CHAPTER 7

TWO-LAYERED BINGHAM PLASTIC MODEL OF BLOOD FLOW WITHIN TAPERED ARTERY WITH ASYMMETRIC STENOSIS

7.1 INTRODUCTION

Atherosclerosis (stenosis) is a wide spread disease that takes a good number of human lives in the long run (Boyd 1963, Guyton 1970). Mild stenosis may cause critical flow conditions, related to artery collapse which leads directly to heart attack and stroke. It also causes pressure changes at the throat of stenosis and equally changes shear stress distribution in the region just distal to stenosis under unsteady conditions (Frank et al. 2002). Different mathematical models have been studied by some researchers to explore the various aspects of blood flow in stenosed artery (Bugliarello and Sevilla, 1970; Cavalcanti, 1995 Hynes, 1960). Many researchers studied the pulsatile flow of blood in stenosed artery (Biswas and Chakraborty 2010, Chaturani and Ponalagusamy 1982, Chaturani and Ponalagusamy, 1990, Chaturani and Biswas, 1984 and Verma and Parihar, 2009), have considered the behaviour of blood in stenosed artery in the presence of magnetic effect. Recently, Biswas and Paul (2013) have dealt with steady blood flow in an inclined non tapering Stenosed artery.

Body fluid blood is made up of different cells or particles in a solution of proteins and electrolytes, called plasma. Among the cells Erythrocytes, leukocytes and platelets are the main constituents of blood. It has been pointed out that plasma behaves as a Newtonian fluid (Schlichting, 1968 ) whereas the whole blood exhibits the non-Newtonian property (Biswas, 2000, Fung, 1981). The Red Blood cells (RBC) are more than thousand times numerous than the white Blood cells (WBC) and much larger than platelets. For this reason, the flow properties of blood mainly involve the RBC’s. When the flow of blood to a part of the body is reduced, the oxygen supply to that part of the body is cut off and cells begin to die (Boyd, 1963).

The wall shear stress distribution is an important diagnostic factor for examining the flow characteristics of the blood through the arteries. There is no doubt that tapering is a significant aspect of mammalian arterial system. Keeping these views in consideration, in this study an attempt has been made to investigate the effects of the
stenosis on the axial velocity, flow rate and wall shear stress for Newtonian fluid through a tapered artery, in a two-layered blood flow.

The study of blood flow through tapered tubes (Verma and Parihar, 2009, 2010) is important not only for an understanding of the flow behavior of the marvelous body fluid blood in arteries, but also for the design of prosthetic blood vessels (How and Black, 1987). In the blood flow modeling, a number of studies of blood flow in particular both theoretical (Vand; 1948 Bloch, 1962 and Nubar; 1967 ) and experimental (Bugliarello and Hayden; 1962,Bennett; 1967) have reported the presence of slip at the flow boundaries. It seems that consideration of a velocity slip at the vessel wall be quite rational in blood flow modeling. In most of the abovementioned studies, zero slip velocity has been considered at the wall. But it seems appropriate if a slip velocity at the interface of fluids is considered.

It has been reported by many authors theoretically that body fluid blood while flowing through circulatory channels leaves a peripheral layer (ppl) of plasma and a core region of red cell suspension which is supported by experimental results. It seems therefore rational to consider a cell free layer near the vessel wall and a core region consisting of red cell suspension. In view of this fact, in the present study we have considered a two-fluid model or a two-layered model with the ppl i.e., a cell poor region near the vessel wall and a core region of cells and investigated the behaviour of flow variables in this constricted artery channel.

It has been pointed out by many investigators that under certain flow situations, blood possesses a finite yield stress (Fung, 1981; Kapur et al, 1982). One interesting and specialized case of material with yield stress, is known as Bingham plastic whose consistency curve or flow behaviour, is shown by a straight line (Fung, 1981, Kapur et al, 1982). This type of material deform elastically, until the yield stress is reached, but once the stress is exceeded, it flows as a Newtonian fluid, with shear stress being linearly related to strain rate (Schlichting, 1968). It therefore seems to be realistic, in considering blood to behave as a Bingham plastic, in red cell region (Fung, 1981; Kapur et al, 1968).

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also for the design of prosthetic blood vessels (How and Black, 1987). In the blood flow modeling, a number of studies of blood flow in particular both theoretical (Vand, 1948; Bloch, 1962; Brum, 1975; Nubar, 1967) and experimental (Bugliarello and Hayden, 1962; Bennett, 1967) have reported in the presence of slip at the flow boundaries. It seems that consideration of a velocity slip at the vessel wall be quite rational in blood flow modeling. In most of the abovementioned literatures, constant velocity slip has been considered at the wall. In the present study, slip velocity has been taken at the innermost wall which varies with radial distance due to non-symmetric stenosis.

