{\tilde{u}_1(t) - bt\} e^{-\omega s} + \sum_{n=2}^{\infty} e^{-n\omega s} L\{\tilde{u}_n(t)\}\right]
\]

\[
= \frac{a}{s} \left(1 - \frac{c}{s}\right) + \left(\frac{bc}{s^3} + \frac{1}{s^2} L\{\tilde{A}_0(t)\}\right) e^{-\omega s} + \frac{1}{s^2} \sum_{n=2}^{\infty} e^{-n\omega s} L\{\tilde{A}_{n-1}(t)\}, \quad (3.2.9)
\]

where

\[ L\{\tilde{u}_0(t)\} = \frac{a}{s}, \]

\[ L\{\tilde{u}_1(t) - bt\} = \left(1 - \frac{c}{s}\right)^{-1} \left(\frac{bc}{s^3} + \frac{1}{s^2} L\{\tilde{A}_0(t)\}\right). \]

\[ L\{\tilde{u}_n(t)\} = \left(1 - \frac{c}{s}\right)^{-1} \left(\frac{1}{s^2} L\{\tilde{A}_{n-1}(t)\}\right), \quad n = 2, 3, 4, \ldots \]

and \( \tilde{A}_n \) is \( n^{th} \) degree Adomian Polynomial [28] of \( f(u(t)) \) in the powers of \( \tilde{u}_1(t), \tilde{u}_2(t), \ldots, \tilde{u}_n(t) \) and it is similar to \( A_n \)'s of Laplace decomposition method.
Again by applying inverse Laplace transform for the modified Laplace decomposition series (3.2.8), we get

\[ u(t) = \tilde{u}(t) = \sum_{n=0}^{\infty} \tilde{u}_n(t - n\omega) e(t - n\omega), \]  

where \( e(t - n\omega) \) is a unit step function given by,

\[ e(t - n\omega) = \begin{cases} 
0, & t < n\omega, \\
1, & t > n\omega. 
\end{cases} \]

Hence the approximate solution for each interval \( N = 0, 1, 2, \ldots \) is given by,

\[ \tilde{u}(t) = \sum_{n=0}^{N} \tilde{u}_n(t - n\omega), \quad N\omega \leq t \leq (N + 1)\omega, \]  

3.3 Test Problems

Test Problem - 1

Let us consider the following linear differential-difference equation with differential order two and difference of order one:

\[ u''(t) - 2u'(t) + u(t - \omega) = 1, \quad t > \omega, \]  

with the initial interval condition

\[ u(t) = 2 + t, \quad 0 \leq t \leq \omega. \]

First, we note that as \( \omega \to 0 \), the equation (3.3.1) becomes linear second order differential equation with exact solution \( u(t) = 1 + e^t \). Hence we have selected the initial interval condition (3.3.2).
Laplace transform method for the differential equation

\[ u''(t) - 2u'(t) + u(t) = 1, \quad u(0) = 2, \quad u'(0) = 1. \]

By applying Laplace transform we obtain,

\[
\begin{align*}
(s^2 L\{u(t)\} - 2s - 1) &- 2(sL\{u(t)\} - 2) + L\{u(t)\} = \frac{1}{s} \\
(s^2 - 2s + 1)L\{u(t)\} &\quad = 2s - 3 + \frac{1}{s} \\
L\{u(t)\} &\quad = \frac{2s - 1}{s(s - 1)} \\
\Rightarrow u(t) &\quad = L^{-1}\left\{ \frac{1}{s} + \frac{1}{s - 1} \right\} = 1 + e^t.
\end{align*}
\]

Laplace decomposition method for (3.3.1) - (3.3.2)

First we note that,

\[
\begin{align*}
\int_\omega^\infty u'(t) e^{-st} \, dt &= L\{u'(t)\} - \int_0^\omega 1 e^{-st} \, dt = L\{u'(t)\} + \frac{1}{s} (e^{-\omega s} - 1) \\
\text{and} \quad \int_0^\omega u''(t) e^{-st} \, dt &= 0 \Rightarrow \int_\omega^\infty u''(t) e^{-st} \, dt = L\{u''(t)\}.
\end{align*}
\]

