4.1. Introduction:

The Euclidean functional integral gives a very convenient representation of the wave function in the quantum mechanics of closed cosmologies as shown by Hartle [41], and Holliwell [90], where the wave function reads:

\[(4.1) \quad \psi[h_g, \phi] = \int Dg_{\mu\nu} D\phi \exp \left[ g_{\mu\nu} A_m[g_{\mu\nu}, \phi] \right].\]

where \(A_m[g_{\mu\nu}, \phi]\) be the matter action and \(A_g[g_{\mu\nu}]\) be the Euclidean Einstein-Halbert action for gravity.

\[(4.2) \quad A_g[g_{\mu\nu}, \phi] = \frac{1}{16\pi G} \int_M d^4x \ g^{\frac{1}{2}} (-R + 2\Lambda) \]
\[\quad - \frac{1}{8\pi G} \int_{\partial M} d^3x h^{\frac{1}{2}} K.\]
The path integral expression (4.1) generates solutions to the wheeler- Dewitt equation and momentum constraints

\[ H\psi = 0, \]  
\[ H_i \psi = 0, \]  

as presented by Halliwell [65], Halliwell and Hartle [91], Barvinsky [49].

There is one particularly appealing theory of cosmological initial conditions namely, the no- boundary proposal of Hartle and Hawking which is presented in terms of the form (4.1) as given by Hawking [25], Hartle and Hawking [29], Hawking [34]. Their proposal is take the functional integral to be over four- metric \( g_{\mu\nu} \) and matter fields \( \phi \) on compact manifolds \( M \) whose only boundary is a three- surface \( \partial M \), on which \( g_{\mu\nu} \) and \( \phi \) should match the argument of the wave function \( h_{ij} \) and \( \phi \). However, there are many technical aspects of a construction such as (4.1) that need to be specified before such an expression may be regarded as properly and uniquely defined. In addition to the usual Gauge fixing, measure etc. It is necessary to specify a contour of integration. The problem of obtaining a
suitable contour is nontrivial for gravity because the action (4.2) is not bounded from below on the space of real Euclidean four metrics. Hence, the convergent contours are necessarily complex. In a very different situation of the quantum theory of asymptotically flat space times, prescriptions such as the conformal rotation of Gibbons, Hawking and Perry [18] have been very successful in defining suitable complex integration contours. Such prescription is supported by explicit evaluation relating it to the Hamiltonian theory of the physical degrees of freedom as given by Hartle and Schleinich [56] and by the positive action theorem of Schoen and Yau [67].

However, for several reasons as shown by Horowitz [43], Halliwell and Hartle [93], none of this successfully explains, the quantum theory of closed cosmologies. Hence the contour remains unspecified. The lack of any proper prescription for the contour in the case of closed cosmologies suggests that a general investigation is needed. Such an investigation has been presented by Halliwell and Hartle [93], Halliwell and Louko [78], Hartle [80], Halliwell and Myers [81]. The present investigation is part of that investigation.

Halliwell and Louko [78, 79] presented in two papers I and II, they considered the problem of finding convergent contours of integration in
some simple quantum cosmological models. In paper I, the model was a spatially homogeneous and isotropic minisuperspace model with a positive cosmological constant referred to as the de Sitter mini super space model. The path integral consisted of functional integrals over the lapse and a single scale factor \(a(t)\). In paper II, the model consisted of a family of four-geometries, with the topology of the four-ball, and satisfying the Einstein equations, but with a fictitious value of the cosmological constant. This model was referred to as the de Sitter microsuperspace model. The functional integral reduced to a single ordinary integral over the fictitious cosmological constant.

4.2 General Technique:

We are mainly interested in the approximate evaluation of a path integral of the form (4.1). A frequently used procedure is to approximate such a path integral by

\[
\int Dg_{\mu\nu}\exp(-A[g_{\mu\nu}]) = \sum_{k} \exp(-A_k),
\]

Where \(A_k\) be the actions of solutions to the Euclidean-Einstein equations satisfying the same boundary conditions the path integral satisfies. Typically, there is more than one saddle point because there exist more than one four-
metric satisfying the Einstein equations, with the given boundary conditions. Furthermore, the Einstein equations as a boundary-value problem rarely have real solutions, and more often than note, the solutions are complex. More importantly, the validity of in approximation of the from (4.5) is very much dependent on the integration contour. From a naive application of eq. (4.5) one might have thought that the path integral is always dominated by the saddle point whose action has least real part. It will not be the case, however, unless the contour may be distorted into a steepest-descent contour along which this saddle point is the global maximum.

