Chapter 5

GENERALIZED REED-MULLER CODES

Reed-Muller (RM) codes are one of the best studied error correcting codes. They are easy to decode and first order Reed-Muller codes are especially efficient. In the year 1972 the importance of RM codes was realised when they were used for sending photographs from Mars by Mariner series of space crafts. They have presently become more useful as the application of RM codes have expanded manifolds because telecommunications have expanded and developed active use of self-correcting codes. The practical applicability of RM-codes encouraged researchers to study their variants in detail. In this chapter we study dual of Generalized Reed-Muller (DGRM) Codes with respect to a new construction technique.

This chapter is organised in two sections. In section 5.1, we have studied duals of GRM-codes. Their existence is shown with the help of examples in section 5.2.

5.1 CONSTRUCTION OF DGRM CODES

We begin by giving some definitions:

**Definition 5.1:** For any positive integers $r$ and $m$ ($r < m$) there is a RM-code of order $r$ formed by using as a basis the vectors $v_0$, $v_1$, $v_2$, ..., $v_m$ and all vector products of $v_1$, $v_2$, ..., $v_m$ taken $r$ or fewer at a time, where $v_1$, $v_2$, ..., $v_m$ are the rows of a matrix that has columns which are binary representations of $0$, $1$, $2$, ..., $2^m - 1$ as $m$-tuples and the vector $v_0$ has all its components as 1's.

**Definition 5.2:** For a given $m$, we define the **punctured vector** of the vector $v_i$ to be the vector product $v_1v_2v_3...v_{i-1}v_{i+1}...v_m$ of the $m - 1$ vectors. Similarly
the punctured vector of the vector $v_i v_j$ is defined to be the vector $v_1 v_2 \ldots v_{i-1} v_{i+1} \ldots v_{j-1} v_{j+1} \ldots v_m$ and so on.

Dass and Muttoo (1981) introduced a new class of Generalized Reed-Muller codes known as GRM codes of order $r + (r + 1)m, s$ by extending $r$th order Reed-Muller code. Their duals were studied by Dass and Tyagi (1988) and it was proved that the duals are also GRM-codes of order $(m - r - 2) + (m - r - 1)$.

Definition 5.3: A GRM code of order $r + (r + 1)m, s$ is one formed by using as basis the vectors $v_0, v_1, v_2, \ldots, v_m$ and all vector products of $v_1, v_2, \ldots, v_m$ taken $r$ or fewer at a time along with some $s$ vector products $1 \leq s < \binom{m}{r+1}$ of these vectors taken $(r + 1)$ at a time.

So, if $G(r, m)$ denotes the generator matrix of a $RM(r, m)$ code, then generator matrix of $GRM$-code of order $r + (r + 1)m, s$ may be written as

$$
\begin{bmatrix}
G(r, m) \\
X
\end{bmatrix},
$$

where $X$ is a matrix containing some $s$ vector products of $v_1, v_2, \ldots, v_m$ vectors taken $(r + 1)$ at a time. These codes and their duals are also studied by Charpin (1984), Dass and Wasan (1983) and Muttoo and Rana (2008).

Recently Tyagi and Rani (2010) have developed a new category of GRM-codes by combining two or more GRM-codes as follows:

If $G_1, G_2$ and $G_3$ be the generator matrices of the GRM codes of order $(r + 2) + ((r + 2) + 1)m, s_1$, $(r + 1) + ((r + 1) + 1)m, s_2$ and $r + (r+1)m, s_3$, respectively,
then the code $C$ generated by $G = \begin{bmatrix} G_1 & G_1 & G_1 & G_1 \\ O & G_2 & O & G_2 \\ O & O & G_2 & G_2 \\ O & O & O & G_3 \end{bmatrix}$ is also a GRM code of order $(r + 2) + ((r + 2) + 1)_{m+2,n_1+2s_2+s_3}$.