In view of the above considerations, we are interested to study the two-layered flow of blood through tapered artery with asymmetric stenosis. In this case, blood is taken as Bingham plastic in the core region and slip velocity is employed at the interface. The motivation behind this theoretical study is to get an insight of the complicated situations or to throw some light in the flow characteristics, arising in blood flow through the constricted artery channel.

7.2 PROBLEM FORMULATION

Consider two regions, of which one region consists of a core of red blood cell suspension in the middle layer and the peripheral plasma in the outer layer. The flow of blood is assumed to be steady, laminar and axi-symmetric through a tapered artery in which non-symmetric stenosis has been considered. In this model, the body fluid blood is assumed to behave like Bingham plastic fluid.

![Schematic diagram of two-layered Bingham plastic fluid model in a tapered artery with asymmetric stenosis.](image)
Where \( R(z) \) and \( R_0(z) \) are the radial distances from stenosis in peripheral and core region respectively. The geometries of two-layered tapered artery with asymmetric stenosis have been given in earlier Chapter 3 (equations (3.3.1) and (3.3.2)).

The governing equation of the fluid flow has the usual form

\[
C + \frac{\mu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = 0,
\]

where, \( C = \frac{dp}{dz} \)  

The constitutive equation of Bingham plastic fluid is in the form

\[
\tau = \pm \tau_0 + \mu e,
\]

where, \( \tau \), \( \tau_0 \), \( \mu \) and \( e \) are the shear stress, yield stress value, viscosity of the fluid and strain rate respectively.

In case of Bingham plastic, it is noticed that the fluid does not flow below the yield stress value \( \tau_0 \) and above this yield value the fluid flow is possible, so the rise in shear stress is proportional to the shear rate and hence equation (7.2.2) can be put in the form

\[
e = \frac{1}{\mu} \left( \tau_{rz} - \tau_0 \right), \quad \tau_{rz} \geq \tau_0,
\]

\[
e = 0 \quad \text{,} \quad \tau_{rz} \leq \tau_0.
\]

7.3 SOLUTIONS

In order, to obtain the strain rate (\( \dot{e} \)) of the fluid in the artery wall with the formula used for Newtonian fluid. The integral of the equation (7.2.1) is
\[ u(r) = -\frac{Cr^2}{4\mu} + A \ln r + B, \quad (7.2.5) \]

where, \( A \) and \( B \) are constants of integration and these constants are to be determined with the help of usual conditions: for symmetry condition at \( r = 0 \) i.e. \( u(0) \) is finite and zero slip at \( r = R(z) \) i.e. \( u(R(z)) = 0 \) the equation (7.2.5) becomes

\[ u(r) = -\frac{C}{4\mu} \left( R^2 - r^2 \right), \quad 0 \leq r \leq R(z). \quad (7.2.6) \]

Now the shear stress component at any distance ‘\( r \)’ from the tube axis is given by (Schlichting 1968)

\[ \tau_{rz} = \mu \frac{\partial u}{\partial r} = \mu e. \quad (7.2.7) \]

With the help of equation (7.2.6) the equation (7.2.7) becomes

\[ \tau_{rz} = -\frac{C}{2} \frac{r}{2} \frac{dp}{dz}. \quad (7.2.8) \]

Which implies, \[ \tau_{rz} \propto r, \quad (7.2.9) \]

where, \( \frac{1}{2} \frac{dp}{dz} \) is a constant of proportionality.

The expression for wall shear stress (\( \tau_w \)) is obtained from equation (7.2.8) as

\[ \tau_w = \tau_{rz} \bigg|_{r=R_1} = \mu \frac{\partial u}{\partial r} \bigg|_{r=R_1} = -\frac{C}{2} \frac{R_1}{2} \left( \frac{dp}{dz} \right). \quad (7.2.10) \]
Using equation (7.2.8), we have obtained yield stress as

$$
\tau_0 = \left. \tau_{rz} \right|_{r = r_0} = \mu \left. \frac{\partial u}{\partial r} \right|_{r = r_0} = -\frac{C}{2} r_0 \frac{dP}{dz}.
$$

(7.2.11)

In between two stresses \( \tau_w \) and \( \tau_0 \) there may arise two cases viz.

