Chapter VI
Stability of a purely growing mode

1 Introduction

We had, in Chapter V, discussed the importance of the purely growing mode in a wide variety of contexts and considered its stability for exactly perpendicular propagation. We had found that this zero frequency mode, which is stable in a Maxwellian plasma for $\beta$-values less than one, can be excited in an anti-loss cone plasma, even for $\beta$-values less than one, provided the plasma is temperature anisotropic with $T_\perp < T_i$. This condition of temperature anisotropy is rather restrictive.

We therefore consider, in this chapter, the stability of this mode again but for parallel propagation. Our plasma again consists of a warm ion component modelled by an ALC distribution; the electrons are again assumed cold and provide only a neutralizing background.

This Chapter is intended to complement the studies of Chapter V in more than one way: one, as mentioned above, we now consider parallel propagation and two, any instability excited will now grow on the time scale of the ion gyro-frequency rather than the electron gyro-frequency. This is because of our assumption that the electrons are cold.

We find that temperature anisotropy is again the source of instability. However, in contrast to the results of Chapter V, the mode can be excited even in a low-$\beta$ Maxwellian plasma provided the temperature anisotropy is in the range $1.0 < \frac{T_\perp}{T_i} < 1.5$. In an ALC plasma this upper limit is extended by the ALC factor $\frac{(1-\rho(T_i))^{1/3}}{(1-\rho(T_i))^{1/4}}$. In a high-$\beta$ anisotropic Maxwellian and an ALC plasma, the condition for stability is more complicated.

2 Expression for Growth/Damping rate

As mentioned above we consider the zero frequency mode in a plasma of warm ions modelled by an ALC distribution and electrons that are assumed cold. This Chapter borrows heavily the expressions from Chapter III; they can be got from the corresponding ones by putting $\delta = 0$ and $\eta = 0$. We reproduce them here for convenience.
2.1 Low-β case

The relevant expressions for ReD(x, k) and ImD(x, k) are:

\[ \text{ReD}(x, k) = \frac{c^2 k^2}{\omega^2_{\text{wup}}} \frac{x^2}{1-x^2} \left( 1 + \frac{1}{3} \beta_{\text{wup}} \frac{T_i}{T_i^*} \frac{x^2}{1-x^2} \right) \]

(1)

and

\[ \text{ImD}(x, k) = -\sqrt{\pi} \frac{\alpha_{\text{wup}}}{k [1 - \rho(T_m^*)]} \left\{ \left[ (A_1 + 1)(1 - x) - 1 \right] \exp(-\zeta^2) - \rho [(A_m + 1)(1 - x) - 1] \exp(-\zeta_m^2) \right\} \]

(2)

where all the terms are as defined in Chapter III. Taking the derivative of (1), we get

\[ \frac{\partial \text{ReD}(x, k)}{\partial x} |_{x=0} = \frac{c^2 k^2}{\omega^2_{\text{wup}}} \beta_{\text{wup}} \left[ \frac{T_i}{T_i^*} - \frac{3(1 - \rho(T_m^*)^1)}{2 \rho(T_m^*)^1} \right] \]

(3)

From (2) and (3), we get the expression for the growth/damping rate of the zero frequency mode as

\[ \gamma |_{x=0} = -\frac{\text{ImD}(x, k)}{\text{ReD}(x, k)} |_{x=0} \]

(4)

\[ = -\frac{\sqrt{\pi}}{(c k)^3 \beta_{\text{wup}}^{3/2} [1 - \rho(T_m^*)]} \frac{\left( \frac{T_i}{T_i^*} - 1 \right) \exp(-\zeta^2) - \rho \left( \frac{T_m}{T_m^*} \right) \exp(-\zeta_m^2)}{\left[ \frac{T_i}{T_i^*} - \frac{3}{2} \left( \frac{1 - \rho(T_m^*)}{\rho(T_m^*)} \right) \right]} \]

where \( \zeta^2 \) and \( \zeta_m^2 \) are now evaluated at \( x = 0 \) and are given by \( \zeta^2 = \frac{1}{(c k)^3 \beta_i} \) and \( \zeta_m^2 = \zeta_i^2 \frac{T_i}{T_m} \).

