Chapter 9

Similarity Solution of Einstein’s Field Equations

It is known that Einstein’s field equations in general relativity are a system of 10 second order non-linear partial differential equations. The dependent variables are metric potentials while four spacetime coordinates are independent variables. This system is very difficult to solve in its original form.

In this chapter, the similarity method discussed in previous chapters is applied to solve Einstein’s field equations for an empty spacetime. A general metric admitting a three-parameter group of isometries with the two-dimensional trajectories \( r = \text{constant} \) and \( t = \text{constant} \) has been considered. The Einstein field equations for empty space have been merged to a system of partial differential equation. By similarity transformations it has been reduced to an ordinary differential equation. Solution of special case of reduced equations is given.

9.1 Introduction

General relativity is a theory of gravity defined on a spacetime manifold, where the force of gravitation is described in terms of the curvature of the spacetime. These curvatures are in turn generated by the matter fields existing in the spacetime, as governed by the Einstein equations. The Einstein equations involve second derivatives
of the metric tensor. So, it is assumed that the metric components are at least $C^2$-differentiable functions of the coordinates.

All matter fields on the spacetime, such as electromagnetic fields, dust, perfect fluids or scalar fields, are assumed to be represented by a second rank tensor $T^{\alpha\beta}$, called the energy-momentum tensor. The central content of general relativity may be described in the following schematic form:

$$
\begin{pmatrix}
\text{a measure of local spacetime curvature} \\
\text{a measure of matter energy density}
\end{pmatrix}
= 
\begin{pmatrix}
\text{a measure of local spacetime curvature} \\
\text{a measure of matter energy density}
\end{pmatrix}
$$

(9.1)

This relation, called the Einstein equation (or Einstein’s equation), are the field equations of general relativity in the way that Maxwell’s equations are the field equations of electromagnetism. Maxwell’s equations relate the electromagnetic field to its sources charges and currents. Einstein’s field equations relate spacetime curvature to its source the mass-energy of matter. The analogy goes further. Maxwell’s equations are eight second-order partial differential equations for the electromagnetic potentials. Einstein’s equation is a set of ten second-order partial differential equations for the metric coefficients $g_{\alpha\beta}(x)$. We note that Maxwell’s equations are linear but the Einstein equation is nonlinear.

Returning to the question (9.1) where, on the right-hand side is the measure of matter energy density (the stress-energy $T^{\alpha\beta}$, the left-hand side is a measure of curvature. (The Ricci curvature $R^{\alpha\beta}$ is one such measure. But another rank-two symmetric tensor that can be formed from it and the metric is $g_{\alpha\beta}R$, where $R = R^\gamma_\gamma = g^{\gamma\delta} R_{\gamma\delta}$, is called the Ricci curvature scalar. A candidate for the relation between curvature and stress-energy is

$$R_{\alpha\beta} + \lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

(9.2)

for the yet to undetermine constants $\kappa$ and $\lambda$. The left-hand side of the Einstein equation consistent with local conservation if $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}R$ giving $\lambda = -\frac{1}{2}$ in (9.2). The constant $\kappa$ must be proportional to the gravitational coupling constant $G$. The Newtonian limit fixes its precise value at $8\pi G$ (in the units where $c = 1$ used throughout
The Einstein equations are thus
\[ R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -8\pi GT_{\alpha\beta} \] (9.3)

- In $c \neq 1$ units the factor $8\pi G$ is replaced by $\frac{-8\pi G}{c^4}$.
- In geometrized units, where mass is measured in units of length and $G = 1$, it is just $-8\pi$.