i. If the value of yield stress is smaller than the shear stress i.e. \( \tau_0 < \tau_w \),
then there will be flow and for that

$$u = u(r), \text{ when } \frac{dP}{dz} > \frac{2}{R} \tau_0.$$  \hspace{1cm} (7.2.12)

ii. If the value of yield stress is greater than the shear stress i.e. \( \tau_0 > \tau_w \),
no flow will be possible then

$$u(r) = 0, \text{ when } \frac{dP}{dz} < \frac{2}{R} \tau_0.$$  \hspace{1cm} (7.2.13)

So, the Bingham equations (7.2.3) and (7.2.4) can be written in the form

$$e = \frac{1}{\mu} (\tau_w - \tau_0), \quad \tau_w \geq \tau_0,$$

(7.2.14)

$$= 0, \quad \tau_w \leq \tau_0.$$  \hspace{1cm} (7.2.15)

From equation (7.2.15), we have seen that the strain rate vanishes i.e. \( e = 0 \)

Implies that,

$$\frac{\partial u}{\partial r} = 0.$$  \hspace{1cm} (7.2.16)

Which on integration, we obtain

$$u(r) = \text{const.} = u_0, \text{ when } \tau_w = \tau_0.$$  \hspace{1cm} (7.2.17)
Where, \( u_0 \) is the core velocity at \( r = r_0 \)

Thus for blood flow, there are three regions viz. \( 0 \leq r \leq r_0 \); \( r_0 \leq r \leq R_1 \)

and \( R_1 \leq r \leq R \) Indicating that velocity profile will be flat for the region

\( 0 \leq r \leq r_0 \) velocity profile will exhibit different for the region \( r_0 \leq r \leq R_1 \) and velocity profile will exhibit deviation from that profile, hence Bingham equation (7.2.8) has to be applied for this region of blood flow (Fung; 1981). For peripheral layer Newtonian motion is applied.

As a consequence of above considerations following equations are made

Using equation (7.2.8) and (7.2.11), the equation (7.2.3) becomes

\[
\frac{du_2}{dr} = \frac{C}{2\mu_2} (r_0 - r), \quad r_0 \leq r \leq R_1 ,
\]  \hspace{1cm} (7.2.18)

On integration the above equation (7.2.8), we obtain

\[
u_2 = \frac{Cr}{4\mu_2} (2r_0 - r) + A_2 ,
\]  \hspace{1cm} (7.2.19)

and the equation (7.2.5) can be written as, for peripheral layer

\[
u_1 = \frac{Cr^2}{4\mu_1} + A_1 \ln r + B_1 , \quad R_1 \leq r \leq R
\]  \hspace{1cm} (7.2.20)

and the equation (7.1.13) can be written as

\[
u_0 = \text{const.} = A_3
\]  \hspace{1cm} (7.2.21)

where \( A_1, A_2, A_3 \) and \( B_1 \) are constants, these are to be determined with the help of the following boundary conditions:
\[ i) \quad u_1 = 0, \quad \text{at } r = R \]
\[ ii) \quad u_2 - u_1 = u_s, \quad \text{at } r = R_1 \]  
(7.2.22)
\[ iii) \quad u_2 = u_0 = A_3, \quad \text{at } r = r_0 \]

iv) \( u_1 \) is finite at the axis

where \( u_s \) is the slip velocity at the interface of fluids at the stenotic region and \( u_1, u_2 \)
are velocities in ppl and core regions respectively.

As a result of employing conditions (7.2.22) in the solutions (7.2.19--7.2.21), we have obtained the expressions for velocity function in axial direction as

for peripheral layer

\[ u_1 = \frac{C}{4\mu_1} (R^2 - r^2), \quad R_1 \leq r \leq R, \quad (7.2.23) \]

for core region

\[ u_2 = u_s + \frac{C}{4\mu_1} (R^2 - R_1^2) + \frac{C}{4\mu_2} (R_1 - r)(R_1 + r - 2r_0), \quad r_0 \leq r \leq R_1, \quad (7.2.24) \]

and

\[ u_0 = u_s + \frac{C}{4\mu_1} (R^2 - R_1^2) + \frac{C}{4\mu_2} (R_1 - r_0)^2, \quad 0 \leq r \leq r_0. \quad (7.2.25) \]

The volumetric flow rate \( Q \) can be obtained by the formula

\[ Q = 2\pi \left[ \int_{r=0}^{r_0} ru_0 dr + \int_{r_0}^{R_1} ru_2 dr + \int_{R_1}^{R} ru_1 dr \right]. \quad (7.2.26) \]
\[ Q = l_1 + l_2 + l_3, \]  