2.2 Discussion

For \( \rho = 0 \), corresponding to an anisotropic Maxwellian plasma, the expression for \( \gamma \) reduces to

\[ \gamma = -\sqrt{\pi} \frac{(\frac{T_i}{T_i^*} - 1) \exp(-\zeta^2)}{(c k)^3 \beta_{\text{wup}}^{3/2} \frac{T_i}{T_i^*} - \frac{3}{2}} \]

(5)

For \( \frac{T_i}{T_i^*} < 1 \), \( (\frac{T_i}{T_i^*} - 1) \) and \( (\frac{T_i}{T_i^*} - \frac{3}{2}) \) are both negative making \( \gamma \) negative; the zero frequency mode is thus damped. On the other hand, for \( \frac{T_i}{T_i^*} > 1 \) but \( < \frac{3}{2} \), they are of the opposite sign giving a \( \gamma \) which is positive. Finally when \( \frac{T_i}{T_i^*} > \frac{3}{2} \), both the terms are once again
positive making $\gamma$ negative again. Thus the zero frequency mode is excited in a Maxwellian plasma for $1.0 < \frac{T_i}{T_m} < 1.5$. In an ALC plasma, however, the $\frac{3}{2}$-term in the denominator of (4) is multiplied by the ALC factor, thus extending the temperature range over which the mode is unstable.

2.3 High-$\beta$ plasmas

The relevant expression for $ReD(x, k)$ and $ImD(x, k)$ are now

\[
ReD(x, k) = \left( \frac{c_s k_s^2}{\omega_{pwp}^2} \right)^2 - \frac{c_s k_s^2}{\omega_{pwp}^2} \left( \frac{A_t - \rho(\frac{T_m}{T_i})^{1/2} A_m}{1 - \rho(\frac{T_m}{T_i})^{1/2}} - \frac{x}{1 - x} + x^2 \left[ \frac{1}{1 - x} \right] \right) + \frac{2}{\beta_{twp}} \frac{(1 - x)^2}{1 - \rho(\frac{T_m}{T_i})^{1/2}} \{A_t - \frac{x}{1 - x} - \frac{\sqrt{T_i}}{T_m} [A_m - \frac{x}{1 - x}] \}
\]

\[
ImD(x, k) = -\frac{\Omega_p c_{twp}^{1/2}}{k} \frac{\sqrt{\pi}}{1 - \rho(\frac{T_m}{T_i})^{1/2}} \{[(A_t + 1)(1 - x) - 1] \exp(-\frac{\zeta_i^2}{x}) - \rho[(A_m + 1)(1 - x) - 1] \exp(-\frac{\zeta_m^2}{x}) \}
\]

From (6), the derivative of $ReD(x, k)$ is

\[
\frac{\partial ReD(x, k)}{\partial x} \bigg|_{x=0} = \frac{2 \omega_{pwp}^2}{c^2 k^2 \beta_{twp}} \frac{1}{1 - \rho(\frac{T_m}{T_i})^{1/2}} \left\{ \left[ \frac{T_i}{T_i} - 1 \right] - \rho(\frac{T_i}{T_m})^{1/2} \right\} \left[ \frac{T_i}{T_m} - 1 \right]
\]

From (7) and (8), we can arrive at the expression for the growth/damping rate in the usual manner. On simplification we arrive at the expression

\[
\gamma \bigg|_{x=0} = \frac{\sqrt{\pi} \left[ \left( \frac{T_i}{T_i} - 1 \right) \exp(-\frac{\zeta_i^2}{x}) - \rho(\frac{T_i}{T_m} - 1) \exp(-\frac{\zeta_m^2}{x}) \right]}{\left( \frac{c_s k_s}{\omega_{pwp}} \right)^{1/2} \beta_{twp} \left[ 1 - \rho(\frac{T_m}{T_i})^{1/2} \right] Q}
\]

where

\[
Q = \left\{ \left[ \frac{T_i}{T_i} - 1 - \rho(\frac{T_i}{T_m})^{3/2} \right] - \left[ 1 - \rho(\frac{T_i}{T_m})^{1/2} \right] \right\} \frac{2 \omega_{pwp}^2}{\beta_{twp} c^2 k^2} - 1
\]

Inspecting (10), the expression for $Q$, we find that it does not reduce to a simple expression even for the case of $\rho = 0$, because of the factor $\left( \frac{2 \omega_{pwp}^2}{\beta_{twp} c^2 k^2} \right)$ and hence (9) has to be evaluated numerically.
3 Results

We now consider the numerical evaluation of the expressions for the growth / damping rates, namely (4) and (9). The $\beta$-values are the same as considered in Chapter III; the growth /damping rate was studied as a function of the ALC index $\rho$; the ratio of temperatures of the missing to the trapped component $\frac{T_m}{T_t}$ and the temperature anisotropy of the warm ion component $\frac{T_i}{T_w}$. We first consider the low-$\beta$ case.