### 9.2 Einstein’s Field Equations and the Similarity Solution

In this section, we consider the situation in which spacetime is empty, that is $T_{ij} = 0$ everywhere, and we will apply Einstein’s field equations to the general form of a four-dimensional spherically symmetric metric,
\[
ds^2 = B(r, t)dt^2 - A(r, t)dr^2 - H(r, t)d\sigma^2 \] (9.4)

where $d\sigma^2$ is a two-dimensional metric with constant Gaussian curvature $K$ and can be written as $d\sigma^2 = d\theta^2 + f^2(\theta)d\phi^2$ where $f(\theta) = \sin(\theta), \sinh(\theta)$ for $K = 1, 0$ and $-1$, respectively. These cases correspond to spherical, plane and hyperbolic symmetries respectively. This is a metric admitting a three-parameter group of isometries with the two-dimensional trajectories $r =$constant and $t =$constant. Gupta et al [68]. Choosing coordinate labeling $x^1 = r, x^2 = \theta, x^3 = \varphi, x^4 = t$ the non-vanishing components of Ricci tensor for the metric (9.4) read as,

\[
R_{11} = -A_t^2 - \frac{A_t B_t}{4A} + \frac{1}{2} A_{tt} + \frac{A_t B_t}{4A} + \frac{B_t^2}{4B} - \frac{1}{2} B_{tt} + \frac{A B_t H_t}{2AH} + \frac{B H_t^2}{2H^2} \] (9.5)

\[
R_{14} = -\frac{H_t B_t}{2B} - \frac{A H_t}{2A} - \frac{H_t H_t}{2H} + H_{tt} \] (9.6)

\[
R_{33} = \sin(\theta)^2 R_{22} = -1 - \frac{A_t H_t}{4AB} + \frac{B_t H_t}{4B^2} - H_{tt} - \frac{A_t H_t}{4A^2} + \frac{B_t H_t}{4AB} + \frac{H_{tt}}{2A} \] (9.7)

\[
R_{44} = -\frac{A_t^2}{4A} - \frac{A_t B_t}{4B} + \frac{1}{2} A_{tt} + \frac{A_t B_t}{4A} + \frac{B_t^2}{4B} - \frac{1}{2} B_{tt} - \frac{A B_t H_t}{2BH} + \frac{A H_t^2}{2H^2} \] (9.8)
which is calculated using Mathematica program by Hasmani and Rathva [52]. Einstein’s field equations for empty spacetime reduce to $R_{a\beta} = 0, (\alpha, \beta = 1, 2, 3, 4)$. Thus, merging and reordering the equations $R_{a\beta} = 0$ we get the following partial differential equations:

\[- \left( \frac{ABH_r}{B} + \frac{BAH_r}{A} \right) - \frac{1}{H} (AH_r^2 + BH_r^2) - (B, H_r + A_rH_t) + 2(AH_{tt} + BH_{rr}) = 0 \]  
(9.9)\[ - \frac{H_r B_r}{2B} \frac{A_r H_r}{2A} - \frac{H_r H_{tt}}{2H} + H_{rr} = 0 \]  
(9.10)\[ - 1 - \frac{A_r H_t}{4AB} + \frac{B_r H_t}{4B^2} - \frac{H_{tt}}{2B} - \frac{A_r H_r}{4A^2} + \frac{B_r H_r}{4AB} + \frac{H_{rr}}{2A} = 0 \]  
(9.11)

### 9.2.1 Similarity Variables and Similarity Equations

We introduce the following new variables:

\[ \eta = \frac{r}{t^{\frac{\alpha 2}{\alpha 1}}} \]  
(9.12a)\[ A(r, t) = \frac{1}{t^{\frac{\alpha 3}{\alpha 1}}} F_1(\eta) \]  
(9.12b)\[ B(r, t) = \frac{1}{t^{\frac{\alpha 3}{\alpha 1}}} F_2(\eta) \]  
(9.12c)\[ H(r, t) = \frac{1}{t^{\frac{\alpha 3}{\alpha 1}}} F_3(\eta) \]  
(9.12d)

