Chapter 1

INTRODUCTION
1.1 Motivation

Principles of Physics such as Newton’s second law of motion, principle of least action, Hamilton’s principle and so on can be expressed in a beautiful manner using the mathematical language of differential equations. Differential equations also help Scientists, Engineers and Technologists to understand many practical problems such as problems of chemical kinetics, problems of ecology, problems of finance and so on, because they serve as mathematical models built on meaningful mathematical principles. There are several aspects of studies of differential equations such as

(i) Mathematical modelling of practical problems using differential equations [15, 18, 51, 52, 72, 76, 92].

(ii) Existence and Uniqueness of solutions [14, 19, 20, 39].

(iii) Stability and Controllability of the dynamical system [14, 20, 24, 76].

(iv) Bifurcation and Chaos [14, 51].

(v) Numerical solutions of differential equation, their Stability and Convergence [36, 74].

(vi) Series solutions yielding useful approximate solution when subjected to Pade approximation and/or Asymptotic approximation [1, 2, 3, 66, 87] and so on. The aim of the present thesis is to work on the (vi) and the last objective mentioned above.

For the purpose of getting a good motivation, let us describe a problem of mechanics
based on Newton’s second law of motion in the language of differential equation and
describe three methods to compute exact solution, namely, Adomian decomposition
series method, Laplace transform method and Laplace decomposition series method.

1.1.1 An Illustration for the application of the methods for
an Ordinary Differential Equation (ODE)

A natural application of Newton’s second law of motion to a simple problem
of finding position of a damped harmonic oscillator [51, 76] as a function of time
subjected to an external force when initial position and initial velocity are given can
be formulated mathematically as an initial value problem for a second order ordinary
differential equation.

Description of physical quantities:

\[ t \rightarrow \text{time variable, } t \geq 0. \]
\[ x(t) \rightarrow \text{position or displacement of the oscillator, a bounded} \]
\[ \text{and infinitely differentiable function of } t. \]
\[ x'(t) \rightarrow \text{velocity of the oscillator.} \]
\[ x''(t) \rightarrow \text{acceleration of the oscillator.} \]
\[ m \rightarrow \text{mass of the oscillator.} \]
\[ -2mx \rightarrow \text{restoring force acting on the oscillator.} \]
\[ -2mx'(t) \rightarrow \text{damped force acting on the oscillator.} \]
\[ me^{-t} \rightarrow \text{external force acting on the oscillator.} \]
\[ x(0) = 1 \quad \rightarrow \quad \text{initial displacement.} \]

\[ x'(0) = 0 \quad \rightarrow \quad \text{initial velocity.} \]

Then the initial value problem is

\[
m x''(t) = -2mx'(t) - 2mx(t) + me^{-t}, \quad t \geq 0
\]

\[
x(0) = 1, \quad x'(0) = 0
\]

or

\[
x''(t) = -2x'(t) - 2x(t) + e^{-t}, \quad t \geq 0
\]

\[
x(0) = 1, \quad x'(0) = 0.
\]

Since coefficients of \(x'(t)\) as well as \(x(t)\) and \(e^{-t}\) are continuous functions, by Picard’s theorem, the initial value problem has a unique solution [76].

First, it is easy to reduce the problem to a simpler problem without damping term as follows:

Put \(y(t) = e^t x(t)\) in

\[
x''(t) + 2x'(t) + 2x(t) = e^{-t}, \quad x(0) = 1, \quad x'(0) = 0
\]

\[
\Leftrightarrow \frac{d^2 y}{dt^2} + y = 1, \quad y(0) = 1, \quad y'(0) = 1.
\]

