Chapter 4

NON-EXISTENCE OF SOME OPTIMAL AND PERFECT CODES

The question of the existence of perfect codes has been a matter of intense research activity for many years. The investigations have been done with a view to have perfect codes for random and burst error correction. The only known perfect codes, in addition to the class of single error correcting codes, (popularly known as Hamming codes (1950)), and the binary repetition codes are Golay's (1949) (23, 12, 7) binary and (11, 6, 5) ternary codes. After several attempts to establish the non-existence of perfect codes, it has finally been proved by Tietavainen (1973) and Van Lint (1975) that there are no perfect Hamming metric codes over prime-power alphabets other than the above mentioned codes. The non-existence of perfect codes has been proved by several authors. In this connection, Etzion (2001) has mentioned that (8, 4) 2-burst correcting perfect code does not exist.

The Non-Existence of Certain Binary Linear \((n, k)\)-Codes with different values of the parameters \(n\) and \(k\) was given by Hill and Traynor (1990). Also, the non-existence of ternary linear codes has been given by Van Eupen (1995), Dasakalov and Metodieva (2004) and others.

In this chapter, we study the non-existence of some perfect and optimal codes. The chapter is divided into two sections. In section 4.1, we have studied non-existence of some \((1, 2)\)-optimal codes. Section 4.2 deals with the non-existence of the famous \((8, 4)\)-code with respect to burst of length 2 or less over \(GF(2)\).
4.1 NON-EXISTENCE OF SOME (1,2)-OPTIMAL CODES

In this section we present the non-existence of certain blockwise burst error correcting (1, 2)-Optimal Linear Codes over GF(q) (q prime), where the code length n is divided into two sub-blocks of lengths \( n_1 \) and \( n_2 \); \( n = n_1 + n_2 \).

An \((n = n_1 + n_2, k)\) linear code that corrects all bursts of length \( b_1 = 1 \) in the first sub-block of length \( n_1 \) and all bursts of length \( b_2 = 2 \) (fix) in the second sub-block of length \( n_2 \), and no other error pattern, will be called as (1, 2)-optimal linear code.

After closely observing the table 2.1 (in Chapter 2) for different values of \( q \), it was found in many cases that (1, 2)-optimal codes do not exist even though the values of the parameters \( n_1, n_2 \) and \( k \) satisfy equation (2.2). These values of the parameters are listed below in Table-4.1.

<table>
<thead>
<tr>
<th>( q )</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( k )</th>
<th>Corresponding Codes which do not exist</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>(1 + 5, 3)</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>7</td>
<td>5</td>
<td>(1 + 7, 5)</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>9</td>
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</tr>
<tr>
<td></td>
<td>11</td>
<td>13</td>
<td></td>
<td>(11 + 5, 13)</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>9</td>
<td>7</td>
<td>(1 + 9, 7)</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>13</td>
<td></td>
<td>(8 + 8, 13)</td>
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<tr>
<td></td>
<td>15</td>
<td>19</td>
<td></td>
<td>(15 + 7, 19)</td>
</tr>
<tr>
<td></td>
<td>22</td>
<td>25</td>
<td></td>
<td>(22 + 6, 25)</td>
</tr>
<tr>
<td></td>
<td>29</td>
<td>31</td>
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<tr>
<td>11</td>
<td>1</td>
<td>13</td>
<td>11</td>
<td>(1 + 13, 11)</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>21</td>
<td></td>
<td>(12 + 12, 21)</td>
</tr>
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<td></td>
<td>23</td>
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<td>45</td>
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<td></td>
<td>(45 + 9, 51)</td>
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<td>56</td>
<td>61</td>
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<td>(56 + 8, 61)</td>
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<td>67</td>
<td>71</td>
<td></td>
<td>(67 + 7, 71)</td>
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<td>78</td>
<td>81</td>
<td></td>
<td>(78 + 6, 81)</td>
</tr>
<tr>
<td></td>
<td>89</td>
<td>91</td>
<td></td>
<td>(89 + 5, 91)</td>
</tr>
</tbody>
</table>

Table Contd.
It has been observed that such a situation arises mainly when \( n_2 \geq 5 \) and \( n - k = 3 \). Further, it may be noted that there seems to be a possibility for \(((q^2 - 3q + 1) + 5, q^2 - 3q + 3)\) code over \( GF(q) \) to exist (Table 4.1), however, we illustrate in our next theorem that it is not possible to construct a parity check matrix for such codes.

