Chapter 1

Introduction

Partial differential equations (PDEs) are widely used to describe complex phenomena in various fields of science and technology such as physics, engineering and geometry. The mathematical modeling of these systems are usually governed by a single differential equation or a system of differential equations. Therefore solving such equations is of prime importance from both mathematical and application points of view.

Most of such physical phenomena include the motion of complex flows which be governed by highly nonlinear systems of partial differential equations and so exact solutions are not always possible. Thus it is globally accepted that in study the nonlinear partial differential equations, finding explicit solutions are almost more difficult, if not impossible to find, and is considered to be a difficult and confusing endeavor. Hence there is no existing general theory for completely solving nonlinear partial differential equations. Many methods are used by the scientists for study these equations such as Approximation methods, Perturbation Methods, Advanced Numerical Methods etc. These methods are applicable to very limited classes of problems; and only provide particular solutions. That is a reason for searching of new methods to deal with nonlinear problems.

Similarity methods are one of the most powerful analytic tools currently available in the study of nonlinear partial differential equations and which have the property of
simplifying the analysis of the problems. This is why similarity analysis is a subject becoming more and more important. It is the only universal and effective method for solving nonlinear differential equations analytically.

The transformations which reduce number of independent variables in partial differential equations at least one less than that of the original equation by exploiting an inherent symmetry of the problem are designated as similarity transformations, the methods used to search such a transformations are known as similarity methods and the solutions obtained by employing similarity transformations are known as similarity solutions.

There are many methods by which the similarity transformations can be obtained, those methods are classified into two categories. The first category makes no use of group theory such as

- Dimensional Analysis Method
- Free Parameter Method given by Gies (1955)
- Separation of Variables Methods proposed by Abbott and kline (1960).

while the second depend on invoked invariance of differential equations under group of transformations (Group theoretic methods) such as

- Birkhoff and Morgan’s Method (1952);
- Hellums-Churchill procedure (1964);
- Finite Group Method by Moran and Gaggioli (called also Deductive Group theoretic technique or Deductive Similarity Method) (1968);
- Classical Method by Lie (1922);
- Nonclassical Method by Bluman and Cole (1974);
1.1 Literature Review

Symmetry analysis for DEs was introduced by Marius Sophus Lie in the 1870s as general principles for finding solutions of systems of ODEs and PDEs use the group continuous transformations. After that concept of similarity appeared in physical context with dimensional analysis Helmholtz [55]. Ten years later, Reynolds [122] demonstrated the importance of similarity parameters in physics followed by Boltzmann [33] who used the algebraic symmetry to study the diffusion of particles with a diffusion coefficient proportional to their concentration, but the basic idea to exploit the algebraic symmetry of a partial differential equation is due to the work of Birkhoff [24]. In the first half of the nineteenth century the similarity solutions appear extensively in the work of Prandtl [116] and Blasius [26], on boundary layers in fluid mechanics and extended by Falkner and Skan [64], to the case where the velocity distribution of the inviscid flow is a power law.

Birkhoff [24] is pioneer in developing the similarity solutions of partial differential equations under the appropriate one-parameter group of transformations. Morgan [96] investigated quite thoroughly the theories for developing similarity solutions of partial differential equations. later, Michal [87] extended Morgan’s theorem to similarity transformations which reduce the number of independent variables by more than one.

The classical work of Birkhoff [25] played a key role in bringing attention to Lie’s ideas by clarifying the relationship between group invariance and dimensional analysis as applied to problems in fluid mechanics. Since that fine study, the applications and generalizations of group analysis have blossomed. Manohar [85], Hansen [47] and Ames [16, 123], extend the methods to special forms of n-parameter groups. A variety methods, theories and applications were presented by Moran and his co-workers [89, 95], for reducing systems of partial differential equations where they formulated a simplification of general group theory techniques based upon elementary group theory and earlier methods due to Birkhoff [25] and Morgan [96].

More detailed of techniques for finding the similarity solutions of partial differential equations, is to be found also in a number of literatures. These include Ovsian-
nikov [110], Ames [13,17,123], Seshadri and Na [130], Bluman et al. [27,32], Hansen [47] introduced the concept of similarity through four different types of methods namely Separation of variables method, Free parameter method, Group theory method and dimensional analysis method, Ibragimov [62], Barenblatt [19], Na [99,101], Hill [56]. An excellent general theory of the subject is presented by Olver [105]. For more references see the book of Ames [123].