\textbf{Application of Adomian Decomposition Method to ODE}

Let us apply Adomian decomposition series [1]-[3] \(y(t) = 1 + \sum_{n=1}^{\infty} y_n(t)\) on both sides of \(y''(t) + y(t) = 1\) with the initial condition \(y(0) = 1\) and \(y'(0) = 1\), to get

\[
\sum_{n=1}^{\infty} \frac{d^2 y_n}{dt^2} + \sum_{n=1}^{\infty} y_n(t) = 0 \quad \text{or} \quad \frac{d^2 y_1}{dt^2} + \sum_{n=2}^{\infty} \frac{d^2 y_n}{dt^2} = 0 - \sum_{n=1}^{\infty} y_n(t).
\]
A simple iteration,

\[
\frac{d^2 y_1}{dt^2} = 0, \quad y_1(0) = 0, \quad y_1'(0) = 1.
\]

\[
\frac{d^2 y_k}{dt^2} = -y_{k-1}, \quad y_k(0) = 0 = y_k'(0), \quad k = 2, 3, 4, \ldots,
\]

will readily yield

\[
y_1 = t, \quad y_2 = -\frac{t^3}{3!}, \ldots, \quad y_k = (-1)^{k-1} \frac{t^{2k-1}}{(2k-1)!}, \ldots.
\]

Hence \( y(t) = 1 + t - \frac{t^3}{3!} + \cdots + (-1)^{k-1} \frac{t^{2k-1}}{(2k-1)!} + \cdots = 1 + \sin t \)

and \( x(t) = e^{-t}y(t) = e^{-t}(1 + \sin t) \) is the desired exact solution.

**Application of Laplace Transform Method to ODE**

Let us apply Laplace transform \([11, 13]\) on both sides of

\[
y''(t) + y(t) = 1,
\]

with the initial condition \( y(0) = 1 \) and \( y'(0) = 1 \), to get

\[
s^2 L\{y(t)\} - s - 1 + L\{y(t)\} = \frac{1}{s}
\]

\[
L\{y(t)\} = \frac{s + 1}{s^2 + 1} + \frac{1}{s(s^2 + 1)} = \frac{1}{s} + \frac{1}{s^2 + 1}
\]

\[
y(t) = 1 + \sin t.
\]

Hence we get the desired exact solution

\[
x(t) = e^{-t}y(t) = e^{-t}(1 + \sin t).
\]
Application of Laplace Decomposition Method to ODE

Let us apply decomposition series technique in the Laplace transform method, we take straightaway the equation

$$L \{y(t)\} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^2}L \{y(t)\}.$$ 

The Laplace decomposition series [5] can be taken as

$$L \{y(t)\} = \frac{1}{s} + \sum_{n=1}^{\infty} L \{y_n(t)\}$$

$$\frac{1}{s} + \sum_{n=1}^{\infty} L \{y_n(t)\} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^2} \left[ \frac{1}{s} + \sum_{n=1}^{\infty} L \{y_n(t)\} \right]$$

$$L \{y(t)\} + \sum_{n=2}^{\infty} L \{y_n(t)\} = \frac{1}{s^2} - \frac{1}{s^2} \sum_{n=2}^{\infty} L \{y_{n-1}(t)\}.$$ 

We can obtain an iteration to compute $L \{y_n(t)\}$ as follows:

$$L \{y_1(t)\} = \frac{1}{s^2}$$

$$L \{y_2(t)\} = -\frac{1}{s^2} L \{y_1(t)\} = -\frac{1}{s^4}$$

$$\vdots$$

$$L \{y_n(t)\} = -\frac{1}{s^2} L \{y_{n-1}(t)\} = \frac{(-1)^{n-1}}{s^{2n}}$$

$$\vdots$$

Hence by using Laplace decomposition series for $L \{y(t)\}$, we arrive at

$$L \{y(t)\} = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{s^{2n}}, \quad s > 1$$

$$= \frac{1}{s} + \frac{1}{s^2 + 1}$$

$$y(t) = L^{-1} \left\{ \frac{1}{s} \right\} + L^{-1} \left\{ \frac{1}{s^2 + 1} \right\} = 1 + \sin t$$

and hence the desired solution is $x(t) = e^{-t}y(t) = e^{-t}(1 + \sin t)$.
1.1.2 An Illustration for the application of the methods for a related Differential-Difference Equation (DDE)