In this section, we study the code $\bar{C}$ generated by $\bar{G} = \begin{bmatrix} \bar{G}_1 & \bar{G}_1 & \bar{G}_1 & \bar{G}_1 \\ O & \bar{G}_2 & O & \bar{G}_2 \\ O & O & \bar{G}_2 & \bar{G}_2 \\ O & O & O & \bar{G}_3 \end{bmatrix}$, where $\bar{G}_1$ is the generator matrix of the dual of the GRM code generated by the matrix $G_1$ of order $r + (r+1)_{m_{n_1}}$; $\bar{G}_2$ is the generator matrix of the dual of the GRM code generated by the matrix $G_2$ of order $(r + 1) + ((r + 1) + 1)_{m_{n_2}}$; $\bar{G}_3$ is the generator matrix of the dual of the GRM code generated by the matrix $G_3$ of order $(r + 2) + ((r + 2) + 1)_{m_{n_3}}$ and prove that the code $\bar{C}$ is also a GRM code.

In the following theorem we study the nature of the code $\bar{C}$.

**Theorem 5.1:** The matrix $\bar{G} = \begin{bmatrix} \bar{G}_1 & \bar{G}_1 & \bar{G}_1 & \bar{G}_1 \\ O & \bar{G}_2 & O & \bar{G}_2 \\ O & O & \bar{G}_2 & \bar{G}_2 \\ O & O & O & \bar{G}_3 \end{bmatrix}$ generates a GRM-code of order

$$(m-r-2) + (m-r-1)_{m+2-(m-r-1)-(n_{1}+2s_{2}+s_{3})}.$$ 

**Proof.** In this theorem take $\bar{G}_1$ as GRM code of order

$$(m-r-2) + (m-r-1)_{m-(m-r-1)-n_{1}}.$$
$\tilde{G}_2$ as GRM-code of order

$$(m - r - 3) + (m - r - 2) \binom{m}{m - r - 2} x_2$$

and $\tilde{G}_3$ as GRM-code of order

$$(m - r - 4) + (m - r - 3) \binom{m}{m - r - 3} y_3.$$ 

Then $\tilde{G}$ is a GRM-code of order

$$(m - r - 2) + (m - r - 1) \binom{m + 2}{m - r - 1} \left[\binom{m}{(m - r - 1)} y_1 + 2 \binom{m}{(m - r - 2)} y_2 + 3 \binom{m}{(m - r - 3)} y_3\right]$$

i.e.

$$(m - r - 2) + (m - r - 1) \binom{m + 2}{m - r - 1} ^{(m + 2, m - r - 1)}.$$ 

**Corollary:** (i) The code $\tilde{C}$ with generator matrix

$$\tilde{G} = \begin{bmatrix} \tilde{G}_1 & \tilde{G}_1 & \tilde{G}_1 \\ O & \tilde{G}_2 & O \\ O & O & \tilde{G}_2 \\ O & O & O & \tilde{G}_3 \end{bmatrix}$$

is a GRM-code of order $(m - r - 2) + (m - r - 1) \binom{m + 2}{m - r - 1} ^{(m + 2, m - r - 1)}.$

**Proof:** In the above theorem take $s_1 = s$, $s_2 = s$ and $s_3 = s$, we get the required result.

(ii) The generator matrix $\tilde{G}$ may be written as

$$\begin{bmatrix} \tilde{G}_1' & \tilde{G}_1' \\ 0 & \tilde{G}_2' \end{bmatrix},$$

where $\tilde{G}_1' = \begin{bmatrix} \tilde{G}_1 & \tilde{G}_1 \\ O & \tilde{G}_2 \end{bmatrix}$ and $\tilde{G}_2' = \begin{bmatrix} \tilde{G}_2 & \tilde{G}_2 \\ O & \tilde{G}_3 \end{bmatrix}.$

It may be observed that the results given by Muttoo and Rana (2008) are particular cases of the above theorem.
5.2. ILLUSTRATIONS

We now illustrate the dual code $\overline{C}$, generated by $\overline{G} = \left[ \begin{array}{cccc} \overline{G}_1 & \overline{G}_1 & \overline{G}_1 & \overline{G}_1 \\ \overline{G}_2 & 0 & \overline{G}_2 & 0 \\ 0 & \overline{G}_2 & \overline{G}_2 & 0 \\ 0 & 0 & 0 & \overline{G}_3 \end{array} \right]$ using different values of $r$ and $m$ with the help of examples.