**Theorem 4.1:** \((1, 2)\)-optimal \(((q^2 - 3q + 1) + 5, q^2 - 3q + 3)\) code over \( GF(q) \), does not exist.

**Proof:** We prove this theorem by assuming that such codes exist.

Suppose that \( H \) denotes parity-check matrix for \(((q^2 - 3q + 1) + 5, q^2 - 3q + 3)\) code. The matrix \( H \) is formed by first constructing 2\(^{nd}\) sub-block. Once 2\(^{nd}\) sub-block is constructed, we will be left with exactly \((q - 1)(q^2 - 3q + 1)\) number of non-zero 3-tuples (since there are 5 columns in 2\(^{nd}\) sub-block, therefore the number of error-patterns corresponding to this sub-block are \(4(q^2 - q)\). Since the total number of non-zero 3-tuples is \((q^3 - 1)\), therefore the number of unused
non-zero 3-tuples is \( [(q^3 - 1) - 4(q^2 - q)] = (q - 1)(q^2 - 3q + 1) \), and hence \( (q^2 - 3q + 1) \) number of 3-tuples can be placed as the columns in the first sub-block of \( H \) in any order. Therefore main problem is to construct the second sub-block of \( H \).

Suppose \( H = [H_1 \ H_2] \) (where \( H_1 \) and \( H_2 \) denote the sub-matrices corresponding to the 1\(^{\text{st}}\) and 2\(^{\text{nd}}\) sub-blocks of \( H \) respectively), and

\[
H_2 = [H'_2 \ H''_2],
\]
where \( H'_2 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \) and \( H''_2 \) denotes the matrix form of the last two columns of 2\(^{\text{nd}}\) sub-block of \( H \).

Since \( H = [H_1 \ H_2] \) is a parity-check matrix of the given code and required properties of the code ensure that 2\(^{\text{nd}}\) column of \( H'_2 \) must be independent of the first and then that the third must be independent of the first two (i.e. all the three columns are linearly independent), therefore, \( |H'_2| \neq 0 \) (\( \because \) a necessary and sufficient condition for the column vectors of an \( (n \times n) \) matrix to be linearly independent is that the matrix be non-singular). Also, we know that every non-singular matrix of order \( n \) is row equivalent to a unit matrix \( I_n \) (by elementary row operations), therefore,

\[
H'_2 \sim I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

As the elementary row operations which are applied on \( H'_2 \), would apply on \( H \) and since the parity-check matrices for the same code can be obtained by using elementary row operations, therefore the first three columns of the 2\(^{\text{nd}}\) sub-block of \( H \) can be chosen in the following way without loss of generality.
Suppose fourth column in the 2\textsuperscript{nd} sub-block of \( H \) is \(
\begin{pmatrix}
  x \\
  y \\
  z 
\end{pmatrix}
\). Fifth column can be taken as one of the first three columns. Let this be \((1\ 0\ 0)\). Then the complete 2\textsuperscript{nd} sub-block of \( H \) becomes:

\[
\begin{pmatrix}
  1 & 0 & 0 & x & 1 \\
  0 & 1 & 0 & y & 0 \\
  0 & 0 & 1 & z & 0 \\
\end{pmatrix}
\]

Suppose \( S \) denotes the set of all the non-zero 3-tuples over \( GF(q) \). Then

\[
S = GF(q)^3 \sim \{(0, 0, 0)\}.
\]

\[
|S| = (q^3 - 1).
\]

Let \( A \) denote the set of all the syndromes corresponding to the error pattern of first three columns (i.e. 1\textsuperscript{st}-2\textsuperscript{nd}, 2\textsuperscript{nd}-3\textsuperscript{rd}, including the vectors \((0\ 0\ 1)\), \((0\ 0\ 2)\), ..., \((0\ 0\ q - 1)\)) of second sub-block in \( H \). Then

\[
A = \{(0, 0, 0) \neq (a, b, c) \in GF(q)^3 \mid a = 0 \text{ or } c = 0\}.
\]

Also, if \( B = S \sim A \), then

\[
B = \{(a, b, c) \in GF(q)^3 \mid a \neq 0 \text{ and } c \neq 0\}.
\]

Therefore

\[
|A| = (q - 1) (2q + 1) \text{ and } |B| = q(q - 1)^2.
\]

Clearly, \( S = A \cup B \) and \( A \cap B = \phi \), so that

\[
|S| = |A| + |B|.
\]
Then for $H$ to be the parity check matrix of $((q^2 - 3q + 1) + 5, q^2 - 3q + 3))$ code, each of the $(q-1)$ elements in the set $C$ given by

$$C = \{(ax, ay, az + 1) \mid 0 \neq a \in GF(q)\}$$

should not belong to $A$.

But, there is an element $\left(\frac{q-1}{z}x, \frac{q-1}{z}y, 0\right)$ of $C$ which belongs to $A$,

($\because$ for each $0 \neq z \in GF(q)$, $\frac{q-1}{z}z + 1 = 0$ in $GF(q)$)

which is a contradiction to the above assumption.