The Einstein-Hilbert Euclidean action (4.2), in (3+1) form, in usual notation reads

\[(4.6)\]

\[A[g_{\mu\nu}] = \frac{1}{16\pi G} \int \! d^3 x d^3 N \frac{1}{\hbar} \]

\[(k_{ij} k^{ij} - k^2 - 3R + 2\Lambda).\]

We are interested in minisuper space models, in which the lapse function N be homogeneous and three-metric components \(h_{ij}\) and possibly matter field contribution also, so that they may be described by
$q^\alpha(\tau), \alpha = 1, 2, \ldots, n$. In terms of these functions, the action \((4.6)\) assumes the form

\[
(4.7) \quad A[q(\tau)] = \int_{\tau'}^{\tau''} d\tau N \left[ \frac{1}{2N^2} f_{\alpha\beta} q^\alpha q^\beta + U(q) \right],
\]

Where $f_{\alpha\beta}$ be the minisupper space metric and has indefinite signature (-, +, +, +, .................) . By scaling $N$ and shifting $\tau$, one may take

\[
(4.8) \quad \tau' = 0, \quad \tau'' = 1.
\]

The action \((4.7)\) describes the path- integral construction between fixed initial and final $[q^\alpha]$ for system. Halliwell [66] has shown that in the Gauge

\[
(4.9) \quad \dot{N} = 0,
\]

the propagation amplitude reads

\[
(4.10) \quad G(q^\alpha / q^\alpha') = \int dN \int Dq^\alpha \exp\{-A[q(\tau)]\}.
\]

Hence, the integral over all times $N$ of an ordinary quantum- mechanical propagator:
(4.11) \[ <q^\alpha, N/q^\alpha, 0> = \int Dq^\alpha \exp \{-A[q(\tau)]\}. \]

If one assumes N to have infinite range and by using eq. (4.11), we may show that eq. (4.10) is a solution to the Wheeler-Dewitt equation.

(4.12) \[ \hat{H} G(q^\alpha | q') = \left[-\frac{1}{2} \nabla^2 + \xi R + U(q)\right] G(q^\alpha | q') \]

\[ = 0. \]

Where \( \nabla^2 \), R and \( \xi \) are the Laplacian, curvature scalar in the metric \( f_{\alpha\beta} \) and conformal coupling. Again if one assumes N to have a half-infinite range, with one of the end points \( N = 0 \), then eq. (4.10) be a green function of the Wheeler-Dewitt operator i.e. we obtain a \( \delta \)– function on the right hand side of eq. (4.12). Another possibility is to integrate N over a non-trivial closed loop in complex N plane. The closed contour gives a solution to the wheeler-Dewitt equation. These, considerations concerning the derivation of wheeler-Dewitt equation are dependent on the problem of convergence. We have two problems concerning convergence. First, since \( A \) is not positive definite, (4.11) may not converge integrate over real metric components \( q^\alpha \). Hence, we may
assume that complex contour will be obtained for the $q^\alpha$ which makes (4.11) well defined. Second, (4.10) will not converge unless $N$ is integrated along a complex contour.

The saddle points of the path integral (4.10) are:

\[
(4.13) \quad \delta A/\delta q^\alpha = 0 ,
\]

\[
(4.14) \quad \delta A/\delta N = 0 ,
\]

i.e.

\[
(4.15) \quad \frac{q^\alpha}{N^2} + \frac{1}{N^2} \Gamma_{\beta\gamma}^{\alpha} q^\beta q^\gamma - f_{\alpha\beta \gamma} \frac{\partial U}{\partial q^\beta} = 0 ,
\]

and

\[
(4.16) \quad \int_0^1 d\tau \left[ -\frac{1}{2N^2} f_{\alpha\beta \gamma} q^\beta q^\gamma - U(q) \right] = 0,
\]

with boundary condition
Let us put

\begin{equation}
\alpha \equiv q(0) = q
\end{equation}

\begin{equation}
\alpha' \equiv q(1) = q'
\end{equation}

Where \( q^{-\alpha}(\tau) \) be the solution i.e. saddle point, of eq. (4.15) and \( Q^\alpha(\tau) \) vanishes at the both ends. Let us insert eq. (4.19) in eq. (4.7), and expanding to quadratic order in Q, we get.