Example 5.1. $m = 4$

\[ \begin{align*}
G(3,4) &= \begin{bmatrix}
\nu_0 \\
\nu_4 \\
\nu_3 \\
\nu_2 \\
\nu_1 \\
\nu_1 \nu_2 \\
\nu_1 \nu_3 \\
\nu_1 \nu_4 \\
\nu_2 \nu_3 \\
\nu_2 \nu_4 \\
\nu_3 \nu_4 \\
\nu_1 \nu_2 \nu_3 \\
\nu_1 \nu_2 \nu_4 \\
\nu_2 \nu_3 \nu_4 \\
\nu_1 \nu_3 \nu_4
\end{bmatrix} \\
\text{Case 1. Let } m = 4, r = 0, s = 1.
\end{align*} \]

The generator matrices $G_i$ for $GRM$-codes of order $0 + (0 + 1)_{4,1}$, are four in number as follows:

\[ G_{i,i} = \begin{bmatrix} \nu_0 \\ \nu_i \end{bmatrix}, \ i = 1 \text{ to } 4. \]

The generator matrices $\overline{G}_1$ of the dual of these codes are
We observe that the punctured vector of $v_i$ is absent in the corresponding generator matrices of the duals of $GRM$-codes.

The generator matrices $G_2$ for $GRM$-codes of order $(0 + 1) + ((0 + 1) + 1)_{4,1}$, are six in number as follows:

$$G_{2,n} = \begin{bmatrix} G(1,4) \\ v_iv_j \end{bmatrix}, \quad i, j = (1, 2, 3, 4); \quad i < j$$

$$n = 1 \text{ to } 6.$$

The generator matrices $\bar{G}_2$ for the duals of these codes are

$$\bar{G}_{2,1} = \begin{bmatrix} v_i v_2 \\ v_i v_3 \\ v_i v_4 \\ v_2 v_3 \\ v_2 v_4 \end{bmatrix}, \quad \bar{G}_{2,2} = \begin{bmatrix} v_i v_2 \\ v_i v_4 \\ v_2 v_3 \\ v_2 v_4 \end{bmatrix}, \quad \bar{G}_{2,3} = \begin{bmatrix} v_i v_2 \\ v_i v_3 \\ v_2 v_3 \\ v_2 v_4 \end{bmatrix}, \quad \bar{G}_{2,4} = \begin{bmatrix} v_i v_2 \\ v_i v_4 \\ v_2 v_4 \\ v_3 v_4 \end{bmatrix}.$$
\[
\begin{pmatrix}
G(1, 4)\\
v_1v_2 \\
v_1v_3 \\
v_1v_4 \\
v_2v_3 \\
v_2v_4
\end{pmatrix},
\begin{pmatrix}
G(1, 4)\\
v_1v_2 \\
v_1v_3 \\
v_1v_4 \\
v_2v_3 \\
v_2v_4
\end{pmatrix}.
\]

We observe that the punctured vector of \(v_i\) \(v_j\) is absent in the corresponding generator matrices of the duals of GRM-codes.

The generator matrices \(G_3\) for GRM-codes of order \((0 + 2) + ((0 + 2) + 1)_{4,1}\) are four in number which may be written as

\[
G_{3,n} = \begin{bmatrix}
G(2, 4) \\
v_i v_j v_k
\end{bmatrix}; \quad i, j, k = (1, 2, 3, 4); \quad i < j < k
\]

\[
n = 1 \text{ to } 4.
\]

The generator matrices \(\bar{G}_3\) of the dual of these codes are

\[
\bar{G}_{3,n} = \begin{bmatrix}
v_0 \\
v_k \\
v_j \\
v_i
\end{bmatrix}; \quad i, j, k = (1, 2, 3, 4); \quad i < j < k
\]

\[
n = 1 \text{ to } 4.
\]

We observe that the punctured vector of \(v_i\) \(v_j\) \(v_k\) is absent in the corresponding generator matrices of the duals of GRM-codes.