Therefore it is not possible to add the fourth column in the second sub-block of $H$ and hence we are unable to find the second sub-block in $H$ which may yield different syndromes in the respective error pattern – syndrome table. Hence the result. \[\Box\]

### 4.2 NON-EXISTENCE OF (8, 4) 2-BURST ERROR CORRECTING PERFECT CODE OVER $GF(2)$

Fire (1959) gave the lower bound on the number of parity check digits required in a linear code over $GF(q)$ that corrects all bursts of length $b$ or less as follows:

$$q^{n-k} \geq q^{b-1}[(q-1)(n-b+1)+1]. \quad (4.1)$$

On the lines of the definition of perfect codes, we may consider a code to be $b$-burst error correcting perfect if it corrects all the burst errors of length $b$ or less and no others. Such a code, if exists, shall satisfy the equation obtained from (4.1) viz.

$$q^{n-k} = q^{b-1}[(q-1)(n-b+1)+1]. \quad (4.2)$$
In this section, we prove, that (8, 4) 2-burst error correcting perfect code does not exist by showing that a parity check matrix cannot be constructed.

To explore the possibility of the existence of burst error correcting perfect codes, it is obvious that equation (4.2) should have integral solutions for \( q, n, k \) and \( b \). Though equation (4.2) is satisfied for \( b = 2 \) and \( q = 2, n = 8 \) and \( k = 4 \), however, we prove that such a code does not exist as it is not possible to form a parity check matrix for this case.

**Result:** (8, 4)-2 burst error correcting code over \( GF(2) \) does not exist.

**Proof:** We prove this result by assuming that such codes exist.

Suppose that \( H \) denotes the parity check matrix for (8, 4)-2 burst error-correcting code,

where \( H = [H_1 H_2]_{4 \times 8} \), \( H_1 \) and \( H_2 \) denote the square sub-matrices of \( H \) each of order 4.

Since required properties of the code ensure that first four columns must be linearly independent, therefore \( |H_1| \neq 0 \) (\( \because \) a necessary and sufficient condition for the Column (or row) vectors of an \((n \times n)\) matrix to be linearly independent is that the matrix be non-singular). Also, we know that every non-singular \((n \times n)\) matrix is row equivalent to \( I_n \) (by elementary row operations), therefore

\[
H_1 \sim I_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

As the elementary row operations which are applied on \( H_1 \) would apply on \( H \) and since parity check matrices for the same code can be obtained by
using elementary row operations, therefore the first four columns of $H$ can be chosen in the following way without loss of generality

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

Suppose we write $H$ as

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & x_1 & y_1 & z_4 & w_1 \\
0 & 1 & 0 & 0 & x_2 & y_2 & z_2 & w_2 \\
0 & 0 & 1 & 0 & x_3 & y_3 & z_3 & w_3 \\
0 & 0 & 0 & 1 & x_4 & y_4 & z_4 & w_4 \\
\end{bmatrix},
\]

where $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)$ and $(w_1, w_2, w_3, w_4)$ belong to $S$, where $S$ denotes the set of all non-zero 4-tuples over $GF(2)$.

\[
|S| = 2^4 - 1 = 15.
\]

Let $A$ denote the set of all the syndromes corresponding to the error pattern of first four columns (i.e. 1st-2nd, 2nd-3rd, 3rd-4th, including the vector $(0 \ 0 \ 0 \ 1)$) of $H$. Then

\[
A = \{(1000), (0100), (0010), (0001)\}.
\]

Also if $B = S - A$, then

\[
B = \{(1010), (0101), (1110), (1101)\}.
\]

Thus $S = A \cup B$ and $A \cap B = \phi$.

Then for $H$ to be the parity check matrix of (8, 4)-2 burst error correcting code, each element in the set $C$ (set of all the syndromes corresponding to the error patterns of 4th-5th, 5th-6th, 6th-7th, 7th-8th, columns

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of \( H \) including the vectors \((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4), (w_1, w_2, w_3, w_4)) \) given by

\[
C = \left\{ \begin{array}{l}
(x_1, x_2, x_3, x_4), \\
(y_1, y_2, y_3, y_4), \\
(z_1, z_2, z_3, z_4), \\
(w_1, w_2, w_3, w_4),
\end{array} \right\}
\]

should not belong to \( A \). So, in order to prove that the above code exists, we must have

\[
C \subseteq B \quad (\because S = A \cup B \text{ and } A \cap B = \emptyset)
\]

Now we show that for all possible eligible values of \((x_1, x_2, x_3, x_4)\) in \( B \) and corresponding suitable choices of \((y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4), (w_1, w_2, w_3, w_4)\) in \( B \), we arrive at a contradiction:

If \((x_1, x_2, x_3, x_4) \in B\), we get that

\[
(x_1, x_2, x_3, x_4) \in B \Rightarrow (x_1, x_2, x_3, x_4 + 1) \in B.
\]

(\(\because\) the presence of 5th column implies that the syndrome of vector having 1 at 5th component and 0 elsewhere and that of the syndrome of vector having 1 at 4th and 5th components (syndrome of burst of length 2) and 0 elsewhere should belong to \( B \))

We have the following possibilities:

(i) \((1 \ 0 \ 1 \ 0) \in B \Rightarrow (1 \ 0 \ 1 \ 1) \in B.\)

(ii) \((0 \ 1 \ 0 \ 1) \in B \Rightarrow (0 \ 1 \ 0 \ 0) \in B \quad \text{but} \quad (0100) \in A \text{ and } A \cap B = \emptyset.\)

(iii) \((1 \ 1 \ 1 \ 0) \in B \Rightarrow (1 \ 1 \ 1 \ 1) \in B.\)

(iv) \((1 \ 1 \ 0 \ 1) \in B \Rightarrow (1 \ 1 \ 0 \ 0) \in B \quad \text{but} \quad (1100) \in A \text{ and } A \cap B = \emptyset.\)

(v) \((1 \ 0 \ 1 \ 1) \in B \Rightarrow (1 \ 0 \ 1 \ 0) \in B.\)
(vi) \((0\ 1\ 1\ 1)\in B \Rightarrow (0\ 1\ 1\ 0)\in B\) but \((0110)\in A\) and \(A\cap B = \emptyset\).

(vii) \((1\ 0\ 0\ 1)\in B \Rightarrow (1\ 0\ 0\ 0)\in B\) but \((1000)\in A\) and \(A\cap B = \emptyset\).

(viii) \((1\ 1\ 1\ 1)\in B \Rightarrow (1\ 1\ 1\ 0)\in B\).

Therefore only possibilities of the 5th column are \((1010), (1110), (1011), (1111)\) and not \((0101), (1101), (0111), (1001)\).

**Case I:** If \((x_1\ x_2\ x_3\ x_4) = (1\ 0\ 1\ 0)\in B\)

then \((x_1\ x_2\ x_3\ x_4 + 1) = (1\ 0\ 1\ 1)\in B\).

Now to add the 6th column, \((y_1\ y_2\ y_3\ y_4)\) should not belong to \(A_1\), where

\[A_1 = A \cup \{(1\ 0\ 1\ 0), (1\ 0\ 1\ 1)\}\]

\[= \{(1000), (0100), (0010), (0001),
\{(1100), (0110), (0011), (1010), (1011)\}\}

and \((y_1\ y_2\ y_3\ y_4)\) should belong to \(B_1\), where

\[B_1 = B \sim \{(1010), (1011)\} = \{(0101), (1110), (1101),
\{(0111), (1001), (1111)\}\}.

If \((y_1\ y_2\ y_3\ y_4)\in B_1\), we get that

\[(y_1\ y_2\ y_3\ y_4)\in B_1 \Rightarrow (x_1 + y_1\ x_2 + y_2\ x_3 + y_3\ x_4 + y_4)\in B_1.

We have the following possibilities:

(i) \((0\ 1\ 0\ 1)\in B_1 \Rightarrow (1\ 1\ 1\ 1)\in B_1\).

(ii) \((1\ 1\ 1\ 0)\in B_1 \Rightarrow (0\ 1\ 0\ 0)\in B_1\) but \((0100)\in A_1\) and \(A_1\cap B_1 = \emptyset\).

(iii) \((1\ 1\ 0\ 1)\in B_1 \Rightarrow (0\ 1\ 1\ 1)\in B_1\).

(iv) \((0\ 1\ 1\ 1)\in B_1 \Rightarrow (1\ 1\ 0\ 1)\in B_1\).

(v) \((1\ 0\ 0\ 1)\in B_1 \Rightarrow (0\ 0\ 1\ 1)\in B_1\) but \((0011)\in A_1\) and \(A_1\cap B_1 = \emptyset\).
(vi) \((1 1 1 1) \in B_1 \Rightarrow (0 1 0 1) \in B_1\).

Therefore only possibilities of \(6^{th}\) column are \((0101), (1101), (0111), (1111)\) and not \((1110), (1001)\).

**Sub-case 1.1:** If \((y_1 \ y_2 \ y_3 \ y_4) = (0101) \in B_1\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (1 1 1 1) \in B_1\).