3.1 Low $\beta$ case

We plot in Figure 1 the growth rate versus $\frac{c_k}{\omega_p}$ for $\beta = 0.04$, $\rho = 0.2$ and $\frac{T_m}{T_t} = \frac{1.0}{1.75}$ for two values of temperature anisotropy $\frac{T_i}{T_w} = 1.8$ and 2.0. This figure has been plotted to show the effect of the ALC function in increasing the temperature anisotropy range for which the mode is unstable: it was proved in section 2.2 that this mode is unstable in a Maxwellian plasma, only for the temperature range $1.0 < \frac{T_i}{T_w} < 1.5$. Thus we find that the mode is unstable in a plasma, described by the ALC distribution, for temperature ratios even greater than 1.5. The growth rate of the mode increases very sharply with $\frac{c_k}{\omega_p}$ and falls off gradually. The growth rate for $\frac{T_i}{T_w} = 2.0$ is smaller than that for $\frac{T_i}{T_w} = 1.8$; this indicates that the upper limit of $\frac{T_i}{T_w}$ is being approached; any further increase would make the mode a damped one. However, within the range of $1.0 < \frac{T_i}{T_w} < 1.5$ the growth rate is for an anisotropic Maxwellian plasma ($\rho = 0$) than for an ALC plasma. This is depicted in Figure 2 as a plot of $\gamma$ versus $\frac{c_k}{\omega_p}$ for $\frac{T_i}{T_w} = 1.2$, $\beta = 0.04$ for two values of $\rho = 0$ (for an anisotropic Maxwellian plasma) and $\rho = 0.1$ ($\frac{T_m}{T_t} = \frac{1.0}{1.25}$).

Figure 3 is also a plot of $\gamma$ versus $\frac{c_k}{\omega_p}$ for $\frac{T_i}{T_w} = 1.8$, $\beta = 0.04$, $\frac{T_m}{T_t} = \frac{1.0}{1.25}$ for two values of ALC index $\rho = 0.1$ and 0.2. While for $\rho = 0.1$, the wave is damped over most of the wavelength range studied, it is strongly unstable over this range for $\rho = 0.2$ (the growth rate for $\rho = 0.2$ is more than an order of magnitude more compared to $\gamma$ for $\rho = 0.1$). Thus the growth rate of the zero frequency mode is sensitive to the ALC index $\rho$ and increases with increasing $\rho$.

Figure 4 is intended to study the variation of the growth rate as a function of $\frac{T_m}{T_t}$; it is therefore a plot of $\gamma$ versus $\frac{c_k}{\omega_p}$ for $\rho = 0.4$ and $\frac{T_m}{T_t} = 1.2$ for two values of $\frac{T_m}{T_t} = \frac{1.0}{1.05}$ and $\frac{1}{1.25}$. We find that the growth rate increases with increasing $\frac{T_m}{T_t}$ or when the trapped
and missing components have approximately the same temperature.

The variation in the growth rate was also studied as a function of $\beta_{i,wp}$ for $T_{ni} = 1.2$, $T_{ni} = \frac{10}{1.05}$ and $\rho = 0.4$ with $\beta_{i,wp}$ taking values of 0.2 and 0.4. We found the variation in growth rate to be very similar except for a shift in the peak value of the growth rate which moved to a lower $\frac{ck}{\omega_p}$ as $\beta_{i,wp}$ increased.

### 3.2 High - $\beta$ case

We now consider the numerical results for the high-$\beta$ case. The $\beta$-value was held a constant at 1.26 and the variation in the growth rate was again studied as a function of $\rho$, $T_{ni}$ and $T_{ni}$.

Figure 5 is thus a plot of $\gamma$ versus $\frac{ck}{\omega_p}$ for an anisotropic Maxwellian plasma ($\rho = 0$) as a function of the temperature anisotropy parameter $\frac{T_{ni}}{T_i}$. We find that the growth rate increases with increasing $\frac{T_{ni}}{T_i}$. Since, unlike the low-$\beta$ case, there is no range for an instability, we find that the mode is unstable for all values of anisotropy studied. Thus temperature anisotropy is the source of instability for this purely growing mode.

We plot, in Figures 6 (a) and (b), $\gamma$ versus $\frac{ck}{\omega_p}$, for two values of $\rho$ (0.0 and 0.6). $T_{ni} = \frac{10}{1.05}$ for the case $\rho = 0.6$. When the temperature anisotropy is low ($\frac{T_{ni}}{T_i} = 1.1$) we find that the growth rate is less for $\rho = 0.6$ as compared to the of $\rho = 0$; however, for a larger $\frac{T_{ni}}{T_i}$ ($=1.7$) the reverse holds and the growth rate increases with increasing $\rho$. This is because of the complicated way in which $\rho$ and $T_{ni}$ enter into the expression for the growth rate.