**Subcase:** Consider

\[
\bar{G} = \begin{bmatrix}
\bar{G}_{1,1} & \bar{G}_{1,1} & \bar{G}_{1,1} & \bar{G}_{1,1} \\
o & \bar{G}_{2,1} & o & \bar{G}_{2,1} \\
o & o & \bar{G}_{2,1} & \bar{G}_{2,1} \\
o & o & o & \bar{G}_{3,1}
\end{bmatrix}
\]
\[
\begin{bmatrix}
    v_0 & v_0 & v_0 & v_0 \\
    v_4 & v_4 & v_4 & v_4 \\
    v_3 & v_3 & v_3 & v_3 \\
    v_2 & v_2 & v_2 & v_2 \\
    v_1 & v_1 & v_1 & v_1 \\
    v_1 v_2 & v_1 v_2 & v_1 v_2 & v_1 v_2 \\
    v_1 v_3 & v_1 v_3 & v_1 v_3 & v_1 v_3 \\
    v_1 v_4 & v_1 v_4 & v_1 v_4 & v_1 v_4 \\
    v_2 v_3 & v_2 v_3 & v_2 v_3 & v_2 v_3 \\
    v_2 v_4 & v_2 v_4 & v_2 v_4 & v_2 v_4 \\
    v_3 v_4 & v_3 v_4 & v_3 v_4 & v_3 v_4 \\
    v_4 v_5 & v_4 v_5 & v_4 v_5 & v_4 v_5 \\
    v_5 v_6 & v_5 v_6 & v_5 v_6 & v_5 v_6 \\
\end{bmatrix}
= 
\begin{bmatrix}
    \bar{v}_0 \\
    \bar{v}_4 \\
    \bar{v}_3 \\
    \bar{v}_2 \\
    \bar{v}_1 \\
    \bar{v}_1 \bar{v}_2 \\
    \bar{v}_1 \bar{v}_3 \\
    \bar{v}_1 \bar{v}_4 \\
    \bar{v}_2 \bar{v}_3 \\
    \bar{v}_2 \bar{v}_4 \\
    \bar{v}_3 \bar{v}_4 \\
    \bar{v}_4 \bar{v}_5 \\
    \bar{v}_5 \bar{v}_6 \\
    \bar{v}_6 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0 & 0 & v_2 v_3 & v_2 v_3 \\
    0 & 0 & v_2 v_4 & v_2 v_4 \\
    0 & 0 & 0 & v_0 \\
    0 & 0 & v_2 v_3 & v_2 v_3 \\
    0 & 0 & v_2 v_4 & v_2 v_4 \\
    0 & 0 & 0 & v_1 \\
    0 & 0 & v_2 v_3 & v_2 v_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
    v_2 v_3 v_6 \\
    v_2 v_4 v_6 \\
    v_3 v_6 \\
    v_2 v_3 v_6 \\
    v_2 v_4 v_6 \\
    v_3 v_6 \\
    v_3 v_6 \\
\end{bmatrix}
\]
which is dual of $GRM$-code of order $(0 + 2) + ((0 + 2) + 1)_{6,4}$ generated by

$$G = \begin{bmatrix}
G_{3,1} & G_{3,1} & G_{3,1} \\
0 & G_{2,1} & O \\
0 & O & G_{2,1} \\
0 & O & O & G_{1,1}
\end{bmatrix}.$$ 

Similarly, if we take either of $G_{1,1}, G_{1,2}, G_{1,3}, G_{1,4}$ as $G_1$, and either of $G_{2,1}, G_{2,2}, G_{2,3}, G_{2,4}, G_{2,5}, G_{2,6}$ as $G_2$, and either of $G_{3,1}, G_{3,2}, G_{3,3}, G_{3,4}$ as $G_3$, then $C$ is a $GRM$ code of order $2 + (3)_{6,16}$ which is dual of $GRM$-code of order $2 + (3)_{6,4}$.