Now to add the \(7^{th}\) column, \((z_1 \ z_2 \ z_3 \ z_4)\) should not belong to \(A_{1.1}\), where

\[
A_{1.1} = A_1 \cup \{(0101), (1111)\}
\]

\[
= \{(1000), (0100), (0010), (0001), (1100), (0110), (0111), (1010), (1011), (0101), (1111)\}
\]

and \((z_1 \ z_2 \ z_3 \ z_4)\) should belong to \(B_{1.1}\),

where \(B_{1.1} = B_1 \sim \{(0101), (1111)\} = \{(1110), (1101)\} \cup \{(0111), (1001)\}\).

If \((z_1 \ z_2 \ z_3 \ z_4) \in B_{1.1}\), we get that

\((z_1 \ z_2 \ z_3 \ z_4) \in B_{1.1} \Rightarrow (y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) \in B_{1.1}\).

we have the following possibilities :

(i) \((1110) \in B_{1.1} \Rightarrow (1011) \in B_{1.1}\)
(ii) \((1101) \in B_{1.1} \Rightarrow (1000) \in B_{1.1}\)
(iii) \((0111) \in B_{1.1} \Rightarrow (0010) \in B_{1.1}\)
(iv) \((0011) \in B_{1.1} \Rightarrow (1100) \in B_{1.1}\)

but \((1011), (1000), (0010), (1100) \in A_{1.1}\) and \(A_{1.1} \cap B_{1.1} = \phi\).

**Sub-case 1.2:** If \((y_1 \ y_2 \ y_3 \ y_4) = (1 1 0 1) \in B_1\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (0 1 1 1) \in B_1\).
Now to add the 7th column, \((z_1 \ z_2 \ z_3 \ z_4)\) should not belong to \(A_{1,2}\), where

\[ A_{1,2} = A_1 \cup \{(1101), (0111)\} = \{(1000), (0100), (0010), (1100), (0110), (0011), (1010), (1011), (1101), (0111)\} \]

and \((z_1 \ z_2 \ z_3 \ z_4)\) should belong to \(B_{1,2}\),

where \(B_{1,2} = B_1 \sim \{(1101), (0111)\} = \{(0101), (1110), (0101), (1111)\}\).

If \((z_1 \ z_2 \ z_3 \ z_4) \in B_{1,2}\), we get that

\[(z_1 \ z_2 \ z_3 \ z_4) \in B_{1,2} \Rightarrow (y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) \in B_{1,2}.\]

We have the following possibilities:

(i) \((0101) \in B_{1,2} \Rightarrow (1000) \in B_{1,2}\)

(ii) \((1110) \in B_{1,2} \Rightarrow (0011) \in B_{1,2}\) but \((1000), (0011), (0100), (0010) \in A_{1,2}\)

(iii) \((1001) \in B_{1,2} \Rightarrow (0100) \in B_{1,2}\) and \(A_{1,2} \cap B_{1,2} = \emptyset\).

(iv) \((1111) \in B_{1,2} \Rightarrow (0010) \in B_{1,2}\)

**Sub-case 1.3:** If \((y_1 \ y_2 \ y_3 \ y_4) = (0 \ 1 \ 1 \ 1) \in B_1\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (1 \ 1 \ 0 \ 1) \in B_1\).

Now to add the 7th column, \((z_1 \ z_2 \ z_3 \ z_4)\) should not belong to \(A_{1,3}\),

where \(A_{1,3} = A_1 \cup \{(0111), (1101)\} = \{(1000), (0100), (0010), (1100), (0110), (0011), (1010), (1011), (0111), (1101)\}\)

and \((z_1 \ z_2 \ z_3 \ z_4)\) should belong to \(B_{1,3}\),

where \(B_{1,3} = B_1 \sim \{(0111), (1101)\} = \{(0101), (1110), (0101), (1111)\}\).
If \((z_1 z_2 z_3 z_4) \in B_{1,3}\), we get that

\[(z_1 z_2 z_3 z_4) \in B_{1,3} \Rightarrow (y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) \in B_{1,3}.\]

We have the following possibilities:

(i) \((0101) \in B_{1,3} \Rightarrow (0010) \in B_{1,3}\) but \((0010) \in A_{1,3}\) and \(A_{1,3} \cap B_{1,3} = \phi\).

(ii) \((1110) \in B_{1,3} \Rightarrow (1001) \in B_{1,3}\).

(iii) \((1001) \in B_{1,3} \Rightarrow (1110) \in B_{1,3}\).

(iv) \((1111) \in B_{1,3} \Rightarrow (1000) \in B_{1,3}\) but \((1000) \in A_{1,3}\) and \(A_{1,3} \cap B_{1,3} = \phi\).

Therefore only possibilities of 7\(^{th}\) column are \((1110), (1001),\) and not \((0101), (1111)\).

**Sub-case 1.3.1:** If \((z_1 z_2 z_3 z_4) = (1110) \in B_{1,3}\)

then \((y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) = (1001) \in B_{1,3}\).