Let us consider the following linear differential-difference equation or delay-differential equation with differential order two and difference of order one [8, 10]:

\[ y''(t) + y(t - \omega) = 1, \quad t > \omega, \]  

(1.1.1)

with the initial interval condition

\[ y(t) = 1 + t, \quad 0 \leq t \leq \omega. \]  

(1.1.2)

Application of Laplace Transform Method to DDE [10]

Now multiplying both sides of (1.1.1) by \( e^{-st} \) and integrate between \( \omega \) and \( \infty \), we obtain

\[
\begin{align*}
\int_{\omega}^{\infty} y''(t) e^{-st} dt + \int_{\omega}^{\infty} y(t - \omega) e^{-st} dt &= \int_{\omega}^{\infty} 1 \cdot e^{-st} dt \\
\int_{0}^{\infty} y''(t)e^{-st} dt + e^{-\omega s} \int_{0}^{\infty} y(t)e^{-st} dt &= \frac{e^{-\omega s}}{s} \\
L \{y''(t)\} + e^{-\omega s} L \{y(t)\} &= \frac{e^{-\omega s}}{s} \\
s^2 L \{y(t)\} - s - 1 + e^{-\omega s} L \{y(t)\} &= \frac{e^{-\omega s}}{s}
\end{align*}
\]

(1.1.3)

In step two, we have used \( y''(t) = 0, \ 0 \leq t \leq \omega \) and in step four, we have used \( y(0) = 1, \ y'(0) = 1 \).

\[
\left(1 + \frac{e^{-\omega s}}{s^2}\right) L \{y(t)\} = \frac{1}{s} + \frac{1}{s^2} + \frac{e^{-\omega s}}{s^3}
\]
Now by applying inverse Laplace transform, we get

\[
y(t) = 1 + t + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} e(t - n\omega).
\]

(1.1.4)

Hence the exact solution in each interval is given by

\[
y(t) = 1 + t + \sum_{n=1}^{N} \frac{(-1)^n}{(2n+1)!} \left( t - n\omega \right)^{2n+1} e(t - n\omega), \quad N\omega \leq t \leq (N+1)\omega, \quad N = 1, 2, 3, \ldots .
\]

(1.1.5)

In (1.1.4), when we allow \(\omega \rightarrow 0\), we get back the exact solution of the ODE, namely, \(y(t) = 1 + \sin(t)\).

**Application of Laplace Decomposition Method to DDE**

Let us again consider (1.1.1) and (1.1.2)

\[
y''(t) + y(t - \omega) = 1, \quad t > \omega,
\]

with the initial interval condition

\[
y(t) = 1 + t, \quad 0 \leq t \leq \omega.
\]

Borrowing initial steps from the Laplace transform method, we shall directly consider (1.1.3) and rewrite as:

\[
L \{y(t)\} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^2 (1 + \frac{e^{-\omega s}}{s^2})} - \frac{e^{-\omega s}}{s^2} L \{y(t)\}.
\]
On applying \( L \{ y(t) \} = \sum_{n=0}^{\infty} e^{-n \omega s} L \{ y_n(t) \}, \quad t > 0 \), which may be regarded as Laplace decomposition [5].

\[
\sum_{n=0}^{\infty} e^{-n \omega s} L \{ y_n(t) \} = \frac{1}{s} + \frac{1}{s^2} + \frac{e^{-\omega s}}{s^3} - \frac{e^{-\omega s}}{s^2} \sum_{n=0}^{\infty} e^{-n \omega s} L \{ y_n(t) \}. \tag{1.1.6}
\]

For \( n = 0, 1, 2, \ldots \), equate the co-efficient of \( e^{-n \omega s} \) on both sides of (1.1.6) to get \( L \{ y_n(t) \} \).