**Case II:** For $m = 4, r = 0, s = 2$, the generator matrices $G_1$ for $GRM$-codes of order $0 + (0 + 1)_{4,2}$ are six in number which may be written as

$$G_{1,n} = \begin{bmatrix}
v_i \\
v_j
\end{bmatrix} ; \quad i,j = (1,2,3,4); \quad i < j ; \quad n = 1 \text{ to } 6.$$ 

The generator matrices $\bar{G}_1$ of the dual of these codes are

$$\bar{G}_{1,1} = \begin{bmatrix} G(2,4) \\ v_1v_2v_3 \\ v_1v_2v_4 \end{bmatrix}, \quad \bar{G}_{1,2} = \begin{bmatrix} G(2,4) \\ v_1v_2v_3 \\ v_1v_3v_4 \end{bmatrix}, \quad \bar{G}_{1,3} = \begin{bmatrix} G(2,4) \\ v_1v_2v_4 \\ v_1v_3v_4 \end{bmatrix}, \quad \bar{G}_{1,4} = \begin{bmatrix} G(2,4) \\ v_1v_2v_3 \\ v_2v_3v_4 \end{bmatrix},$$

$$\bar{G}_{1,5} = \begin{bmatrix} G(2,4) \\ v_2v_3v_4 \\ v_1v_2v_4 \end{bmatrix}, \quad \bar{G}_{1,6} = \begin{bmatrix} G(2,4) \\ v_1v_3v_4 \\ v_2v_3v_4 \end{bmatrix}.$$ 

We observe that the punctured vectors of $v_i, v_j$ is absent in the corresponding generator matrices of the duals of $GRM$-codes.
The generator matrices $G_2$ for $GRM$-codes of order $(0 + 1) + ((0 + 1) + 1)_{4,2}$ are fifteen in number which may be written as

\[
G_{2,1} = \begin{bmatrix}
G(1,4) \\
v_1v_2 \\
v_1v_3 \\
v_1v_4
\end{bmatrix},
G_{2,2} = \begin{bmatrix}
G(1,4) \\
v_1v_2 \\
v_1v_4 \\
v_2v_3
\end{bmatrix},
G_{2,3} = \begin{bmatrix}
G(1,4) \\
v_1v_2 \\
v_2v_3 \\
v_2v_4
\end{bmatrix},
G_{2,4} = \begin{bmatrix}
G(1,4) \\
v_1v_2 \\
v_2v_4 \\
v_3v_4
\end{bmatrix},
G_{2,5} = \begin{bmatrix}
G(1,4) \\
v_1v_2 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
G_{2,6} = \begin{bmatrix}
G(1,4) \\
v_1v_3 \\
v_2v_3 \\
v_2v_4
\end{bmatrix},
G_{2,7} = \begin{bmatrix}
G(1,4) \\
v_1v_3 \\
v_2v_4 \\
v_3v_4
\end{bmatrix},
G_{2,8} = \begin{bmatrix}
G(1,4) \\
v_1v_3 \\
v_2v_4 \\
v_3v_4
\end{bmatrix},
G_{2,9} = \begin{bmatrix}
G(1,4) \\
v_1v_3 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
G_{2,10} = \begin{bmatrix}
G(1,4) \\
v_1v_4 \\
v_2v_3 \\
v_2v_4
\end{bmatrix},
G_{2,11} = \begin{bmatrix}
G(1,4) \\
v_2v_4 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
G_{2,12} = \begin{bmatrix}
G(1,4) \\
v_2v_4 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
G_{2,13} = \begin{bmatrix}
G(1,4) \\
v_2v_4 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
G_{2,14} = \begin{bmatrix}
G(1,4) \\
v_2v_4 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
G_{2,15} = \begin{bmatrix}
G(1,4) \\
v_2v_4 \\
v_3v_4 \\
v_3v_4
\end{bmatrix}.
\]