Due to the similarity transformations (9.12a)-(9.12d), equations (9.9)-(9.11) readily reduce to

\[- t^{(2 + \frac{\alpha 2}{\alpha 1})} \alpha_1^2 F_2^2 F_3 F_1' F_3' - F_1 F_2 \left( t^{2 + \frac{\alpha 3}{\alpha 1}} \alpha_2 \eta F_3 F_1' (\alpha_3 F_3 + \alpha_2 \eta F_3') + t^{2 + \frac{\alpha 3}{\alpha 1}} \alpha_1^2 F_3 F_1' F_3 (F_2 F_3' + F_3 (F_2' F_3 + 2F_2 F_3')) \right) + t^{2 + \frac{\alpha 3}{\alpha 1}} \alpha_2 \eta F_3 F_1' \left( \alpha_3 F_3 + \alpha_2 \eta F_3' \right) + F_2 (\alpha_3 (2\alpha_1 - \alpha_3 - \alpha_4 + \alpha_5) F_3^2 + \alpha_2 \eta F_3' (2\alpha_1 + 2\alpha_2 - \alpha_3 - \alpha_4 + 2\alpha_5) F_3' + 2\alpha_2 \eta F_3') \right) = 0 \]  
(9.13)\[ F_1 F_2 \left( t^{2 + \frac{\alpha 3}{\alpha 1}} \alpha_2 \eta F_3 F_1' \left( \alpha_3 F_3 + \alpha_2 \eta F_3' \right) + F_2 (\alpha_3 (2\alpha_1 - \alpha_3 - \alpha_4 + \alpha_5) F_3^2 + \alpha_2 \eta F_3' (2\alpha_1 + 2\alpha_2 - \alpha_3 - \alpha_4 + 2\alpha_5) F_3' + 2\alpha_2 \eta F_3') \right) \]  
(9.14)
\[
\alpha_2 \eta F_2 F_3' F_3'' + F_1 \left( \alpha_5 F_3^2 F_2' + \alpha_2 \eta F_2 F_3'' \right) + 
F_3 \left( (\alpha_2 - 2 \alpha_5) F_2 + \alpha_2 \eta F_2' F_3' - 2 \alpha_2 \eta F_2 F_3'' \right) = 0
\] (9.15)

\[
-t^{(2 + \frac{2\alpha_1}{\alpha_1})} \alpha_1^2 F_2^2 F_1' F_3' + F_1 F_2 \left( t^{(2 + \frac{\alpha_1}{\alpha_1})} F_1' F_3' + 2 F_2 F_3'' \right) - t^{(2 + \frac{\alpha_1}{\alpha_1})} \alpha_2 \eta F_1' (\alpha_5 F_3 + \alpha_2 \eta F_3') \right)

\[
- t^{2 + \frac{\alpha_1}{\alpha_1}} \alpha_1^2 F_2^2 F_1' F_3' + F_2 \left( \alpha_5 (2 \alpha_1 + \alpha_3 - \alpha_4 + 2 \alpha_5) F_3 
+ \alpha_2 \eta (2 \alpha_1 + 2 \alpha_2 + \alpha_3 - \alpha_4 + 4 \alpha_5) F_3' + 2 \alpha_2 \eta F_3'' \right) \right) = 0
\] (9.16)

which express in terms of single variable \( \eta \) if the powers of the \( t \) are equal to zero. i.e.,

\[
\frac{2 \alpha_2 + \alpha_4}{\alpha_1} = 0, \quad (9.17)
\]

\[
2 + \frac{\alpha_3}{\alpha_1} = 0, \quad (9.18)
\]

\[
\frac{2 \alpha_2 + 2 \alpha_1 + \alpha_5}{\alpha_1} = 0 \quad (9.19)
\]

Solving this system and let \( m = \frac{\alpha_5}{\alpha_1} \) we get:

\[
\frac{\alpha_2}{\alpha_1} = -\frac{1}{2} m, \quad \frac{\alpha_3}{\alpha_1} = -2, \quad \frac{\alpha_5}{\alpha_1} = m - 2 \quad (9.20)
\]
So that, with these values the reduced system (9.13)-(9.16) become

\[ 4F_2^2F_3F_1'F_3' + F_1F_3'\left(2m(2 - m)\eta F_3^2F_1' + 4F_2^2F_3' + F_3((m^2\eta^2F_1' + 4F_2^2)F_3' - 8F_2F_3'') \right) \]
\[ + F_1^2[\eta F_3F_2'(2(2 - m)F_3 + mnF_3') + F_2(8(2 - m)F_3^2 + m^2\eta^2(F_3'^2 - 2F_3F_3''))] = 0 \] \hspace{1cm} (9.21)