Following initial steps of the Laplace decomposition method for (3.3.1)-(3.3.2), we obtain

\[
L\{u''(t)\} - 2L\{u'(t)\} + \frac{2}{s} (e^{-\omega s} - 1) + e^{-\omega s} L\{u(t)\} = \frac{e^{-\omega s}}{s}.
\]

After applying formula of Laplace transform for first and second order derivative, finally, we arrive at

\[
\begin{align*}
s(s - 2)L\{u(t)\} &= \left( 2s - 3 - \frac{2}{s} \right) + \frac{3}{s} e^{-\omega s} - e^{-\omega s} L\{u(t)\} \\
\left( 1 - \frac{2}{s} \right) L\{u(t)\} &= \left( 1 - \frac{2}{s} \right) \left( \frac{2}{s} + \frac{1}{s^2} \right) + \frac{3}{s^3} e^{-\omega s} - \frac{e^{-\omega s}}{s^2} L\{u(t)\}.
\end{align*}
\]
Now we seek the following type of decomposition for $L \{u(t)\}$:

$$L \{u(t)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L \{u_n(t)\},$$

which may be regarded as Laplace decomposition. Now the main idea of Laplace decomposition is to set an iteration as follows:

$$\left(1 - \frac{2}{s}\right) \sum_{n=0}^{\infty} e^{-n\omega s} L \{u_n(t)\} = \left(1 - \frac{2}{s}\right) \left(\frac{2}{s} + \frac{1}{s^2}\right) + \left(\frac{3}{s^3} - \frac{L \{u_0(t)\}}{s^2}\right) e^{-\omega s} - \frac{1}{s^2} \sum_{n=2}^{\infty} e^{-n\omega s} L \{u_{n-1}(t)\}. \quad (3.3.4)$$

For $n = 0, 1, 2, \ldots$, equate the co-efficient of $e^{-n\omega s}$ on both sides of (3.3.4), we get

$$L \{u_0(t)\} = \frac{2}{s} + \frac{1}{s^2};$$

$$L \{u_1(t)\} = \left(1 - \frac{2}{s}\right)^{-1} \left(\frac{3}{s^3} - \frac{L \{u_0(t)\}}{s^2}\right);$$

$$L \{u_n(t)\} = \left(1 - \frac{2}{s}\right)^{-1} \left(-\frac{L \{u_{n-1}(t)\}}{s^2}\right), \quad n \geq 2.$$

For $n = 1$, we have

$$L \{u_1(t)\} = \left(1 - \frac{2}{s}\right)^{-1} \left(\frac{3}{s^3} - \frac{L \{u_0(t)\}}{s^2}\right)$$

$$= \left(\frac{1}{s^3} - \frac{1}{s^4}\right) \left(1 + \frac{2}{s} + \frac{4}{s^2} + \cdots + \frac{2^n}{s^n} + \cdots\right)$$

$$= \left(\frac{1}{s^3} - \frac{1}{s^4}\right) + \frac{1}{8} \left(\frac{2^4}{s^5} + \frac{2^5}{s^6} + \cdots + \frac{2^{n+4}}{s^{n+5}} + \cdots\right)$$

$$= \left(\frac{1}{s^3} - \frac{1}{s^4}\right) + \frac{1}{8} \left(\frac{1}{s-2} - \frac{1}{s} - \frac{2}{s^2} - \frac{4}{s^3} - \frac{8}{s^4}\right).$$
Again for \( n = 2 \), we have

\[
L \{ u_2(t) \} = \left( 1 - \frac{2}{s} \right)^{-1} \left( - \frac{L \{ u_1(t) \}}{s^2} \right)
\]

\[
= - \left( \frac{1}{s^5} + \frac{1}{s^6} + \frac{2}{s^7} + \frac{4}{s^8} + \cdots + \frac{2^n}{s^{n+6}} + \cdots \right)
\times \left( 1 + \frac{2}{s} + \frac{4}{s^2} + \cdots + \frac{2^n}{s^n} + \cdots \right)
\]