The generator matrices $\bar{G}_2$ for the duals of these codes are

\[
\bar{G}_{2,1} = \begin{bmatrix}
G(1,4) \\
v_1v_2 \\
v_1v_3 \\
v_1v_4
\end{bmatrix},
\bar{G}_{2,2} = \begin{bmatrix}
G(1,4) \\
v_1v_2 \\
v_1v_4 \\
v_2v_3
\end{bmatrix},
\bar{G}_{2,3} = \begin{bmatrix}
G(1,4) \\
v_1v_2 \\
v_2v_3 \\
v_2v_4
\end{bmatrix},
\bar{G}_{2,4} = \begin{bmatrix}
G(1,4) \\
v_1v_2 \\
v_2v_4 \\
v_3v_4
\end{bmatrix},
\bar{G}_{2,5} = \begin{bmatrix}
G(1,4) \\
v_1v_2 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
\bar{G}_{2,6} = \begin{bmatrix}
G(1,4) \\
v_1v_3 \\
v_2v_3 \\
v_2v_4
\end{bmatrix},
\bar{G}_{2,7} = \begin{bmatrix}
G(1,4) \\
v_1v_3 \\
v_2v_4 \\
v_3v_4
\end{bmatrix},
\bar{G}_{2,8} = \begin{bmatrix}
G(1,4) \\
v_1v_3 \\
v_2v_4 \\
v_3v_4
\end{bmatrix},
\bar{G}_{2,9} = \begin{bmatrix}
G(1,4) \\
v_1v_3 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
\bar{G}_{2,10} = \begin{bmatrix}
G(1,4) \\
v_1v_4 \\
v_2v_3 \\
v_2v_4
\end{bmatrix},
\bar{G}_{2,11} = \begin{bmatrix}
G(1,4) \\
v_2v_4 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
\bar{G}_{2,12} = \begin{bmatrix}
G(1,4) \\
v_2v_4 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
\bar{G}_{2,13} = \begin{bmatrix}
G(1,4) \\
v_2v_4 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
\bar{G}_{2,14} = \begin{bmatrix}
G(1,4) \\
v_2v_4 \\
v_3v_4 \\
v_3v_4
\end{bmatrix},
\bar{G}_{2,15} = \begin{bmatrix}
G(1,4) \\
v_2v_4 \\
v_3v_4 \\
v_3v_4
\end{bmatrix}.
\]
We observe that the punctured vectors of $v_1v_2$, $v_1v_3$; $v_1v_2$, $v_1v_4$; $v_1v_2$, $v_2v_4$; $v_1v_2$, $v_3v_4$; $v_1v_3$, $v_1v_4$; $v_1v_3$, $v_2v_4$; $v_1v_3$, $v_3v_4$; $v_1v_4$, $v_2v_4$; $v_2v_3$, $v_3v_4$; $v_2v_4$, $v_3v_4$ are absent respectively in the corresponding generator matrices of the duals of $GRM$-codes.

The generator matrices $G_3$ for $GRM$-codes of order $(0 + 2) +((0 + 2) + 1)_{4,2}$ which are six in number which may be written as

$$
G_{3,1} = \begin{bmatrix} G(2,4) \\ v_1v_2v_3 \\ v_1v_2v_4 \end{bmatrix},
G_{3,2} = \begin{bmatrix} v_1v_2v_3 \\ v_2v_3v_4 \end{bmatrix},
G_{3,3} = \begin{bmatrix} G(2,4) \\ v_1v_2v_3 \\ v_1v_3v_4 \end{bmatrix},
G_{3,4} = \begin{bmatrix} G(2,4) \\ v_1v_2v_4 \\ v_2v_3v_4 \end{bmatrix},