For \( n = 0 \),

\[
L \{ y_0(t) \} = \frac{1}{s} + \frac{1}{s^2} \Rightarrow y_0(t) = 1 + t.
\]

For \( n = 1 \),

\[
L \{ y_1(t) \} = \frac{1}{s^3} - \frac{1}{s^2} L \{ y_0(t) \} = -\frac{1}{s^4}.
\]

In general, for \( n \geq 1 \) we have,

\[
L \{ y_n(t) \} = -\frac{1}{s^2} L \{ y_{n-1}(t) \} = (-1)^n \frac{1}{s^{2n+2}}. \tag{1.1.7}
\]

By using (1.1.6) and (1.1.7), we have

\[
L \{ y(t) \} = \sum_{n=0}^{\infty} e^{-n \omega s} L \{ y_n(t) \}
= \frac{1}{s} + \frac{1}{s^2} + \sum_{n=1}^{\infty} \frac{(-1)^n e^{-n \omega s}}{s^{2n+2}}.
\]

Now by applying inverse Laplace transform, we get

\[
y(t) = 1 + t + \sum_{n=1}^{\infty} (-1)^n \frac{(t - n \omega)^{2n+1}}{(2n + 1)!} e(t - n \omega).
\]

Hence the exact solution \( y(t) \) takes the following form in each interval which is same as (1.1.8):

\[
y(t) = 1 + t + \sum_{n=1}^{N} (-1)^n \frac{(t - n \omega)^{2n+1}}{(2n + 1)!} e(t - n \omega), \quad N \omega \leq t \leq (N + 1) \omega,
\]

\[
N = 1, 2, 3, \ldots .
\]
With this motivation we work on the main objective, namely, to develop Laplace decomposition series solutions yielding useful interval wise exact or approximate solutions for linear or nonlinear DDEs when subjected to Pade approximation and/or Asymptotic approximation [6, 7, 12, 68].

1.2 Some Basic Definitions and Theoretical Results on Differential-Difference Equations

We quote two basic definitions given in [10]:

**Definition 1.2.1.** By a differential-difference equation, we mean an equation in an unknown function and certain of its derivatives, evaluated at arguments which differ by any of a fixed number of values. The differential order of an equation is the order of the highest derivative appearing and by the difference order one less than the number of distinct arguments appearing in the equation.

The general form of a linear differential-difference equation with constant coefficient of differential order \( n \) and difference of order \( m \) is

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} u^{(j)}(t - \omega_i) = f(t),
\]

where \( m \) and \( n \) are positive integers, \( 0 = \omega_0 < \omega_1 < \cdots < \omega_m \) and \( a_{ij} \) are real constants and \( f(t) \) is given real valued function defined for \( t > 0 \).
**Definition 1.2.2.** The set of all real functions having \( k \) continuous derivatives on an open interval \( t_1 < t < t_2 \) is denoted by \( C^k(t_1, t_2) \). If \( f \) is a member of this set, that is \( f \in C^k(t_1, t_2) \) for every \( t_2 > t_1 \), then \( f \in C^k(t_1, \infty) \).

We quote two standard theorems on differential-difference equations of order \((1, 1)\) with an initial interval condition [10] for existence of solution, continuity of the derivatives of the solution and applicability of Laplace transform method:

**Theorem 1.2.1.** Suppose that \( f \) is of class \( C^1 \) on \([0, \infty)\) and that \( g \) is of class \( C^0 \) on \([0, \omega]\). Then there exists one and only one function for \( t \geq 0 \) which is continuous for \( t \geq 0 \), which satisfies \( u(t) = g(t) \) for \( 0 \leq t \leq \omega \) and which satisfies the equation in

\[
a_0 u'(t) + b_0 u(t) + b_1 u(t-\omega) = f(t), \quad t > \omega.
\]