\[
= \left( \frac{1}{s^5} + \frac{1}{s^6} \right) - \frac{2}{s^6} - \frac{4}{s^7} - \frac{8}{s^8} - \frac{20}{s^9} - \frac{48}{s^{10}} - \cdots
\]

\[
= \left( \frac{1}{s^5} + \frac{1}{s^6} \right) - \frac{1}{2^3} \left( \frac{2^4}{s^5} + \frac{2^5}{s^6} + \frac{2^7}{s^7} + \cdots \right)
\]

\[
- \frac{1}{2^5} \sum_{n=2}^{\infty} \left( \frac{2^{n+5}}{s^{n+6}} + \frac{2^{n+6}}{s^{n+7}} + \frac{2^{n+7}}{s^{n+8}} + \cdots \right)
\]

\[
= \left( \frac{1}{s^5} + \frac{1}{s^6} \right) - \frac{1}{2^3} \left( \frac{1}{s - 2} - \frac{1}{s} - \frac{2}{s^2} - \frac{4}{s^3} - \frac{8}{s^4} \right)
\]

\[
- \frac{1}{2^5} \sum_{n=2}^{\infty} \left( \frac{1}{s - 2} - \frac{1}{s} - \frac{2}{s^2} - \frac{4}{s^3} - \cdots - \frac{2^n}{s^{n+4}} \right).
\]

By using (3.2.6), we obtain an approximate solution of \( u(t) \), for \( t > 0 \).

For \( 2\omega \leq t \leq 3\omega \), the exact solution is

\[
u(t) = \sum_{n=0}^{2} u_n(t - n\omega)
\]

\[
= 2 + t + \left[ \frac{(t - \omega)^2}{2!} + \frac{(t - \omega)^3}{3!} \right]
\]

\[
+ \frac{1}{2^3} \left( e^{2(t-\omega)} - 1 - 2(t - \omega) - \frac{4(t - \omega)^2}{2!} - \frac{8(t - \omega)^3}{3!} \right)
\]

\[
+ \left[ \frac{(t - 2\omega)^4}{4!} + \frac{(t - 2\omega)^5}{5!} \right]
\]

\[
- \frac{1}{2^3} \left( e^{2(t-2\omega)} - 1 - 2(t - 2\omega) - \frac{4(t - 2\omega)^2}{2!} - \frac{8(t - 2\omega)^3}{3!} \right)
\]

\[
- \frac{1}{2^5} \sum_{n=2}^{\infty} \left( e^{2(t-2\omega)} - 1 - 2(t - 2\omega) - \cdots - \frac{2^n}{(n+3)!} \right).
\]

(3.3.5)
When we truncate the solution at third term, we obtain

\[ u(t) \approx \sum_{n=0}^{2} u_n(t - n\omega) e(t - n\omega) \]
\[ u(t) \sim 2 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} \sim 1 + e^t \quad \text{as} \quad \omega \to 0 \quad \text{and} \quad t \to 0. \]

**Test Problem - 2**

Let us consider the following nonlinear differential-difference equation with both differential and difference of order one:

\[ u''(t) + \sin(u(t - \omega)) = 0, \quad t > \omega, \quad (3.3.6) \]

with the initial interval condition

\[ u(t) = 1 + t, \quad 0 \leq t \leq \omega. \quad (3.3.7) \]

**Modified Laplace decomposition method**

Following initial steps of the modified Laplace decomposition method for (3.3.6) - (3.3.7), we directly arrive at

\[ L\{u(t)\} = \frac{1}{s} + \frac{1}{s^2} - \frac{e^{-\omega s}}{s^2} L\{\sin(u(t))\}. \quad (3.3.8) \]