A careful look at the these literature reveals that direct methods which do not exploit group invariance are straightforward and simple to apply. The drawback of such methods is that most of them lead to single solution of a given problem. In contrast, group theoretic methods that exploit group invariance under the infinitesimal groups of transformations contain systematic algebraic manipulation. So the tremendous amount of work necessary to derive a solution of a given differential equations, this is the main deficiency of such methods. While group theoretic methods which exploiting group invariance under finite groups of transformations are less difficult and broadly applicable because a simple group is assumed at the outset of the analysis. All the group theoretic methods require invariance of the given problem for determining the similarity transformation and then the reduction of the problem, this is lengthy and tedious procedures especially in case the differential equations which are nonlinear, huge and complicated in nature.

The work carried out in this thesis is devoted to find a systematic method to achieve invariance and reduction of a given problem in one step; and thus to overcome the long and arduous processes.

1.2 Thesis Outline

This thesis is divided into ten chapters, including this introductory chapter and the last chapter on conclusion. In addition to that all chapters also have their own introduction and conclusion. The content of each chapter is briefly summarized below:

Chapter 1:
The first chapter begins with the general introduction of differential equations and similarity methods. It is followed by a literature review and the outline of this thesis.

Chapter 2:

In this chapter, we provide a preliminary review of notations, definitions and theorems used throughout the thesis, and which be needed for the subsequent chapters and forms a basis for these chapters. Concepts of the symmetry and the invariant of differential equations (DE) are presented. Finally, general definitions of finite and infinitesimal group of transformations are given with explain the relation between them.

Chapter 3:

Some mathematical techniques to determining the similarity transformations which reduce the number of independent variables in partial differential equations are reported in this chapter. The similarity methods which makes no use to group theory (Direct Methods); and which depend on invoked invariance of differential equations under group of transformations (Group theoretic methods) are discussed briefly. Some of them are explained with the help of an illustrative example.

Chapter 4:

In this chapter, we formulate a suitable systematic procedure (new method) for similarity analysis which achieves the invariance and the reduction of the problem at one time. This systematic method depend on proceed in a manner converse to that of Morgan’s (Theorem 2.5.1). More precisely, if a given problem reduces to a form in terms of new variables which be generated by invariants of certain group, then these new variables are similarity variables and the problem is invariant under the group.

Chapter 5:

This chapter is devoted to deriving the absolute invariants of somewhat general group of transformations, which be needed to apply the proposed method which presented in chapter 4. The method which depend on reducing the problem to solving
single linear homogeneous PDEs, namely, the narrowing transformations method for solving a system of linear first-order homogeneous partial differential equations is discussed and explained with the help of the illustrative examples.

Chapter 6:

This chapter deals with an application of proposed method which introduced in chapter 4 to determining the similarity equations for two types of the problem; the flow of an incompressible second-order fluid past a stretching sheet and the laminar boundary layer equations for heat convection on a vertical plate. Number of the variables in system of partial differential equations and a set of auxiliary conditions are reduced. Further, some similarity equations are solved numerically using Matlab BVP solver bvp4c.

Chapter 7:

The problem of three-dimensional non-linear flow incompressible second-order fluid past a stretching sheet is considered in the present chapter. Again, an application of improved method reduces the number of independent variables; also this converts partial differential equations with the boundary conditions to ordinary differential equations with appropriate corresponding conditions. Set of five similarity representations are obtained.

Chapter 8:

This chapter deals with the highly nonlinear system of partial differential equations governing incompressible three-dimensional MHD boundary layer flows of Newtonian and Non-Newtonian fluids. By invoking the transformations which be generated from the invariants of two-parameter group, these equations are transformed to similarity representations. About thirty sets of similarity equations are obtained. Further, the resulting non-linear system of ordinary differential equations is numerically solved for various numerical values of parameters by using Matlab BVP solver bvp4c.
Chapter 9:

In this chapter we have considered the highly nonlinear system of the general form of Einstein’s field equations for empty space. The system of partial differential equations is transformed into system of ordinary differential equations (similarity representation) by applying the new method. An attempt is made to solve the transformed equations.

Chapter 10: Contains the conclusions and some recommendations for future work.