\begin{equation}
A[q(\tau)] = A_o(q^{\alpha''}, \frac{N}{q^{\alpha'}}, 0) + A_2
\end{equation}

\begin{equation}
[q(\tau), Q(\tau)] + \ldots
\end{equation}

Where \( A_o \) be the action of the solution \( q^{-\alpha}(\tau) \) and satisfies the time-dependent Euclidean Hamilton-Jacobi equation

\begin{equation}
\frac{1}{2} f^{\alpha\beta} \frac{\partial A_o}{\partial q^\alpha} \frac{\partial A_o}{\partial q^\beta} - U(q) = - \frac{\partial A_o}{\partial N},
\end{equation}

and \( A_2 \) be quadratic in Q. By inserting (4.19) and (4.20) in eq. (4.10), we obtain.
\[ G(q^{\alpha''} | q^{\alpha'}) \approx \int dN \ \exp \ (-A_{o}) \int DQ^{\alpha} \]

\[ \exp \ (-A_{2}) . \]

Now one may perform the functional integral over Q. As \( A_{2} \) will not be positive definite, it will be necessary to obtain complex contours for Q to ensure convergence. However, \( A_{2} \) is quadratic Q, the integrals are Gaussians for which convergent contours are easily obtained and are unique. In view of standard result for Q integral, we get.

\[ G(q^{\alpha''} | q^{\alpha'}) \approx \int dN \left[ \frac{\partial^{2} A_{o}}{\partial q^{\alpha''} \partial q^{\alpha'}} \right]^{\frac{1}{2}} \]

\[ \cdot \exp \ [-A_{o} (q^{\alpha''}, N q^{\alpha'}, o)]. \]

Saddle points at the points in the complex plane

\[ N = N_{K} , \]

for which

\[ \partial A_{o} / \partial N = 0 \]

If follows from eq. (4.21) that the action at these saddle points
must satisfy the time-dependent Hamilton-Jacobi equation

\[ \frac{1}{2} f^{\alpha \beta} \frac{\partial A_k}{\partial q^\alpha} \frac{\partial A_k}{\partial q^\beta} - U(q) = 0. \]  

Let us now investigate the steepest-descent contours which run between these saddle points. These are the curves on which \( I_m(A_0) \) remains constant. One may then enumerate the distinct contours for which the path integral converges and observe which saddle points dominate along each contour.

It is an important to note that the method presented above does not as much constitute a prescription for the contour. Rather, it is a very convenient method of enumerating all the possible contours, in that any reasonable contour may be distorted into a sequence of steepest-descent contours.

(4.3) **Propagation Amplitude:**

Let us consider a class of minisuper-space models presented by a Euclidean metric of the from
\[ \begin{align*}
\mathbf{ds}^2 & = \sigma^2 \left[ \frac{N^2(\tau)}{a^2(\tau)} d\tau^2 + a^2(\tau) dr^2 + b^2(\tau) d\Omega^2(k) \right], \\
\end{align*} \]

Where \( d\Omega^2_2(k) \) be the metric on a compact orientable two- surface with constant Ricci Scalar.

\[ \begin{align*}
2\mathbf{R} & = 2k, \\
\end{align*} \]

and \( r \) coordinate as peridotic with period \( 2\pi \). Let us select

\[ \begin{align*}
k & = \pm 1 \quad \text{or } k = 0 \\
\end{align*} \]

where \( K = +1 \) for the two surfaces be the unit two sphere, and the model is termed as the Kantowski- Sachs model. For \( k = -1 \), the two surface are compactified unit two- psedosphpher, and the model is known as the locally rotationally symmetric case of Bianchi type III. Again for \( k=0 \) gives the two-surface is a flat tow- torus and the model is known as a special case of Bianchi type I. The prefactor reads

\[ \begin{align*}
\sigma^2 & = 2G/V_2 \\
\end{align*} \]

where \( V_2 \) be the volume of the two surface with the metric \( d\Omega^2_2(k) \).
One may obtain the nimi super space action for fixed initial and final values of the scale factors $a$ and $b$ by inserting the eq. (4.28) into the Euclidean action of general relativity (4.6) and integrating over the spatial slice let us put

(4.32) \[ c = a^2 b , \]

we get

(4.33) \[ A = \frac{1}{2} \int_0^1 d\tau \left( \frac{\dot{b}c}{N} + N(\lambda b^2 - k) \right) . \]

where

(4.34) \[ \lambda = \sigma^2 \Lambda , \]

and $\Lambda$ as the cosmological constant. Let us discuss the cases $\lambda \geq 0$ for $k = \pm 1$ and $\lambda > 0$ for $k = 0$ .