Now we seek the following type of modified Laplace decomposition for \( L\{u(t)\} : \)

\[ L\{u(t)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{\tilde{u}_n(t)\}. \quad (3.3.9) \]
Now let us expanding $L \{ \sin(u(t)) \} \text{ at } u = \tilde{u}_0$, by using Laplace decomposition as follows

$$
L \{ \sin(u(t)) \} = L \{ \sin(\tilde{u}_0(t)) \}
+ e^{-\omega s} L \{ \tilde{u}_1(t) \cos(\tilde{u}_0(t)) \}
+ e^{-2\omega s} L \left\{ \tilde{u}_2(t) \cos(\tilde{u}_0(t)) - \frac{1}{2} \tilde{u}_1^2(t) \sin(\tilde{u}_0(t)) \right\}
+ e^{-3\omega s} L \left\{ \tilde{u}_3(t) \cos(\tilde{u}_0(t)) - \tilde{u}_1(t) \tilde{u}_2(t) \sin(\tilde{u}_0(t)) - \frac{1}{6} \tilde{u}_1^3(t) \cos(\tilde{u}_0(t)) \right\}
+ \ldots
+ \sum_{n=0}^{\infty} e^{-n\omega s} L \left\{ \tilde{B}_n(t) \right\},
$$

(3.3.10)

where $\tilde{B}_i$’s are Adomian Polynomials [28] given below,

$$
\tilde{B}_0(t) = \sin(\tilde{u}_0(t)).
$$
$$
\tilde{B}_1(t) = \tilde{u}_1(t) \cos(\tilde{u}_0(t)).
$$
$$
\tilde{B}_2(t) = \tilde{u}_2(t) \cos(\tilde{u}_0(t)) - \frac{1}{2} \tilde{u}_1^2(t) \sin(\tilde{u}_0(t)).
$$
$$
\tilde{B}_3(t) = \tilde{u}_3(t) \cos(\tilde{u}_0(t)) - \tilde{u}_1(t) \tilde{u}_2(t) \sin(\tilde{u}_0(t)) - \frac{1}{6} \tilde{u}_1^3(t) \cos(\tilde{u}_0(t)).
$$
$$
\tilde{B}_4(t) = \tilde{u}_4(t) \cos(\tilde{u}_0(t)) - \frac{1}{2} \tilde{u}_1^2(t) \sin(\tilde{u}_0(t)) - \tilde{u}_1(t) \tilde{u}_2(t) \sin(\tilde{u}_0(t))
- \frac{1}{2} \tilde{u}_1(t) \tilde{u}_2(t) \cos(\tilde{u}_0(t)) + \frac{1}{24} \tilde{u}_1^4(t) \sin(\tilde{u}_0(t))
$$

and so on.

Now by using (3.3.9) and (3.3.10) in (3.3.8), we get,

$$
\sum_{n=0}^{\infty} e^{-n\omega s} L \{ \tilde{u}_n(t) \} = \frac{1}{s} + \frac{1}{s^2} - \frac{e^{-\omega s}}{s^2} \sum_{n=0}^{\infty} e^{-n\omega s} L \left\{ \tilde{B}_n(t) \right\}
$$
\[ L \{\tilde{u}_0(t)\} + L \{\tilde{u}_1(t) - t\} e^{-\omega_s} + \sum_{n=2}^{\infty} e^{-n\omega_s} L \{\tilde{u}_n(t)\} \]
\[ = \frac{1}{s} - \frac{1}{s^2} \sum_{n=1}^{\infty} e^{-n\omega_s} L \{\tilde{B}_{n-1}(t)\}. \quad (3.3.11) \]

For \(n = 0, 1, 2, \ldots\), equate the co-efficient of \(e^{-n\omega_s}\) on both sides of (3.3.11) to get
\(L \{\tilde{u}_n(t)\}\) and apply inverse Laplace transform to obtain \(\tilde{u}_n(t)\).

For \(n = 0\), \(L \{\tilde{u}_0(t)\} = \frac{1}{s} \Rightarrow \tilde{u}_0(t) = 1\).

For \(n = 1\), \(L \{\tilde{u}_1(t) - t\} = -\frac{1}{s^2} L \{B_0(t)\}\)
\[= -\frac{1}{s^2} L \{\sin(\tilde{u}_0(t))\} = -\frac{\sin(1)}{s^3}\]
\[\Rightarrow \tilde{u}_1(t) = t - \sin(1)\frac{t^2}{2!}.\]