Now to add the 8\(^{th}\) column, \((w_1 \ w_2 \ w_3 \ w_4)\) should not belong to \(A_{1,3.1}\),

where \(A_{1,3.1} = A_{1,3} \cup \{(1110), (1001)\}\)

\[
= \{(1000), (0100), (0010), (0001), (1100), (0110), (0011),
\( (1010), (1011), (0111), (1101), (1110), (1001)\}\}
\]

and \((w_1 \ w_2 \ w_3 \ w_4)\) should belong to \(B_{1,3.1}\), where

\(B_{1,3.1} = B_{1,3} \sim \{(1110), (1001)\} = \{(0101), (1111)\}\).

If \((w_1 \ w_2 \ w_3 \ w_4) \in B_{1,3.1}\), we get that

\[(w_1 \ w_2 \ w_3 \ w_4) \in B_{1,3.1} \Rightarrow (z_1 + w_1 \ z_2 + w_2 \ z_3 + w_3 \ z_4 + w_4) \in B_{1,3.1}.\]

We have the following possibilities:

(i) \((0101) \in B_{1,3.1} \Rightarrow (1011) \in B_{1,3.1}\) but \((1011), (0001) \in A_{1,3.1}\)

(ii) \((1111) \in B_{1,3.1} \Rightarrow (0001) \in B_{1,3.1}\) and \(A_{1,3.1} \cap B_{1,3.1} = \phi\).
Sub-case 1.3.2: If \((z_1 \ z_2 \ z_3 \ z_4) = (1 \ 0 \ 0 \ 1) \in B_{1.3}\)

then \((y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) = (1 \ 1 \ 1 \ 0) \in B_{1.3}.

Now to add the 8\(^{th}\) column, \((w_1 \ w_2 \ w_3 \ w_4)\) should not belong to \(A_{1.3.2}\),
where \(A_{1.3.2} = A_{1.3} \cup \{(1001), (1110)\} = A_{1.3.1}\).

and \((w_1 \ w_2 \ w_3 \ w_4)\) should belong to \(B_{1.3.2}\),
where \(B_{1.3.2} = B_{1.3} \sim \{(1001), (1110)\} = B_{1.3.1}\).

If \((w_1 \ w_2 \ w_3 \ w_4) \in B_{1.3.2}\), we get that
\[ (w_1 \ w_2 \ w_3 \ w_4) \in B_{1.3.2} \implies (z_1 + w_1 \ z_2 + w_2 \ z_3 + w_3 \ z_4 + w_4) \in B_{1.3.2}. \]

We have the following possibilities:

(i) \((0101, 1100) \in B_{1.3.2} \implies (1100) \in B_{1.3.2}\) but \((1100), (0110) \in A_{1.3.2}\)

(ii) \((1111) \in B_{1.3.2} \implies (0110) \in B_{1.3.2}\) and \(A_{1.3.2} \cap B_{1.3.2} = \phi.\)

Sub-case 1.4 : If \((y_1 \ y_2 \ y_3 \ y_4) = (1 \ 1 \ 1 \ 1) \in B_{1}\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (0 \ 1 \ 0 \ 1) \in B_{1}.

Now to add the 7\(^{th}\) column, \((z_1 \ z_2 \ z_3 \ z_4)\) should not belong to \(A_{1.4}\),
where \(A_{1.4} = A_{1} \cup \{(1111), (0101)\}\)

\[ = \{(1000), (0100), (0010), (0001), (1100), (0110), \}
\[\{(0011), (1010), (1011), (1111), (0101)\}\]

and \((z_1 \ z_2 \ z_3 \ z_4)\) should belong to \(B_{1.4}\),
where \(B_{1.4} = B_{1} \sim \{(1111), (0101)\} = \{(1110), (1101), (0111), (1001)\}.\)

If \((z_1 \ z_2 \ z_3 \ z_4) \in B_{1.4}\), we get that
\[(z_1, z_2, z_3, z_4) \in B_{1.4} \Rightarrow (y_1 + z_1, y_2 + z_2, y_3 + z_3, y_4 + z_4) \in B_{1.4}.\]

We have the following possibilities:

(i) \[(1110) \in B_{1.4} \Rightarrow (0001) \in B_{1.4}\]
(ii) \[(1101) \in B_{1.4} \Rightarrow (0010) \in B_{1.4}\]
(iii) \[(0111) \in B_{1.4} \Rightarrow (1000) \in B_{1.4}\]
(iv) \[(1001) \in B_{1.4} \Rightarrow (0110) \in B_{1.4}\]

Case II: If \((x_1, x_2, x_3, x_4) = (1\ 1\ 1\ 0) \in B\)

then \((x_1, x_2, x_3, x_4 + 1) = (1\ 1\ 1\ 1) \in B.\)