(1.2.2)

Moreover, this function \( u \) is of class \( C^1 \) on \((\omega, \infty)\) and of class \( C^2 \) on \((2\omega, \infty)\). If \( g \) is of class \( C^1 \) on \([0, \omega]\), \( u' \) is continuous at \( \omega \) if and only if

\[
a_0 g'(\omega - 0) + b_0 g(\omega) + b_1 g(0) = f(\omega).
\]

(1.2.3)

If \( g \) is of class \( C^2 \) on \([0, \omega]\), \( u'' \) is continuous at \( 2\omega \) if either (1.2.3) holds or else \( b_1 = 0 \) and only in these cases.

**Theorem 1.2.2.** Let \( u(t) \) be a solution of the equation

\[
L(u) = a_0 u'(t) + b_0 u(t) + b_1 u(t-\omega) = f(t)
\]

which is of class \( C^1 \) on \([0, \infty)\). Suppose that \( f \) is of class \( C^0 \) on \([0, \infty)\) and that

\[
|f(t)| \leq c_1 e^{c_2 t}, \quad t \geq 0,
\]

(1.2.5)
where \( c_1 \) and \( c_2 \) are positive constants. Let 
\[ m = \max_{0 \leq t \leq \omega} |u(t)|. \]
Then there are positive constants \( c_3 \) and \( c_4 \), depending only on \( c_2 \) and the coefficients in (1.2.4), such that
\[ |u(t)| \leq c_3 (c_1 + m) e^{c_4 t}, \quad t \geq 0. \]

(1.2.6)

For higher order equations, a similar theory on system of differential-difference equations with differential of order one is applied. For more details please refer [8, 10, 26, 27, 31, 41, 49, 50, 56, 58, 77].

1.3 Literature Survey

Ordinary differential equations (ODE), related integral equations (IE) and integro-differential equations (IDE) have been used to model physical phenomena since the concept of differentiation and integration were first developed. With the advent of modern computing facilities, nowadays complicated ODE, IE and IDE models can be solved numerically with a high degree of confidence. There is vast literature on ODE, IE and IDE including the following quoted references-[4, 9, 13], [14]-[16], [18]-[24], [26, 29, 32, 34, 39, 40, 42], [46]-[48], [51]-[55], [59]-[64], [66, 67], [69]-[73], [76], [79]-[82], [87, 89, 92]. It was recognized in the early part of the previous century by many mathematicians around the world in general and Russian mathematicians in particular that many physical phenomena may have a delayed effect in a differential equation, leading to what is called a delay differential equation or differential-difference equation (DDE). One of the simple and well known model of this kind is,
\[ u'(t) = u(t) - \alpha u(t - \omega), \]

(1.3.1)
for constant $\alpha > 0$, is simple enough to study analytically. It is the term involving a constant lag or delay $\omega > 0$ in the independent variable that makes this a DDE.

With reference to the above example, there are two important differences between DDE and ODE:

1. An obvious distinction between this DDE and an ODE is that specifying the initial value $u(0)$ is not enough to determine the solution for $t \geq 0$; it is necessary to specify the initial interval condition for $u(t)$ in $-\omega \leq t \leq 0$ or in $0 \leq t \leq \omega$.

2. A fundamental technique for solving a system of DDEs is to reduce it to a sequence of ODEs. Whereas a fundamental technique for solving linear ODE is to convert it into a polynomial equation with a substitution $u = e^{mt}$ to solve for $m$.