For \(n = 2\), \(L \{\tilde{u}_2(t)\} = -\frac{1}{s^2} L \{\tilde{B}_1(t)\}\)
\[= -\frac{1}{s^2} L \{\tilde{u}_1(t) \cos(\tilde{u}_0(t))\}\]
\[= -\frac{\cos(1)}{s^4} + \frac{\sin(1) \cos(1)}{s^5}\]
\[\Rightarrow \tilde{u}_2(t) = -\cos(1)\frac{t^3}{3!} + \sin(1) \cos(1)\frac{t^4}{4!}.\]

For \(n = 3\), \(L \{\tilde{u}_3(t)\} = -\frac{1}{s^2} L \{\tilde{B}_2(t)\}\)
\[= -\frac{1}{s^2} L \left\{\tilde{u}_2(t) \cos(\tilde{u}_0(t)) - \frac{1}{2} \tilde{u}_1^2(t) \sin(\tilde{u}_0(t))\right\}\]
\[= \frac{\sin(1)}{s^5} + \frac{\cos^2(1) - 3 \sin^2(1)}{s^6} + \frac{3 \sin^3(1) - \sin(1) \cos^2(1)}{s^7}\]
\[\Rightarrow \tilde{u}_3(t) = \sin(1)\frac{t^4}{4!} + \left[\cos^2(1) - 3 \sin^2(1)\right]\frac{t^5}{5!} + \left[3 \sin^3(1) - \sin(1) \cos^2(1)\right]\frac{t^4}{4!}.\]
We note that, by applying inverse Laplace transform for the Laplace decomposition series (3.3.9), we get

\[ u(t) = \sum_{n=0}^{\infty} \tilde{u}_n(t - n\omega)e(t - n\omega) \]

\[ \approx \sum_{n=0}^{3} \tilde{u}_n(t - n\omega)e(t - n\omega) \]

\[ = 1 + \left[ (t - \omega) - \sin(1)\frac{(t - \omega)^2}{2!} \right] e(t - \omega) \]

\[ + \left[ -\cos(1)\frac{(t - 2\omega)^3}{3!} + \sin(1)\cos(1)\frac{(t - 2\omega)^4}{4!} \right] e(t - 2\omega) \]

\[ + \left[ \sin(1)\frac{(t - 3\omega)^4}{4!} + [\cos^2(1) - 3\sin^2(1)]\frac{(t - 3\omega)^5}{5!} \right. \]

\[ + \left. [3\sin^3(1) - \sin(1)\cos^2(1)]\frac{(t - 3\omega)^6}{6!} \right] e(t - 3\omega) \]

Hence as \( \omega \to 0 \), we get

\[ u(t) \sim 1 + t - \sin(1)\frac{t^2}{2!} - \cos(1)\frac{t^3}{3!} + [\sin(1)\cos(1) + \sin(1)]\frac{t^4}{4!} \]

\[ + \left[ \cos^2(1) - 3\sin^2(1) \right] \frac{t^5}{5!} + \left[ 3\sin^3(1) - \sin(1)\cos^2(1) \right] \frac{t^6}{6!} \]

\[ \sim \sum_{n=0}^{6} \frac{u^{(n)}(0)}{n!} t^n \text{ as } \omega \to 0 \text{ and as } t \to 0. \]

**Test Problem - 3**

Let us consider the following nonlinear differential-difference equation with differential order two and difference of order one:

\[ u''(t) = u^2(t - \omega), \quad t > \omega, \quad (3.3.12) \]

with the initial interval condition:

\[ u(t) = \frac{2}{3} - \frac{4}{9} t, \quad 0 \leq t \leq \omega. \quad (3.3.13) \]
First, we note that as $\omega \to 0$, the equation (3.3.12) becomes nonlinear second order differential equation

$$u''(t) = u^2(t), \quad u(0) = \frac{2}{3}, \quad u'(0) = -\frac{4}{9}. \quad (3.3.14)$$