(7.2.27)

where, \[ l_1 = \int_{r=0}^{r_0} ru_0 dr, \]  

(7.2.28)

\[ l_2 = \int_{r_0}^{R_1} ru_2 dr, \]  

(7.2.29)

\[ l_3 = \int_{R_1}^{R} ru_3 dr. \]  

(7.2.30)

Integral of the above integrations are obtained (using equations 7.2.23—7.2.25) as

\[ l_1 = \pi r_0^2 u_s + \frac{\pi C}{4\mu_1} r_0^2 \left( R^2 - R_1^2 \right) + \frac{\pi C}{4\mu_2} R_1^4 \alpha^2 \left( 1 - \alpha \right)^2, \]  

(7.2.31)

\[ l_2 = \pi u_s \left( R_1^2 - r_0^2 \right) + \frac{\pi C}{4\mu_1} \left( R^2 - R_1^2 \right) \left( R_1^2 - r_0^2 \right) \]

\[ + \frac{\pi C}{8\mu_2} R_1^4 \left( 1 - \frac{4}{3} \alpha - 2\alpha^2 + 4\alpha^3 - \frac{5}{3} \alpha^4 \right), \]  

(7.2.32)

where, \[ \alpha = \frac{r_0}{R_1} \]

and

\[ l_3 = \frac{\pi C}{8\mu_1} \left( R^2 - R_1^2 \right)^2. \]  

(7.2.33)
The total volumetric flow rate is obtained (using equations 7.2.31—7.2.33) as

\[ Q = \pi R_1^2 u_s + \frac{\pi C}{8 \mu_1} R^4 \left\{ 1 - \left( \frac{R_1}{R} \right)^4 \right\} + \frac{\pi C R_1^4}{8 \mu_2} \phi(\alpha) \quad , \quad (7.2.34) \]

where

\[ \phi(\alpha) = 1 - \frac{4}{3} \alpha + \frac{1}{3} \alpha^4 \quad . \quad (7.2.35) \]

Following equations (7.2.12) and (7.2.13) with their consideration, expression

for flow rate can be rewritten in different form as

\[ Q = \pi R_1^2 u_s + \frac{\pi C}{8 \mu_1} R^4 \left\{ 1 - \left( \frac{R_1}{R} \right)^4 \right\} - \frac{\pi R_1^4}{8 \mu_2} \left( \frac{dp}{dz} \right) Y(\beta) \quad , \quad (7.2.36) \]

Where,

\[ Y(\beta) = 1 - \frac{4}{3} \beta + \frac{1}{3} \beta^4 \quad \text{and} \quad \beta = \frac{\left( \frac{2 \tau_0}{R_1} \right)}{\left( \frac{dp}{dz} \right)} \quad . \quad (7.2.37) \]

The pressure gradient can be taken from the equation (7.2.34)

\[ \frac{dp}{dz} = 8 \left( R_1^2 u_s - \pi^{-1} Q \right) \left[ \mu_1^{-1} \left( R^4 - R_1^4 \right) + \mu_2^{-1} R_1^4 \phi(\alpha) \right]^{-1} \quad . \quad (7.2.38) \]

On integration the above equation(7.2.38) between the limits \( p = p_i \) at \( z = 0 \)

and \( p = p_0 \) at \( z = L \), where L is the length of the tube.
We obtain, the pressure drop equation in the form

\[ p_i - p_0 = 8\mu_i \int_0^L \left[ R^4 - \left( 1 - \frac{\mu_1}{\mu_2} \phi(\alpha) \right) R_1^4 \right]^{-1} \left( \pi^{-1} Q - R_1^2 u_s \right) dz. \quad (7.2.39) \]

The resistance to flow (\( \lambda \)), defined by

\[ \lambda = \frac{p_i - p_0}{Q} \quad (7.2.40) \]

It may be obtained with the help of equation (7.2.39) in the following form

\[ \lambda = 8\mu_i \left[ R_0^4 \left\{ 1 - \left( 1 - \frac{\mu_1}{\mu_2} \phi \left( \frac{r_0}{\alpha R_0} \right) \right) \alpha^4 \right\} \right]^{-1} \left\{ \pi^{-1} - Q_1^{-1} (\alpha R_0)^2 u_s \right\} (L - L_0) \]

\[ + 8\mu_i \int_{z=a}^{d+z_0} \left[ R^4 - \left( 1 - \frac{\mu_1}{\mu_2} \phi(\alpha) \right) R_1^4 \right]^{-1} \left( \pi^{-1} - Q_1^{-1} R_1^2 u_s \right) dz, \quad (7.2.41) \]

where, \( Q_1 = Q_{|z=R=r_0} \)