For instance, the simple model (1.3.1) is a constant coefficient, homogeneous DDE. Suppose if it were an ODE, we might solve it by looking for solutions of the form $u(t) = e^{\lambda t}$. Substituting this form into the ODE leads to an algebraic equation, the characteristic equation, for values $\lambda$ that provide a solution. For a first-order equation, there is only one such value. The same approach can be applied to DDEs. Here it leads first to

$$\lambda e^{\lambda t} = -\alpha e^{\lambda (t-\omega)} + e^{\lambda t}$$

and then to the characteristic equation

$$\lambda = -\alpha e^{-\lambda \omega} + 1.$$
In contrast to the situation with a first-order ODE, this algebraic equation has infinitely many roots $\lambda$. Asymptotic expressions for the roots of large modulus which show that the equation can have solutions that oscillate rapidly (For more detailed account please refer, chapter 9, [8] ).

There are several aspects of studies of DDE on the same lines as that of ODE such as

(i) Existence and Uniqueness of solutions.

(ii) Stability and Controllability of the dynamical system.

(iii) Numerical solutions of DDE, their Stability and Convergence.

(iv) Series solutions yielding useful approximate solution when subjected to Pade approximation and/or Asymptotic approximation

and so on, which can be surveyed with the current literature on DDE including the following references-[8, 10, 26, 27, 31, 33, 35, 41, 49, 50, 56, 58, 77].

Adomian recently developed the so called Adomian decomposition method or simply the decomposition method. The method proved to be reliable and effective for a wide class of equation, differential, integral and integro-differential equation for both linear and nonlinear models. The method provide the solution in the form of a series of functions. The method has the following interesting features:

1. An advantage of the method is that it can provide analytical approximation to a wide class of nonlinear problems without involving linearization or perturbation or descretization which lead to massive numerical computation.
2. When compared to other methods, the method does not change the original problem and therefore it is often physically more realistic.

3. It shows flexibility for many modifications for efficient computation. Such analytical solutions are guided by power series expansion and related Padé approximation as well as Asymptotic approximation [6, 7, 12, 68, 82, 86] for solving many nonlinear problems.

4. There is a good amount of evidence to show that the method is widely applicable, produces good approximations and the convergence is faster.

The literature on Adomian decomposition method and modified Adomian decomposition method can be surveyed in the current literature including the following references-[1]-[3], [12, 17, 28, 37, 65, 78], [84]-[86], [91].

Laplace transform method is a very powerful and effective analytic method to solve many linear equations such as ordinary as well as partial differential equations, integral equations, integro-differential equations with initial conditions. It has two important features:

1. Laplace transform converts a problem of mechanics to an algebraic problem. This physically means, it changes the problem from time domain to frequency domain.

2. With the help of unit step function it manages the discontinuities in the derivatives very well. As a result one can apply it to solve linear differential-difference equations or integro-differential-difference equations with initial interval condition.
There are many well written books on Laplace transform available in the literature including in the following references-[10, 13, 24, 25, 75, 76, 88].

The Laplace decomposition method is indeed fine blend of both Laplace transform method and Adomian decomposition methods. Laplace decomposition method shows flexibility as well as provides convenience in the computation of analytical solutions for both linear and nonlinear problems. The literature on Laplace transform method, Laplace decomposition method and modified Laplace decomposition method can be surveyed in the current literature including the following references-[5, 11, 13, 25, 30, 38], [43]-[45], [57, 75, 83, 88, 90].

In the present thesis with the help of the current literature, we study and apply two recent and very powerful analytic methods, namely, Laplace decomposition method and modified Laplace decomposition method. In the next section we present the outline of the thesis.

1.4 Outline of the Thesis

The thesis is divided into five chapters. The first chapter provides good motivation, sufficient survey of recent literature and outline of the thesis.

The second chapter is devoted to find the exact or approximate solution for the following linear as well as nonlinear differential-difference equations of order (1, 1) with appropriate initial interval condition by using Laplace decomposition method guided by Laplace transform method and Adomian decomposition method. The following simple types of differential-difference equations are considered for the study:
1. Linear differential-difference equation of order \((1, 1)\):

\[
\begin{align*}
u'(t) + c u(t - \omega) &= f(t), \quad t > \omega, \\
u(t) &= a + bt, \quad 0 \leq t \leq \omega,
\end{align*}
\]

where \(c \neq 0\), \(a\) and \(b\) are real constants, \(f(t)\) is a given function of exponential order and \(\omega\) is a positive difference parameter.