**Exact Solution of (3.3.14)**

$$u''(t) = u^2(t), \quad u(0) = \frac{2}{3}, \quad u'(0) = -\frac{4}{9}$$

$$\Rightarrow \quad u'(t)u''(t) = u^2(t)u'(t)$$

$$\Rightarrow \quad \frac{(u'(t))^2}{2} - \left(-\frac{4}{9}\right)^2 = \frac{u^3(t)}{3} - \left(\frac{2}{3}\right)^3$$

$$\Rightarrow \quad u'(t) = -\sqrt{\frac{2}{\sqrt{3}}} (u(t))^\frac{3}{2}, \quad u(0) = \frac{2}{3}$$

$$\Rightarrow \quad (u(t))^{-\frac{1}{2}} = \frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{2} \left[ \frac{\sqrt{2}}{\sqrt{3}} t \right].$$

$$(u(t))^{-\frac{1}{2}} = \frac{1}{\sqrt{6}} (t + 3),$$

so the exact $u(t)$ is given by, $u(t) = \frac{6}{(t + 3)^2}$.

Hence we have selected the initial interval condition (3.3.13).

**Laplace decomposition method**

Following initial steps of the Laplace decomposition method for (3.3.12)-(3.3.13), we directly arrive at

$$L \{u(t)\} = \frac{2}{3} \frac{1}{s} - \frac{4}{9} \frac{1}{s^2} + \frac{e^{-\omega s}}{s^2} L \{u^2(t)\}, \quad (3.3.15)$$
where the nonlinear term \( u^2(t) \) is decomposed in terms of the Adomian Polynomial by using Laplace decomposition as follows:

\[
L\{u^2(t)\} = L\{u_0^2(t)\} + e^{-s}L\{2u_0(t)u_1(t)\} + e^{-2s}L\{2u_0(t)u_2(t) + u_1^2(t)\} + e^{-3s}L\{2u_0(t)u_3(t) + 2u_1(t)u_2(t)\} + \ldots + e^{-ns}L\{u_0(t)u_n(t) + u_1(t)u_{n-1}(t) + \ldots + u_n(t)u_0(t)\} = \sum_{n=0}^{\infty} e^{-ns}L\{C_n(t)\},
\]

where \( C_i \)'s are Adomian Polynomials [28],

\[
C_0(t) = u_0^2(t).
\]

\[
C_n(t) = u_0(t)u_n(t) + u_1(t)u_{n-1}(t) + \ldots + u_n(t)u_0(t), \quad \text{for } n \geq 1.
\]

Using (3.3.16), the equation (3.3.15) becomes

\[
\sum_{n=0}^{\infty} e^{-n\omega s}L\{u_n(t)\} = \left(\frac{2}{3s} - \frac{4}{9s^2}\right) + \frac{1}{s^2} \sum_{n=1}^{\infty} e^{-n\omega s}L\{C_{n-1}(t)\}.
\]

Equating the terms with co-efficient of \( e^{-n\omega s} \) on both sides of (3.3.17), we get \( L\{u_n(t)\} \).

An application of inverse Laplace transform will yield \( u_n(t) \).

\[
u_0(t) = \frac{2}{3} - \frac{4}{9}t, \quad (3.3.18)
\]

\[
u_n(t) = L^{-1}\left\{\frac{1}{s^2}L\{C_{n-1}(t)\}\right\}, \quad n \geq 1. \quad (3.3.19)
\]
The four term truncated approximate solution is

\[ u(t) \approx \sum_{n=0}^{3} u_n(t - n\omega) e(t - n\omega) \]

\[ = \left( \frac{2}{3} - \frac{2 \times 2}{3^2} t \right) e(t) \]

\[ + \left( \frac{2 \times 3}{3^3} (t - \omega)^2 - \frac{2 \times 4}{3^4} (t - \omega)^3 + \frac{2 \times 2}{3^5} (t - \omega)^4 \right) e(t - \omega) \]

\[ + \left( \frac{2 \times 3}{3^5} (t - 2\omega)^4 - \frac{2 \times 6}{3^6} (t - 2\omega)^5 + \frac{2 \times 4}{3^7} (t - 2\omega)^6 \right) e(t - 2\omega) \]

\[ - \frac{2 \times 8}{3^8 \times 7} (t - 2\omega)^7 e(t - 2\omega) \]

\[ + \left( \frac{2 \times 3}{3^7} (t - 3\omega)^6 - \frac{2 \times 48}{3^8 \times 7} (t - 3\omega)^7 + \frac{2 \times 45}{3^9 \times 7} (t - 3\omega)^8 \right) e(t - 3\omega) \]

\[ - \frac{2 \times 20}{3^{10} \times 7} (t - 3\omega)^9 + \frac{2 \times 4}{3^{11} \times 7} (t - 3\omega)^{10} \right) e(t - 3\omega). \quad (3.3.20) \]