Apparent viscosity can be expressed in the following:

\[ \mu_a = \frac{\pi CR^4}{8Q} \quad (7.2.42) \]

The expression for apparent viscosity can be computed with the help of equation (7.2.34) as

\[ \mu_a = \left[ \frac{8u_s}{CR^2} \left( \frac{R_1}{R} \right)^2 + \mu_1^{-1} \left\{ 1 - \left( 1 - \frac{\mu_1}{\mu_2} \phi(\alpha) \right) \left( \frac{R_1}{R} \right)^4 \right\} \right]^{-1}. \quad (7.2.43) \]

Another representation of flow variable for peripheral layer (from equation (7.2.26))

\[ \bar{u}_i = \bar{R}^2 \left\{ 1 - \left( \frac{r}{R} \right)^2 \right\}, \quad \frac{R_i}{R} \leq r \leq 1. \quad (7.2.44) \]
Flow variable for core region (using equation (7.2.27))

\[ \bar{u}_2 = \bar{u}_s + \frac{\mu_1}{\mu_2} \left[ \bar{R}_1 - \bar{R} \left( \frac{r}{R} \right) \right] \left[ \bar{R}_1 + \bar{R} \left( \frac{r}{R} \right) - 2\bar{r}_0 \right] \]

\[ + \left( \bar{R}^2 + \bar{R}_1^2 \right) , \quad \frac{\bar{r}_0}{R} \leq \frac{r}{R} \leq \frac{\bar{R}_1}{R} . \quad (7.2.45) \]

Also from equation (7.2.25) we have

\[ \bar{u}_0 = \bar{u}_s + \frac{\mu_1}{\mu_2} \left( \bar{R}_1^2 + \bar{r}_0^2 \right)^2 + \left( \bar{R}^2 + \bar{R}_1^2 \right) , \quad 0 \leq \frac{r}{R} \leq \frac{\bar{r}_0}{R} . \quad (7.2.46) \]

A second representation of flow rate (Q) is obtained by the following formula

\[ Q = \bar{Q} \frac{Q}{Q_0} , \quad \text{where} \quad Q_0 = \frac{\pi C R_0^4}{8 \mu_1} . \]

The expression for flow rate, can be obtained with the help of equation (7.2.34)

\[ \bar{Q} = 2\bar{R}_1^2 \bar{u}_s + \left[ \bar{R}^4 - \left( 1 - \frac{\mu_1}{\mu_2} \phi(\alpha) \right) \bar{R}_1^4 \right] . \quad (7.2.47) \]

The expression for pressure gradient can be computed (using equation (7.2.38)) as

\[ \left( \frac{dp}{dz} \right) = \left[ \bar{R}^4 - \left( 1 - \frac{\mu_1}{\mu_2} \phi(\alpha) \right) \bar{R}_1^4 \right]^{-1} \left( \bar{Q} - 2\bar{R}_1^2 \bar{u}_s \right) = \left( \frac{dp}{dz} \right)_{av} . \quad (7.2.48) \]

A second representation of resistance to flow (\( \bar{\lambda} \)) is defined as

\[ \bar{\lambda} = \frac{\lambda}{\lambda_0} , \quad \text{where} \quad \lambda_0 = \frac{8 \mu_1 L}{\pi R_0^4} . \]
From equation (7.2.41), we have obtained the expression for resistance to flow

\[
\lambda = \left[1 - \left(1 - \frac{\mu_1}{\mu_2} \phi \left(\frac{r_0}{\alpha R_0}\right)\right) \alpha^4 \right]^{-1} \left(1 - 2\alpha^2 \frac{u_s}{Q_1} \left(\frac{L - L_0}{L}\right)\right) + \frac{1}{L} \int_{r=d}^{d+L_0} \left(\frac{R_1}{R}\right)^4 - \left(1 - \frac{\mu_1}{\mu_2} \phi(\alpha) \right) \frac{R_1}{R}^4 \left(1 - 2u_s \frac{R_1}{Q}\right) \, dz. \tag{7.2.49}
\]

A second representation of apparent viscosity (\(\mu_a\), defined as

\[
\mu_a = \frac{\mu_a}{\mu_1}
\]