2. Nonlinear differential-difference equation of order \((1, 1)\):

\[
\begin{align*}
u'(t) &= f(u(t - \omega)), \quad t > \omega, \\
u(t) &= a + bt, \quad 0 \leq t \leq \omega,
\end{align*}
\]

where \(a\) and \(b\) are real constants, \(f(u)\) is a given nonlinear function of exponential order and \(\omega\) is a positive difference parameter.

The following three test problems are worked out with the appropriate initial interval condition:

(i) Linear differential-difference equation of retarded type guided by the literature [10] with both differential and difference of order one:

\[
u'(t) - u(t - \omega) = 1, \quad t > \omega.
\]

(ii) Nonlinear differential-difference equation with differential order one and difference of order one:

\[
u'(t) = u(t - \omega)(1 - u(t - \omega)), \quad t > \omega.
\]
(iii) Nonlinear differential-difference equation with both differential and difference of order one:

\[ u'(t) = \sin(u(t - \omega)), \quad t > \omega. \]

The **third chapter** is devoted to find the exact or approximate solution for the following linear as well as nonlinear differential-difference equations of order (2, 1) with appropriate initial interval condition by applying Laplace decomposition method and modified Laplace decomposition method which solve any linear or nonlinear second order differential equation with the main idea to make them applicable to solve differential-difference equations.

The following simple types of differential-difference equations are considered for the study:

1. Linear differential-difference equation of order (2, 1):

   \[
   u''(t) + c_1 u'(t) + c_2 u(t - \omega) = f(t), \quad t > \omega, \tag{1.4.3}
   \]

   \[
   u(t) = a + bt, \quad 0 \leq t \leq \omega,
   \]

   where \( c_2 \neq 0, \ c_1, a \) and \( b \) are real constants, \( f(t) \) is a given function of exponential order and \( \omega \) is a positive difference parameter.

2. Nonlinear differential-difference equation of order (2, 1):

   \[
   u''(t) = f(u(t - \omega)), \quad t > \omega, \tag{1.4.4}
   \]

   \[
   u(t) = a + bt, \quad 0 \leq t \leq \omega,
   \]

   where \( a \) and \( b \) are real constants, \( f(u) \) is a given nonlinear function of exponential order and \( \omega \) is a positive difference parameter.
The following three test problems are worked out with the appropriate initial interval condition:

(i) Linear differential-difference equation with differential order two and difference of order one:
\[ u''(t) - 2u'(t) + u(t - \omega) = 1, \quad t > \omega. \]

(ii) Nonlinear differential-difference equation with differential order two and difference of order one:
\[ u''(t) + \sin(u(t - \omega)) = 0, \quad t > \omega. \]

(iii) Nonlinear differential-difference equation with differential order two and difference of order one:
\[ u''(t) = u^2(t - \omega), \quad t > \omega. \]

In continuation with the studies of the previous chapters with a common objective, in the **fourth chapter**, the following types of differential-difference equations are considered for further study:

1. First order differential and second order difference equation of the following simple type:

\[
\begin{align*}
  u'(t) &= f(t) + F_1(u(t - \omega), u'(t - \omega)) \\
  &\quad + F_2(u(t - 2\omega), u'(t - 2\omega)) \quad t > 2\omega, \quad (1.4.5) \\
  u(t) &= k, \quad 0 \leq t \leq 2\omega,
\end{align*}
\]

where \( k \) is a known constant, the functions \( f, F_1 \) and \( F_2 \) are either linear or nonlinear functions and \( \omega \) is a positive difference parameter.
2. Second order differential and second order difference equation of the following simple type:

\[ u''(t) = f(t) + F_1(u(t - \omega), u'(t - \omega)) \]
\[ + F_2(u(t - 2\omega), u'(t - 2\omega)) \quad t > 2\omega, \quad (1.4.6) \]
\[ u(t) = k, \quad 0 \leq t \leq 2\omega, \]

where \( k \) is a known constant, the functions \( f, F_1 \) and \( F_2 \) are either linear or nonlinear functions and \( \omega \) is a positive difference parameter.