We note that, by applying inverse Laplace transform for the Laplace decomposition series (3.3.15), we get

\[ u(t) = \sum_{n=0}^{\infty} u_n(t - n\omega) e(t - n\omega) \]

\[ \approx \sum_{n=0}^{3} u_n(t - n\omega) e(t - n\omega). \]

As \( \omega \to 0 \), (3.3.20) becomes

\[ u(t) \approx u_A(t) = \frac{2}{3} - \frac{2 \times 2}{3^2} t + \frac{2 \times 3}{3^3} t^2 - \frac{2 \times 4}{3^4} t^3 + \frac{2 \times 5}{3^5} t^4 - \frac{2 \times 6}{3^6} t^5 \]

\[ + \frac{2 \times 7}{3^7} t^6 - \frac{2 \times 8}{3^8} t^7 + \frac{2 \times 45}{3^9 \times 7} t^8 - \frac{2 \times 20}{3^{10} \times 7} t^9 + \frac{2 \times 4}{3^{11} \times 7} t^{10}. \quad (3.3.21) \]
Modified Laplace decomposition method

Next let us apply the following modified Laplace decomposition

\[ L\{u(t)\} = L\{\tilde{u}(t)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{\tilde{u}_n(t)\}. \tag{3.3.22} \]

Using (3.3.15), we obtain

\[ L\{\tilde{u}_0(t)\} + L\left\{ \tilde{u}_1(t) + \frac{4}{9} t \right\} e^{-\omega s} + \sum_{n=2}^{\infty} e^{-n\omega s} L\{\tilde{u}_n(t)\} = \frac{21}{3s} + \frac{1}{s^2} \sum_{n=1}^{\infty} e^{-n\omega s} L\left\{ \tilde{C}_{n-1}(t) \right\}, \tag{3.3.23} \]

where \( \tilde{C}_i \)'s are Adomian Polynomials [28],

\[
\tilde{C}_0(t) = \tilde{u}_0^2(t), \\
\tilde{C}_n(t) = \tilde{u}_0(t)\tilde{u}_n(t) + \tilde{u}_1(t)\tilde{u}_{n-1}(t) + \cdots + \tilde{u}_n(t)\tilde{u}_0(t), \quad \text{for } n \geq 1.
\]

For \( n = 0, 1, 2, \ldots \), equate the co-efficient of \( e^{-n\omega s} \) on both sides of (3.3.23), we get

\[ L\{\tilde{u}_0(t)\} = \frac{21}{3s}. \tag{3.3.24} \]

\[ L\left\{ \tilde{u}_1(t) + \frac{4}{9} t \right\} = \frac{1}{s^2} L\left\{ \tilde{C}_0(t) \right\}. \]

\[ L\{\tilde{u}_n(t)\} = \frac{1}{s^2} L\left\{ \tilde{C}_{n-1}(t) \right\}, \quad n \geq 2. \]

We note that, by applying inverse Laplace transform for the Laplace decomposition series (3.3.22), we get

\[ \tilde{u}(t) = \sum_{n=0}^{\infty} \tilde{u}_n(t - n\omega) e(t - n\omega) \]

\[ \approx \sum_{n=0}^{5} \tilde{u}_n(t - n\omega) e(t - n\omega). \]
Since \( \dot{u}_0(t) \) is a constant term when compared to \( u_0(t) \) which is a first degree polynomial, we compute six term truncated approximate solution for \( \ddot{u}(t) \) so that it matches with four term truncated approximate solution for \( u(t) \).