We next study the variation in the growth rate as a function of $\frac{T_{ni}}{T_i}$. Figure 7 is thus a plot of $\gamma$ versus $\frac{ck}{\omega_p}$ as a function of $\frac{T_{ni}}{T_i}$ ($=1.0$, $1.5$ and $1.8$) for $\frac{T_{ni}}{T_i} = 1.5$ and $\rho = 0.9$. As in the low-$\beta$ case we find that $\gamma$ increases with increasing $\frac{T_{ni}}{T_i}$.

Finally in Figure 8, we illustrate an interesting feature in the variation of the growth rate. As in the previous cases, it too depicts $\gamma$ versus $\frac{ck}{\omega_p}$ but as a function of $\frac{T_{ni}}{T_i}$ ($=1.1$, $1.5$ and $1.8$) for $\rho = 0.9$ and $T_{ni} = \frac{10}{1.05}$. For $\frac{T_{ni}}{T_i} = 1.1$, the zero frequency mode is damped from $\frac{ck}{\omega_p} = 0.1$ to 0.3; for $\frac{T_{ni}}{T_i} = 1.5$ it is damped from $\frac{ck}{\omega_p}$ 0.5 to 1.0 and for $\frac{T_{ni}}{T_i} = 1.5$ it is damped from 0.7 to 5.0. Thus there are regions where the mode is damped; this region broadening with increasing $\frac{T_{ni}}{T_i}$ and shifting towards higher $\frac{ck}{\omega_p}$. 85
4 Conclusion

We have, in this Chapter, considered the stability of a zero frequency mode for parallel propagation. The mode is driven unstable by the temperature anisotropy of the ions. The mode can be driven unstable even in a low-\(\beta\) anisotropic Maxwellian provided the temperature anisotropy lies in the range \(1 < \frac{T_{\perp}}{T_i} < 1.5\); the effect of the ALC index is to extend the upper limit of this temperature ratio. For high-\(\beta\) plasma, however, the expressions are more complicated requiring a numerical evaluation to find the range of the instability. The growth rate increases with increasing \(\frac{T_{\perp}}{T_i}\) in both low- and high-\(\beta\) plasmas.
Figure 1: Plot of growth rate versus $ck/\omega_p$ for $\beta_{nip} = 0.04$, $T_m/T_i = 1.0/1.75$ as a function of $T_/T_i = 1.8$ and 2.0.
Figure 2: Plot of growth rate versus $ck/\omega_p$ for $\beta_{m0} = 0.04$, $T_m/T = 1.0/1.25$ as a function of $\rho = 0.0$ and $0.1$, $T_0/T_1 = 1.2$
Figure 3: Plot of growth rate versus $ck/\omega_0$ for $\beta_m = 0.04$, $T_m/T = 1.0/7.5$ as a function of $\rho$ ($=0.1$ and $0.2$) $T_s/T = 1.8$
Figure 4: Plot of growth rate versus $ck/\alpha_p$ for $\beta_{\text{opt}} = 0.04$, $\rho = 0.4$, as a function of $T_m/T_t = (1.0/1.05$ and $1.0/1.25)$, $T_\mu/T_t = 1.2$. 
Figure 5: Plot of growth rate versus $\frac{ck}{\omega_p}$ for $\beta_{kp} = 1.26$, $\rho = 0.0$, $T_\infty/T_1 = 0.9524$ as a function of $T_\infty/T_1 = 1.1$, 1.2, 1.5, 1.7.
Figure 8a: Plot of growth rate versus $ck/\omega_p$ for $\beta_{0,\infty}=1.26$, as a function of $\rho = 0.0$ and 0.6, $T_m/T_i = 0.9524$, $T_i/T_e=1.1$. 
Figure 6b: Plot of growth rate versus $ck/\omega_0$ for $\beta_{\text{mep}} = 1.26$. $T_m/T_i = 0.9524$ as a function of $\rho$ (0.0 and 0.6). $T_s/T_i = 1.7$
Figure 7: Plot of growth rate versus $ck/\alpha_0$ for $\beta_{\text{iso}} = 1.26$, $p = 0.9$, as a function of $T_{\text{m}}/T = 1.0/1.25$, $1.0/1.5$, $1.0/1.75$, while $T_i/T = 1.5$.
Figure 8: Plot of growth rate versus $ck/\omega_p$ for $\beta_s = 1.26$, $\rho = 0.9$ as a function of $T_d/T_i$ (1.1, 1.5 and 1.8) and for $T_m/T_i = 1.0/1.05$