G_{3,5} = \begin{bmatrix} G(2,4) \\ v_1v_2v_4 \\ v_1v_3v_4 \end{bmatrix},
G_{3,6} = \begin{bmatrix} v_2v_3v_4 \end{bmatrix}.
$$

The generator matrices $\overline{G}_3$ of the dual of these codes are

$$
\overline{G}_{3,1} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix},
\overline{G}_{3,2} = \begin{bmatrix} v_0 \\ v_2 \\ v_3 \end{bmatrix},
\overline{G}_{3,3} = \begin{bmatrix} v_0 \\ v_1 \\ v_3 \end{bmatrix},
\overline{G}_{3,4} = \begin{bmatrix} v_0 \\ v_2 \\ v_4 \end{bmatrix},
\overline{G}_{3,5} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix},
\overline{G}_{3,6} = \begin{bmatrix} v_3 \\ v_4 \end{bmatrix}.
$$

We observe that the punctured vectors of $v_1v_2v_3$, $v_1v_2v_4$; $v_1v_2v_3$, $v_2v_3v_4$; $v_1v_2v_3$, $v_1v_3v_4$; $v_1v_2v_4$, $v_2v_3v_4$; $v_1v_2v_4$, $v_1v_3v_4$; $v_2v_3v_4$, $v_1v_3v_4$ are respectively absent in the corresponding generator matrices of the duals of $GRM$-codes.

**Subcase:** Consider

$$
\overline{G} = \begin{bmatrix} \overline{G}_{1,1} & \overline{G}_{1,1} & \overline{G}_{1,1} & \overline{G}_{1,1} \\ O & \overline{G}_{2,2} & O & \overline{G}_{2,2} \\ O & O & \overline{G}_{2,2} & \overline{G}_{2,2} \\ O & O & O & \overline{G}_{3,1} \end{bmatrix}.
$$
\[
\begin{bmatrix}
\nu_0 & \nu_0 & \nu_0 & \nu_0 \\
\nu_4 & \nu_4 & \nu_4 & \nu_4 \\
\nu_3 & \nu_3 & \nu_3 & \nu_3 \\
\nu_2 & \nu_2 & \nu_2 & \nu_2 \\
\nu_1 & \nu_1 & \nu_1 & \nu_1 \\
\nu_1 \nu_2 & \nu_1 \nu_2 & \nu_1 \nu_2 & \nu_1 \nu_2 \\
\nu_1 \nu_3 & \nu_1 \nu_3 & \nu_1 \nu_3 & \nu_1 \nu_3 \\
\nu_1 \nu_4 & \nu_1 \nu_4 & \nu_1 \nu_4 & \nu_1 \nu_4 \\
\nu_2 \nu_3 & \nu_2 \nu_3 & \nu_2 \nu_3 & \nu_2 \nu_3 \\
\nu_2 \nu_4 & \nu_2 \nu_4 & \nu_2 \nu_4 & \nu_2 \nu_4 \\
\nu_3 \nu_4 & \nu_3 \nu_4 & \nu_3 \nu_4 & \nu_3 \nu_4 \\
\nu_1 \nu_2 \nu_3 & \nu_1 \nu_2 \nu_3 & \nu_1 \nu_2 \nu_3 & \nu_1 \nu_2 \nu_3 \\
\nu_1 \nu_2 \nu_4 & \nu_1 \nu_2 \nu_4 & \nu_1 \nu_2 \nu_4 & \nu_1 \nu_2 \nu_4 \\
0 & \nu_0 & 0 & \nu_0 \\
0 & \nu_4 & 0 & \nu_4 \\
0 & \nu_3 & 0 & \nu_3 \\
0 & \nu_2 & 0 & \nu_2 \\
0 & \nu_1 & 0 & \nu_1 \\
0 & \nu_1 \nu_2 & 0 & \nu_1 \nu_2 \\
0 & \nu_1 \nu_3 & 0 & \nu_1 \nu_3 \\
0 & \nu_1 \nu_4 & 0 & \nu_1 \nu_4 \\
0 & \nu_2 \nu_4 & 0 & \nu_2 \nu_4 \\
0 & 0 & \nu_0 & 0 \\
0 & 0 & \nu_4 & \nu_4 \\
0 & 0 & \nu_3 & \nu_3 \\
0 & 0 & \nu_2 & \nu_2 \\
0 & 0 & \nu_1 & \nu_1 \\
0 & 0 & \nu_1 \nu_2 & \nu_1 \nu_2 \\
0 & 0 & \nu_1 \nu_3 & \nu_1 \nu_3 \\
0 & 0 & \nu_1 \nu_4 & \nu_1 \nu_4 \\
0 & 0 & \nu_2 \nu_4 & \nu_2 \nu_4 \\
0 & 0 & 0 & \nu_0 \\
0 & 0 & 0 & \nu_1 \\
0 & 0 & 0 & \nu_2 \\
\end{bmatrix}
\]