And it can be obtained with the help of equation (7.2.43) as

\[
\mu_a = \left[2 \frac{u_s}{R} \left(\frac{R_1}{R}\right)^2 + 1 - \left(1 - \frac{\mu_1}{\mu_2} \phi(\alpha) \right) \frac{R_1}{R}^4 \right]^{-1}. \tag{7.2.50}
\]

where, \(\bar{\mu}_a = \frac{\mu_a}{\mu_1}\), \(\bar{\beta} = \frac{r_s}{R_1}\), \(\bar{R} = \frac{R}{R_0}\), \(\bar{\delta}_s = \frac{\delta_s}{R_0}\),

\[
\bar{u}_1 = \frac{u_1}{u_0}, \quad \bar{u}_s = \frac{u_s}{u_0}, \quad \bar{Q} = \frac{Q}{Q_0}, \quad \bar{\lambda} = \frac{\lambda}{\lambda_0}, \quad \bar{\lambda}_0 = \frac{\lambda_0}{\lambda_0},
\]

\[
Q_0 = \frac{\pi CR_0^4}{8\mu_1}, \quad \bar{\lambda}_0 = \frac{8\mu_1 L_0}{\pi R_0^4}, \quad \left(\frac{dp}{dz}\right)_0 = \frac{8\mu_1 Q_0}{\pi R_0^4}, \quad u_0 = \frac{CR_0^2}{4\mu_1}. \tag{7.2.51}
\]
7.4 RESULTS AND DISCUSSIONS

The two-layered Bingham plastic fluid model has been developed, to study the effect of yield stress, slip velocity at the innermost wall and the effect of asymmetric stenosis due to flow of blood through asymmetric Stenosed tapered artery. In the model, core region is accounted by Bingham Plastic fluid and ppl layer as plasma, is treated as Newtonian fluid. The Bingham Plastic Fluid has yield property. It is revealed that, blood will not flow if yield stress is greater than wall shear stress otherwise blood flow will possible. Analytical expressions for velocity, flow rate, pressure gradient, shear stress, resistance to flow and apparent viscosity are presented graphically(obtained from equations (7.2.44—7.2.50) as follows:

Variation of axial velocity profiles against radial distance, variation of rate of flow against axial distance, variation of pressure gradient against axial distance and variation of wall shear stress against axial distance are plotted in the following figures (7.2—7.26).

In this two-layered investigation for steady flow of blood inside an asymmetric tapered stenosed artery, core is taken to be represented by Bingham fluid with viscosity $\mu=2cp$ and peripheral layer is accounted by Newtonian fluid, having a shear viscosity $\mu_1=1.2cp$ (Fig.7.1), a slip condition at interface of two- fluids and usual zero-slip at stenotic vessel wall, are employed, in the model. An axial slip for velocity has been used at the interface of fluids in case of mild, moderate and severe growth of an arterial stenosis. Further that Bingham fluid which is a number of visco-inelastic, non-Newtonian fluids, inhabits a finite yield stress. It is reported that such a fluid fails to flow if the shear stress $\tau_{rz}$ at a radial distance $r$ is below a certain yield stress and if it is above the yield value $\tau_0 (>0)$, the rise in shear stress is proportional shear rate. As a result of these inherent, fluid properties, there may arise two distinct cases in their flow behaviour viz.

(a) If $\tau_{rz} < \tau_0$, i.e. if shear stress at a critical distance $r$ is not higher than its yield value, then blood will not flow inside the circular system.
(b) If $\tau_{rz} > \tau_0$, i.e. shear stress is not lower than a finite yield stress, blood flow inside the body, will be possible . Therefore in accounting or the Bingham behaviour of body fluid blood in two-layered flow, there may arise three
regions for flow along the constricted tapering region of an artery viz., 0 \leq r \leq r_0, r_0 \leq r \leq R_1(z), R_1(z) \leq r \leq R(z). Analytic expressions for axial velocities, flow rate, pressure gradient, resistance to flow, wall shear stresses and apparent viscosity are obtained. Velocity (in eqs. 7.2.26-7.2.28) is clearly a function of shear viscosities \mu_1, \mu, radial r and axial z coordinates, tube radii R_0, R(z) and R_1(z), pressure gradient c, dimensions of stenosis and velocity slip u_e. In case R_1(z) = R(z), \tau_0 \neq 0 and \alpha = 1, the present model results in one-layered Bingham model of an arterial stenosis with slip or, no-slip at vessel wall. For R(z) = R_0 = R_1(z) and \tau_0 = 0, it reduces to Poiseuille flow model with slip and zero-slip at tube wall (Schlichting, 1968). When \tau_0 = 0 and R_1(z) = R(z), it yields to stenosed models of Newtonian fluid foe one-layered blood flow with slip or zero-slip and for \tau_0 = 0, R_1(z) \neq R_0 \neq R(z) and \alpha \neq 1, it leads to a Newtonian model. If \tau_0 \neq 0, \alpha \neq 1 and R_1(z) \neq R_0 \neq R(z), this analysis examines a two-layered flow of Bingham plastic in a stenosed vessel with slip at interface.