\[
\ddot{u}(t) \approx \sum_{n=0}^{5} \ddot{u}_n(t-n\omega) e(t-n\omega)
\]

\[
= \frac{2}{3} e(t) + \left( -\frac{2 \times 2}{3^2} (t - \omega) + \frac{2 \times 3}{3^3} (t - \omega)^2 \right) e(t - \omega)
\]

\[
+ \left( -\frac{2 \times 4}{3^4} (t - 2\omega)^3 + \frac{2 \times 3}{3^5} (t - 2\omega)^4 \right) e(t - 2\omega)
\]

\[
+ \left( \frac{2 \times 2}{3^5} (t - 3\omega)^4 - \frac{2 \times 6}{3^6} (t - 3\omega)^5 + \frac{2 \times 3}{3^7} (t - 3\omega)^6 \right) e(t - 3\omega)
\]

\[
+ \left( \frac{2 \times 4}{3^7} (t - 4\omega)^6 - \frac{2 \times 48}{3^8} (t - 4\omega)^7 + \frac{2 \times 18}{3^9} (t - 4\omega)^8 \right) e(t - 4\omega)
\]

\[
+ \left( -\frac{2 \times 8}{3^8} (t - 5\omega)^7 + \frac{2 \times 45}{3^9} (t - 5\omega)^8 - \frac{2 \times 50}{3^{10} \times 7} (t - 5\omega)^9 \right.
\]

\[
+ \frac{2 \times 15}{3^{11} \times 7} (t - 5\omega)^{10} \right) e(t - 5\omega). 
\]

\( (3.3.25) \)

As \( \omega \to 0 \), \( (3.3.25) \)

\[
u(t) \approx \ddot{u}_A(t) = \frac{2}{3} - \frac{2 \times 2}{3^2} t + \frac{2 \times 3}{3^3} t^2 - \frac{2 \times 4}{3^4} t^3 + \frac{2 \times 5}{3^5} t^4 - \frac{2 \times 6}{3^6} t^5
\]

\[
+ \frac{2 \times 7}{3^7} t^6 - \frac{2 \times 8}{3^8} t^7 + \frac{2 \times 9}{3^9} t^8 - \frac{2 \times 50}{3^{10} \times 7} t^9 + \frac{2 \times 15}{3^{11} \times 7} t^{10}.
\]

\( (3.3.26) \)

When \( \omega = 0 \), equation \( (3.3.12) \) becomes nonlinear second order differential equation,

\[
u''(t) = u^2(t), \quad u(0) = \frac{2}{3}, \quad u'(0) = -\frac{4}{9}
\]

and the exact solution is given by \( \frac{6}{(t + 3)^2} \).

The following table gives a comparative study of two approximate solutions \( u_A(t) \) given by \( (3.3.21) \) and \( \ddot{u}_A(t) \) given by \( (3.3.26) \) with the exact solution given by \( \frac{6}{(t + 3)^2} \):
The above three test problems illustrate the fact that Laplace decomposition method and modified Laplace decomposition method are indeed applicable and quite suitable for solving linear or nonlinear differential-difference equations of order (2, 1). The first test problem clearly indicate that the computation of exact solution in each interval is not simple. However the asymptotic relations guide in each step to arrive at the exact solution of the related differential equation when time and difference parameter approach to zero. In the second problem, the usual Laplace decomposition method leads to integration of successive iterative sine functions like \( \sin(\sin(u_0)) \) which is not desirable. Instead of that the modified Laplace decomposition is quite

<table>
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<th>( t )</th>
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<th>( u_A(t) )</th>
<th>( \tilde{u}_A(t) )</th>
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Table: 3.1
suitable in this situation. The third test problem indicate that the Laplace
decomposition method shows flexibility in rearranging the terms of the decomposition
series which will lead to modified decomposition method suitable for the computation.
The Numerical results of the test problem three show the following facts:

(i) $u_A(t)$ compares with $\tilde{u}_A(t)$ better near $t = 0$ than $t = 0.5$ and $t = 1.0$ in the
table.

(ii) $u_A(t)$ shows better rate of convergence near $t = 0$ than $t = 0.5$ and $t = 1.0$
in the table.

(iii) $u_A(t)$ shows better rate of convergence than $\tilde{u}_A(t)$ in the table.

The above conclusions give good motivation for the study of both linear and nonlinear
differential-difference equation of order $(1, 2)$ and $(2, 2)$ in the next chapter, with the
help of Laplace decomposition method.