Now to add the 6th column, \((y_1, y_2, y_3, y_4)\) should not belong to \(A_2,\)
where \[A_2 = A \cup \{(1110), \ (1111)\} = \{(1000), \ (0100), \ (0010), \ (0001), \ (1100), \ (0110), \ (0011), \ (1110), \ (1111)\}{\}

and \((y_1, y_2, y_3, y_4)\) should belong to \(B_2,\) where

\[B_2 = B - \{(1110), \ (1111)\} = \{(1010), \ (0101), \ (1101), \ (1011), \ (0111), \ (1001)\}{\}

If \((y_1, y_2, y_3, y_4) \in B_2,\) we get that

\[(y_1, y_2, y_3, y_4) \in B_2 \Rightarrow (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \in B_2.\]

We have the following possibilities:

(i) \[(1\ 0\ 1\ 0) \in B_2 \Rightarrow (0\ 1\ 0\ 0) \in B_2\] but \((0100) \in A_2\) and \(A_2 \cap B_2 = \phi.\)
(ii) \[(0\ 1\ 0\ 1) \in B_2 \Rightarrow (1\ 0\ 1\ 1) \in B_2.\]
(iii) \[(1\ 1\ 0\ 1) \in B_2 \Rightarrow (0\ 0\ 1\ 1) \in B_2\] but \((0011) \in A_2\) and \(A_2 \cap B_2 = \phi.\)
(iv) \[(1\ 0\ 1\ 1) \in B_2 \Rightarrow (0\ 1\ 0\ 1) \in B_2.\]
(v) \[(0\ 1\ 1\ 1) \in B_2 \Rightarrow (1\ 0\ 0\ 1) \in B_2.\]
(vi) \[(1\ 0\ 0\ 1) \in B_2 \Rightarrow (0\ 1\ 1\ 1) \in B_2.\]
Therefore only possibilities of 6th column are (0101), (1011), (0111), (1001) and not (1010), (1101).

**Sub-case 2.1:** If \((y_1 \ y_2 \ y_3 \ y_4) = (0101) \in B_2\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (1011) \in B_2\).

Now to add the 7th column, \((z_1 \ z_2 \ z_3 \ z_4)\) should not belong to \(A_{2.1}\),

where

\[ A_{2.1} = A_2 \cup \{(0101), (1011)\} \]

\[ = \{(0000), (0100), (0010), (0001), (1100), (0110), (0011), (1110), (1111), (0101), (1011)\} \]

and \((z_1 \ z_2 \ z_3 \ z_4)\) should belong to \(B_{2.1}\),

where \(B_{2.1} = B_2 \sim \{(0101), (1011)\} = \{(1010), (1101), (0111), (0011)\}\).

If \((z_1 \ z_2 \ z_3 \ z_4) \in B_{2.1}\), we get that

\[(z_1 \ z_2 \ z_3 \ z_4) \in B_{2.1} \Rightarrow (y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) \in B_{2.1}\].

We have the following possibilities :

(i) \((1010) \in B_{2.1} \Rightarrow (1111) \in B_{2.1}\)

(ii) \((1101) \in B_{2.1} \Rightarrow (1000) \in B_{2.1}\) but \((1111), (1000), (0010), (1100) \in A_{2.1}\)

(iii) \((0111) \in B_{2.1} \Rightarrow (0010) \in B_{2.1}\)

(iv) \((1001) \in B_{2.1} \Rightarrow (1100) \in B_{2.1}\) and \(A_{2.1} \cap B_{2.1} = \emptyset\).

**Sub-case 2.2:** If \((y_1 \ y_2 \ y_3 \ y_4) = (1 \ 0 \ 1 \ 1) \in B_2\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (0101) \in B_2\).

Now to add the 7th column, \((z_1 \ z_2 \ z_3 \ z_4)\) should not belong to \(A_{2.2}\),

where \(A_{2.2} = A_2 \cup \{(1111), (0101)\}\)

\[ = \{(1000), (0100), (0010), (0001), (1100), (0110), (0011), (1110), (1111), (0101)\} \]
and \((z_1 \ z_2 \ z_3 \ z_4)\) should belong to \(B_{2,2}\), where

\[
B_{2,2} = B_2 \setminus \{(1011), (0101)\} = \{(1010), (1101), (0111), (1001)\}.
\]

If \((z_1 \ z_2 \ z_3 \ z_4) \in B_{2,2}\), we get that

\[
(z_1 \ z_2 \ z_3 \ z_4) \in B_{2,2} \Rightarrow (y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) \in B_{2,2}.
\]

We have the following possibilities:

(i) \((1010) \in B_{2,2} \Rightarrow (0001) \in B_{2,2}\)

(ii) \((1101) \in B_{2,2} \Rightarrow (0110) \in B_{2,2}\)

(iii) \((0111) \in B_{2,2} \Rightarrow (1100) \in B_{2,2}\)

(iv) \((1001) \in B_{2,2} \Rightarrow (0010) \in B_{2,2}\)

Sub-case 2.3: If \((y_1 \ y_2 \ y_3 \ y_4) = (0111) \in B_2\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (1 \ 0 \ 0 \ 1) \in B_2\).