In literature barring a few studies, attention is mostly given to viscosity changes whereas another major non-Newtonian parameter relevant to blood viz., an yield stress is almost ignored. In this analysis, an attempt will be made to include the effect and influence of a non-zero yield stress and a slip, on different kinds of stenosis and flow variables.

7.3.1 Velocity profiles

The variations of velocity are shown in Figs. (7.2-7.7), indicate that velocity is constant in the yield stress zone and parabolic in core and PPI regions. Velocity is higher in asymmetric stenosis (n>2) than that in symmetric case (n=2). While considering the growths mild, moderate and severe cases, velocity althoughout attains the highest magnitude for mild case and the lowest one for severe form of stenosis. As yield stress increases, velocity in all three stenosis formation cases accordingly increases. However, velocity increases with a velocity slip in all cases of stenosis.

7.3.2 Flow rate

Flow rate is represented in Figs. (7.8-7.13). It shows that flow rate increases as shape parameter increases from n=2 (symmetric case) to n > 2 (asymmetric growths). As yield stress increases, flow rate increases but due to velocity slip. It attains the maximum flow rate in mild stenosis case and the minimum magnitude for severe growth case. The highest flow rate is attained at the two ends of
stenosis considers and the lowest magnitude at the stenosis throat (for n=2) and near the termination position (for n>2).

7.3.3 Pressure gradient

The behaviour of pressure gradient is exhibited in Figs. (7.14-7.17), reveals that it decreases as shape parameter increases for n=2 to n>2 i.e., from symmetric to asymmetric stenosis. It decreases with an increasing magnitude of yield stress. However, it is lowered with a velocity slip.

7.3.4 Apparent viscosity

The variation of apparent viscosity is presented in Figs. (7.18-7.19). Viscosity decreases with velocity slip. As yield stress increases, it increases accordingly. It is the maximum at the severe stenosis and the minimum for mild growth.
Fig 7.3: Variation of axial velocity against radial distance for asymmetric stenosis, $n = 6$.

$\phi = -1^\circ, k_r = 2cp$
$n = 9, \alpha = 0.8$

$\bar{u}_z = 0.00$
$\bar{u}_z = 0.05$
$\bar{u}_z = 0.1$

Fig 7.4: Variation of axial velocity against radial distance for asymmetric stenosis, $n = 9$. 

$n = 6, \alpha = 0.8$

$\phi = -1^\circ, k_r = 2cp$
Fig 7.5: Variation of axial velocity against radial distance for mild, moderate and severe symmetric stenoses, $n = 2$.

Fig 7.6: Variation of axial velocity against radial distance for mild, moderate and severe asymmetric stenoses, $n = 6$. 
Fig 7.7: Variation of axial velocity against radial distance for mild, moderate and severe asymmetric stenoses, $n = 9$.

Fig 7.8: Variation of flow rate against axial distance.
Fig 7.9: Variation of flow rate against axial distance with slip $\bar{u}_s=0.05$

Fig 7.10: Variation of flow rate against axial distance with slip $\bar{u}_s=0.10$
Fig 7.11: Variation of flow rate against axial distance with slip $\bar{u}_s=0.00$, $n=2$ for mild, moderate and severe

Fig 7.12: Variation of flow rate against axial distance with slip $\bar{u}_s=0.05$, $n=2$ for mild, moderate and severe
Fig 7.13: Variation of pressure gradient against axial distance

\[ \alpha = 0.8, k = 2cp, \bar{u}_z = 0.00 \]

Fig 7.14: Variation of pressure gradient against axial distance

\[ \alpha = 0.8, k = 2cp, \bar{u}_z = 0.05 \]
Fig 7.15: Variation of pressure gradient against axial distance

Fig 7.16: Variation of pressure gradient against axial distance with slip $u_s = 0.00$, $n = 2$ for mild, moderate and severe with different tapering angle
Fig 7.17: Variation of pressure gradient against axial distance with slip $\vec{u}_s=0.00$, $n=2$ for mild, moderate and severe with different tapering angle

Fig 7.18: Variation of pressure gradient against axial distance with slip $\vec{u}_s=0.10$, $n=2$ for mild, moderate and severe with different tapering angle
Fig 7.19: Variation of pressure gradient against axial distance with slip $\bar{u}_s = 0.10$, $n = 2$
for mild, moderate and severe with different tapering angle.