we may obtain the Einstein equations by varying the action (4.33) with respect to $b$, $c$, and $N$. In a gauge in which

(4.35) \[ \dot{N} = 0 , \]
the field equations read

\begin{align*}
\text{(4.36)} & \quad \dot{b} = 0 , \\
\text{(4.37)} & \quad \frac{\ddot{c}}{N^2} + 2\lambda b = 0 ,
\end{align*}

and the constraint equation as

\begin{align*}
\text{(4.38)} & \quad \frac{\dot{b}\dot{c}}{N} + N(\lambda b^2 - k) = 0 .
\end{align*}

The path integral with this boundary value reads:

\begin{align*}
\text{(4.39)} & \quad G(b'', c'' \mid b', c') = \int dN \int Db \int Dc \exp (-A),
\end{align*}

where \(A\) be the action (4.33). The integral is over \((b, (\tau), c(\tau), N)\) satisfying the boundary conditions.

\begin{align*}
\text{(4.40)} & \quad b(0) = b' , \\
\text{(4.41)} & \quad c(0) = c' , \\
\text{(4.42)} & \quad b(1) = b'' ,
\end{align*}
With $N$ unrestricted

Now we obtain the solutions as

\begin{equation}
\tilde{b}(\tau) = (b''-b')\tau + b',
\end{equation}

\begin{equation}
\tilde{c}(\tau) = -\frac{\lambda N^2}{3} (b''-b')\tau^3 - \lambda N^2 b'\tau^2 + \left\{c''-c'+\frac{\lambda N^2}{3} (b''+2b')\right\}\tau + c'.
\end{equation}

Let us now perform shift of integration variables in the path integral as

\begin{equation}
b(\tau) = \tilde{b}(\tau) + x(\tau),
\end{equation}

\begin{equation}c(\tau) = \tilde{c}(\tau) + y(\tau).
\end{equation}

In view of the boundary conditions, $x(\tau)$ and $y(\tau)$ must vanish at both end points. In terms of these new variables the action assumes the form

\begin{equation}A = A_0(b'',c,N|b',c') + A_2[x(\tau),y(\tau),N],
\end{equation}
\[
(4.49) \quad A_0 = \frac{\lambda}{6} (b''^2 + b'' b' + b'^2) - \frac{1}{2} k]N - \frac{(c''-c')(b''-b')}{2N},
\]

\[
(4.50) \quad A_2 = \frac{1}{2} \int_0^1 \left[ -\frac{x y}{N} + N\lambda x^2 \right] \, dx.
\]

hence the path integral (4.39) now assumes the form.

\[
(4.51) \quad G(b'',c'' \mid b',c') = \int dN \exp(-A_0) \cdot \int Dx \, Dy \exp(-A_2).
\]

Now, let us obtain contours for \(N, x\) and \(y\) such that eq. (4.51) converges. The \(x\) and \(y\) integrals are Gaussians and hence the contours for them are unique for fixed \(N\). The contours for them are unique for fixed \(N\). The functional integral over \(x\) an \(y\) is evaluated and giving a result proportional to \(N^{-1}\). Hence the eq. (4.51) takes form

\[
(4.52) \quad G(b'',c'' \mid b',c') = \int \frac{dN}{N} \exp(-A_0).
\]
Let us now perform a steepest-descent analysis of this integral. For all values of $k$ i.e. $k = \pm 1$ and $k = 0$, we have three models. In view of this the integral (4.52) assumes the form

\[
F(\alpha, \beta) = \int \frac{dN}{N} \exp \left\{ \frac{1}{2} \left( \alpha N + \frac{\beta}{N} \right) \right\},
\]

where

\[
\alpha = k = \frac{\lambda}{3} (b''^2 + b''b' + b'^2),
\]

\[
\beta = (c'' - c')(b'' - b').
\]

The saddle points are obtained as

\[
\frac{\partial A_0}{\partial N} = 0.
\]

The $\alpha$ and $\beta$ both are nonvanishing for generic values of the boundary data, hence, the integrand has two saddle points. For $k = \lambda = 0$, then $\alpha = 0$, the integral has no saddle points. For $\alpha$ and $\beta$ having the same sign, the saddle points lie on the real axis, at

\[
N = \pm (\beta/\alpha)^{1/2}.
\]
The steepest-descent path passing through these saddle points are given in Fig. (4.1a), for the case

\[(4.58) \quad \alpha, \beta > 0,\]

when \(\alpha, \beta < 0\), the paths are the same, but the arrows are reversed. for \(\alpha, \beta\) having positive signs, the saddle points lie on imaginary axis, at

\[(4.59) \quad N = \pm i(-\beta/\alpha)^{\frac{1}{2}}.\]