which is dual of GRM-code of order \((0 + 2) + ((0 + 2) + 1)_{6,8}\) generated by
\[
G = \begin{bmatrix}
G_{3.1} & G_{3.1} & G_{3.1} & G_{3.1} \\
O & G_{2.2} & O & G_{2.2} \\
O & O & G_{2.2} & G_{2.2} \\
O & O & O & G_{1.1}
\end{bmatrix}.
\]

Similarly, if we take either of \( G_{1.1}, G_{1.2}, G_{1.3}, G_{1.4}, G_{1.5}, G_{1.6} \) as \( G_1 \),

and either of \( G_{2.1}, G_{2.2}, G_{2.3}, \ldots, G_{2.15} \) as \( G_2 \),

and either of \( G_{3.1}, G_{3.2}, G_{3.3}, G_{3.4}, G_{3.5}, G_{3.6} \) as \( G_3 \),

then \( \bar{C} \) is a \( GRM \)-code of order \((0 + 2) + ((0 + 2) + 1)_{6,12}\) which is dual of \( GRM \)-code of order \(2 + (3)_{6,8}\).
SCOPE FOR FURTHER RESEARCH

In the second chapter of this thesis, we have studied a family of \((b_1, b_2)\)-optimal codes over \(GF(3), GF(5)\) and \(GF(7)\) respectively. This study can be extended to find the possibility of the existence of such codes for higher values of \(q\).

In the third chapter of the thesis, we have obtained lower and upper bounds on the number of parity check digits for \((n = n_1 + n_2 + \ldots + n_m, k)\) codes correcting burst errors in different sub-blocks and discussed different possibilities for both the bounds. We have shown the existence of BECL codes for different values of the parameters \(n_1, n_2, n_3, k, b_1, b_2, b_3\) by constructing appropriate parity-check matrices following the synthesis procedure outlined in the proof of Theorem 3.2. However, the problem needs further investigation to find the possibilities of the existence of \((b_1, b_2, b_3, \ldots, b_m)\)-Optimal Codes both in binary and non-binary cases.

In the fourth chapter of the thesis we have shown the non-existence of some \((1, 2)\)-optimal codes over \(GF(q)\) for \(n - k = 3\). However, the problem needs further investigation to find the possibility of the non-existence of \((1, 2)\)-Optimal \((n = n_1 + n_2, k)\) codes over \(GF(q)\), for \(n - k = 4\) and \(m \leq n_2 \leq q(q + 1) + 2\), where

\[
m = \begin{cases} 
12 & \text{if } q = 3 \\
16 & \text{if } q = 5 \\
28 & \text{if } q = 7 \\
\vdots & \vdots 
\end{cases}
\]

and for higher values of \(n - k\).

In the fifth chapter of the thesis, we have shown the existence of dual of \(GRM\)-codes by taking three generator matrices. This study can be generalized by taking four or more generator matrices, resulting in the existence of \(GRM\)-codes.