Fig 7.20: Variation of wall shear stress against axial distance with slip $\bar{u}_s = 0.00$, for mild, moderate and severe of symmetric stenosis $n = 2$ with different tapering angle.
Fig 7.21: Variation of wall shear stress against axial distance with slip $\bar{u}_s = 0.0$, for mild, moderate and severe of symmetric stenosis $n = 2$ with different tapering angle.

Fig 7.22: Variation of wall shear stress against axial distance with slip $\bar{u}_s = 0.10$, for mild, moderate and severe of symmetric stenosis $n = 2$ with different tapering angle.
Fig 7.23: Variation of wall shear stress against axial distance with slip $\bar{u}_s = 0.10$, for mild, moderate and severe of symmetric stenosis $n = 2$ with different tapering angle.

Fig 7.24: Variation of wall shear stress against axial distance with slip $\bar{u}_s = 0.10$, for mild, moderate and severe of symmetric stenosis $n = 2$ with different tapering angle.
Fig 7.25: Variation of apparent viscosity against axial distance for different heights of stenosis $n = 2$ with different tapering, yield stress and slip velocity.

Fig 7.26: Variation of apparent viscosity against axial distance for different heights of stenosis $n = 2$ with different tapering, yield stress and slip velocity.
7.5 CONCLUSION

In the study of body fluid blood has been considered to act as a non-Newtonian fluid, possessing a finite yield stress i.e. a Bingham plastic fluid which flows through constricted region in three stages of an arterial stenosis, mild, moderate and severe in Figs. (7.1). Analytic expressions are obtained for velocity, flow rate, stresses, pressure gradient, resistance to flow and apparent viscosity. It is observed that velocity is a function of pressure gradient c, radial r and axial z coordinates, slip velocity \( u_s \), shear viscosities \( \mu_1, \mu \), radii \( R_0, R(z) \) and \( R_1(z) \) for tapering and obstructed regions and \( L_0, d, \delta, \delta_i \) relating to geometry of an arterial stenosis.

The following observations can be found from the present model:

I. If \( \tau_{rz} < \tau_0 \), i.e. if shear stress at a critical distance r is not higher than its yield value, then blood will not flow inside the circular system.

II. If \( \tau_{rz} > \tau_0 \), i.e. shear stress is not lower than a finite yield stress, blood flow inside the body, will be possible.

III. Therefore in accounting for the Bingham behaviour of body fluid blood in two-layered flow, there may arise three regions for flow along the constricted tapering region of an artery viz., \( 0 \leq r \leq r_0, r_0 \leq r \leq R_1(z), R_1(z) \leq r \leq R(z) \). and velocity profile flat or bluntness in the former region whereas deviations from the flat profile will be noticed in the latter.

IV. The bluntness profiles in velocity are noticed in all three forms of stenosis development in this two-layered flow.

In this study, apparent viscosity decreases as tube radius increases in this two-layered constricted flow. Therefore in this case \( \mu_s \) exhibits the anomalous behaviour IFLE. Thus the present model could be explained to anomalous in two-layered blood flow i.e. blunted velocity profile and IFLE. This mathematical model clearly establishes the fact that introduction of a slip condition at a diseased or stenosed artery wall/ fluids interface, accelerates the unidirectional two-layered constriction in a tapering channel in one hand and retards the hindrance to flow constricted channel on the other hand. This appreciable rise in axial velocity and flow rate as well as a considerable reduction in impedance to flow and in apparent viscosity could be use for improving the functions of the damages or occluded blood vessels, i.e. in turn, could play a significant role in reviving or restoring a regular blood flow, in general and flow
inside a stenosed tapering artery with the presence of asymmetric stenosis, in particular.

The present uni-directional model could be improved by employing a variable velocity in radial and axial direction for two-dimensional flow, visco-elastic waxy wall for an arterial stenosis, appropriate magnitude for velocity slip, yield stress for blood etc. and for suggesting such an appropriate model, the present analysis could be taken as a base.