The steepest-descent path passing through these saddle points are given in fig. (4.1b) for \(\alpha > \beta > 0\). Again the paths are same when \(\alpha < 0 < \beta\) with reversed arrows. Let us consider the case where \(\alpha\) and \(\beta\) have the same sign, then there two steepest-descent contours along which the integral converges. Let us take this contour to be oriented in the counter-clockwise direction. The integral may be evaluated by using the method of residues and the result is at \(\alpha\beta\) as

\[(4.60) \quad F = 2\pi iJ_o (\alpha\beta)^{\frac{1}{2}} \exp [(\alpha\beta)]^{\frac{1}{2}}\]

The dominate contribution to the circular contour comes from
\begin{equation}
N = +\left(\frac{\beta}{\alpha}\right)^{1/2} \text{ for } \alpha, \beta > 0
\end{equation}

and

\begin{equation}
N = -\left(\frac{\beta}{\alpha}\right)^{1/2} \text{ for } \alpha, \beta < 0.
\end{equation}

The second steepest descent contour is the positive real axis for \( \alpha, \beta < 0 \), and the negative real axis for \( \alpha, \beta > 0 \), along which the integral converges. One may take these contours to be oriented in order to start at the origin. We may have a standard result which gives both results

\begin{equation}
F = 2J_\phi(\alpha\beta)^{1/2} \sim \exp[-(\alpha\beta)^{1/2}].
\end{equation}

Let us consider the case where \( \alpha \) and \( \beta \) are of opposite signs and steepest contours are given by Fig. (4.1b). There may three contours along which the integral converges. The first be the semi-infinite contours starting from the first be the semi-infinite contour starting from the origin, passing through the saddle point at positive imaginary \( N \), and going off to \( \text{Re}(N) \rightarrow -\infty \) for \( \beta < 0 < \alpha \) and to \( \text{Re}(N) \rightarrow +\infty \) for \( \alpha < 0 < \beta \). One may take the orientation, such that the contours starts at the origin. The contour maybe
deformed into the positive imaginary axis, and for $\beta < 0 < \alpha$ a standard result gives

$$F = i\pi H_o^{(1)} (-\alpha\beta)^{\frac{1}{2}} \sim e^{\pi i (-\alpha\beta)^{\frac{1}{2}}},$$

and for $\alpha < 0 < \beta$, one obtains the complex conjugate of eq. (4.64). The second contour may be the complex conjugate of the first one. The third contour consists of the first two joined together at origin, where $H_o^{(1)}$ is taken Hankel Function.

4.4 Concluding Remarks:

We have presented the contour of integration in path-integral approach to quantum cosmology. We have described evaluation a general technique for the approximate evaluation of the path integral for spatially homogeneous minisuperspace models. In this technique the path integral reduces to a single ordinary integration over the lapse after some trivial functional integrals. Then we have studied the lapse integration contours in detail by finding the steepest-descent paths. By selecting different complex contours, different solutions to the wheeler-Dewitt equation, may be obtained. The method may also be very useful for generating and studying the complex solution to the Einstein equations that inevitably arise as saddle
points. One may apply this method to a class of anisotropic minisuperspace models, such as Bianchi type I and III, and the Kantowski-Sachs model. We may regard the amplitude evaluated here as a candidate wave function of the Universe and determine the extent to which the contours satisfy the criteria:

1. The integral should converge, which implies that the contour should be complex.

2. The wave function generated should satisfy the Wheeler-Dewitt equation and momentum constraints.

3. The wave function should be consistent with a prediction of classical space time when the Universe is large.

4. In the limit that space-time become classical i.e. at the saddle points of the functional integral over metrics satisfying the condition (3), conventional quantum field theory for $\phi$ in those classical backgrounds should be recovered. This implies a further restriction on the nature of the dominating saddle points, namely, they should have $\text{Re} \left( g^{1/2} \right) > 0$. 
5. Coleman [69] has suggested that the extent the cosmological constant is a variable assumed, the contour should be consistent with the vanishing of the low-energy effective cosmological constant. The real part $A_g$ of the action of a complex solution, loosely speaking, gives a measure $e^{-A_g}$ on possible values of $\wedge$. Again this implies that the dominating saddle points must satisfy $\text{Re. } (g^{\frac{1}{2}}) > 0$. 

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