\[ mnF_2F_3F_1'F_3' + F_1(2F_2'(2 - m)F_3^2 + mnF_3'^2F_2 + mnF_3(F_2'^2F_3' - 2F_2F_3'')) = 0 \] \hspace{1cm} (9.22)

\[ 4F_2^2F_1'F_3' + F_1F_2'\left(2(2 - m)mnF_1'F_3 + F_3((m^2\eta^2F_1' - 4F_2^2)F_3' - 8F_2F_3'') \right) \]
\[ + F_1^2[16F_2^2 - mnF_2'(2m - 2)F_3 + mnF_3'] \]
\[ + 2F_2(2(m - 4)(m - 2)F_3 - mnF_3' + m^2\eta^2F_3'')] = 0 \] \hspace{1cm} (9.23)

All the above expressions were calculated automatically using Mathematica.

### 9.3 Solutions to the Problem

Consider the case when the function \( H \) independent of \( r \) and take \( m = 1 \) since it is arbitrary constant. Therefore, equation (9.12d) can be written as:

\[ H(t)^{\frac{1}{2}m-1} = F_3(\eta) \] \hspace{1cm} (9.24)

It is evident that the left side is function of \( t \), whereas the right side is function of \( \eta \) which depends on \( r \) too, it follows that \( F_3(\eta) \) should be constant. Consequently, \( F_3' = F_3'' = 0 \). In the light of the above values of \( m, F_3' \) and \( F_3'' \), equations (9.21)-(9.23) take the form

\[ 8F_2^2F_2F_3^2 + 2\eta F_3F_2F_3^2F_1' + 2\eta F_2^2F_3^2F_2' = 0 \] \hspace{1cm} (9.25)

\[ F_1F_2'F_3'(4 - 2m) = 0 \] \hspace{1cm} (9.26)

\[ 16F_1^2F_2^2 + 12F_1^2F_2F_3 + 2\eta F_1F_2F_3F_1' - 2\eta F_1^2F_3F_2' = 0 \] \hspace{1cm} (9.27)
It is obvious from (9.26) that $F'_2 = 0$ and the equations (9.25) and (9.27) with further analysis give

$$\frac{F'_1}{F_1} = -\frac{4}{\eta}$$

(9.28)

$$16F_2 + F_3(12 + 2\eta \frac{F'_1}{F_1}) = 0$$

(9.29)

Inserting (9.28) in (9.29), we get

$$F_3 = -4F_2$$

(9.30)

Since $F'_2 = 0$, then $F_2 = k_1$, where $k_1$ is integration constant. Therefore from (9.30), $F_3 = -4k_1$. Also by integrating equation (9.28), we get

$$F_1 = \frac{k_2}{\eta^4}$$

(9.31)

where $k_2$ is integration constant.

By substituting the value of $F_1$, $F_2$ and $F_3$ back into equations (9.12a)-(9.12d), we get

$$\eta = rt^{\frac{1}{2}}$$

(9.32a)

$$A(r, t) = \frac{k_2}{r^3}$$

(9.32b)

$$B(r, t) = \frac{1}{t}k_1$$

(9.32c)

$$H(t) = -4tk_1$$

(9.32d)

So, at this stage, the line element has been reduced by (9.32a)-(9.32d) to

$$ds^2 = -\frac{k_1}{t}dt^2 - \frac{k_2}{r^6}dr^2 - 4tk_1d\sigma^2$$

(9.33)

### 9.4 Discussion and Conclusion

Einstein’s field equations form a highly coupled system second order non-linear partial differential equations. They are very difficult to solve in the original form.
However, under simplifying assumptions due to observational facts (not ad-hoc assumptions); this system is solved. A beautiful collection of exact solutions is compiled by Stephani et al [144], they have also classified the solutions on the basis of their geometrical properties as well as physical relevance.

We have employed the techniques of similarity transformations to solve EFEs for a comparatively simple spacetime having no material content (empty).

We are aware that the solution \(9.33\) does not demonstrate any remarkable feature. The aim is to highlight the applicability of the technique in the filed of General relativity where the system of differential equations is very complicated. It is hoped that the physically an geometrically relevant situations also.