The following three test problems are worked out with the appropriate initial interval condition:

(i) Linear differential-difference equation with differential order one and difference of order two:
\[ 2u'(t) - u(t - \omega) = u(t - 2\omega), \quad t > 2\omega. \]

(ii) Nonlinear differential-difference equation with differential order one and difference of order two:
\[ u'(t) = 2 - u(t - \omega) + au^3(t - 2\omega), \quad t > 2\omega. \]

(iii) Nonlinear differential-difference equation with both differential and difference of order two:
\[ u''(t) - u'(t - 2\omega) = -1 + u^2(t - \omega) + \cos(u(t - \omega)), \quad t > 2\omega. \]

In continuation with the studies of the previous chapters with a common objective to demonstrate the fact that Laplace decomposition method shows flexibility as well
as provides convenience in the computation of analytical solutions for both linear and nonlinear problems, in the **fifth and the final chapter**, the following types of integro-differential-difference equations are considered for further study:

1. First order differential and first order difference integro-differential-difference equation of the following simple type:

\[
    u'(t) = k \ u'(t - \omega) + a \ f(t - \omega) + b \ g(u(t - \omega)) \\
    + c \ \int_0^t h(u(t_1 - \omega)) \, dt_1, \quad t > \omega, \quad (1.4.7)
\]

\[
    u(t) = \lambda, \quad 0 \leq t \leq \omega,
\]

where \( a, b, k, \lambda \) and \( c \neq 0 \) are known constants and \( f(t), g(u(t)) \) and \( h(u(t)) \) are given linear or nonlinear functions depending upon the particular problem discussed.

2. Second order differential and second order difference integro-differential-difference equation of the following simple type:

\[
    u''(t) = a \ f(t - \omega) + b \ g(u(t - \omega)) \\
    + c \ \int_0^t h(u(t_1 - 2\omega)) \, dt_1, \quad t > 2\omega, \quad (1.4.8)
\]

\[
    u(t) = \lambda, \quad 0 \leq t \leq 2\omega, \quad (1.4.9)
\]

where \( a, b \neq 0, c \neq 0 \) and \( \lambda \) are known constants and \( f(t), g(u(t)) \) and \( h(u(t)) \) are given linear or nonlinear functions depending upon the particular problem discussed.
The following three test problems are worked out with the appropriate initial interval condition:

(i) Linear integro-differential-difference equation with both differential and difference of order one:
\[ u'(t) - u'(t - \omega) = 1 - \omega + t + u(t - \omega) + \int_{\omega}^{t} u(t_1 - \omega) \, dt_1, \quad t > \omega. \]

(ii) Nonlinear integro-differential-difference equation with both differential and difference of order one:
\[ u'(t) - u'(t - \omega) = \sin(u(t - \omega)) + \int_{0}^{t} u^2(t_1 - \omega) \, dt_1, \quad t > \omega. \]

(iii) Nonlinear integro-differential-difference equation with both differential and difference of order two:
\[ u''(t) = u(t - \omega)u'(t - \omega) + \int_{0}^{t} \sin(u(t_1 - 2\omega)) \, dt_1, \quad t > 2\omega. \]

We have the following three common conclusions:

1. The Laplace decomposition method is indeed fine blend of both Laplace transform method and Adomian decomposition methods.

2. Laplace decomposition method shows flexibility as well as provides convenience in the computation of analytical solutions for both linear and nonlinear problems.

3. Laplace decomposition method has produced exact or approximate solution in each interval with smooth computation.