Now to add the 7th column, \((z_1 \ z_2 \ z_3 \ z_4)\) should not belong to \(A_{2,3}\),
where \(A_{2,3} = A_2 \cup \{(0111), (1001)\}\)

\[
= \{ (1000), (0100), (0010), (0001), (1100), (0110), (0011), (1110), (1111), (0111), (1001) \}
\]

and \((z_1 \ z_2 \ z_3 \ z_4)\) should belong to \(B_{2,3}\), where

\[
B_{2,3} = B_2 \setminus \{(0111), (1001)\} = \{(1010), (0101), (1101), (1011)\}.
\]

If \((z_1 \ z_2 \ z_3 \ z_4) \in B_{2,3}\), we get that

\[
(z_1 \ z_2 \ z_3 \ z_4) \in B_{2,3} \Rightarrow (y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) \in B_{2,3}.
\]

We have the following possibilities:

(i) \((1 \ 0 \ 1 \ 0) \in B_{2,3} \Rightarrow (1 \ 1 \ 0 \ 1) \in B_{2,3} \).
(ii) \((0 \ 1 \ 0 \ 1) \in B_{2,3} \Rightarrow (0 \ 0 \ 1 \ 0) \in B_{2,3}\) but \((0010) \in A_{2,3} \text{ and } A_{2,3} \cap B_{2,3} = \phi\).

(iii) \((1 \ 1 \ 0 \ 1) \in B_{2,3} \Rightarrow (1 \ 0 \ 1 \ 0) \in B_{2,3} .

(iv) \((1 \ 0 \ 1 \ 1) \in B_{2,3} \Rightarrow (1 \ 1 \ 0 \ 0) \in B_{2,3}\) but \((1100) \in A_{2,3} \text{ and } A_{2,3} \cap B_{2,3} = \phi\).

Therefore only possibilities of 7th column are \((1010), (1101) \text{ and not (0101), (1011)}.

Sub-case 2.3.1: If \((z_1 \ z_2 \ z_3 \ z_4) = (1 \ 0 \ 1 \ 0) \in B_{2,3}\)

then \((y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) = (1 \ 1 \ 0 \ 1) \in B_{2,3} .

Now to add the 8th column, \((w_1 \ w_2 \ w_3 \ w_4)\) should not belong to \(A_{2,3.1}\), where

\[ A_{2,3.1} = A_{2,3} \cup \{(1010), (1101)\}

and \((w_1 \ w_2 \ w_3 \ w_4)\) should belong to \(B_{2,3.1}\), where

\[ B_{2,3.1} = B_{2,3} \sim \{(1010), (1101)\} = \{(0101), (1011)\} .

If \((w_1 \ w_2 \ w_3 \ w_4) \in B_{2,3.1},\) we get that

\( (w_1 \ w_2 \ w_3 \ w_4) \in B_{2,3.1} \Rightarrow (z_1 + w_1 \ z_2 + w_2 \ z_3 + w_3 \ z_4 + w_4) \in B_{2,3.1} \)

We have the following possibilities:

(i) \((0101) \in B_{2,3.1} \Rightarrow (1111) \in B_{2,3.1}\) but \((1111), (0001) \in A_{2,3.1}\)

(ii) \((1011) \in B_{2,3.1} \Rightarrow (0001) \in B_{2,3.1}\) and \(A_{2,3.1} \cap B_{2,3.1} = \phi\).

Sub-case 2.3.2: If \((z_1 \ z_2 \ z_3 \ z_4) = (1101) \in B_{2,3}\)

then \((y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) = (1010) \in B_{2,3}\).

Now to add the 8th column, \((w_1 \ w_2 \ w_3 \ w_4)\) should not belong to \(A_{2,3.2}\), where

\[ A_{2,3.2} = A_{2,3} \cup \{(1101), (1010)\} = A_{2,3.1}\) and \((w_1 \ w_2 \ w_3 \ w_4)\) should belong to \(B_{2,3.2}\), where
\[ B_{2.3.2} = B_{2.3} \sim \{(1101), (1010)\} = B_{2.3.1}. \]

If \((w_1 \ w_2 \ w_3 \ w_4) \in B_{2.3.2}\) we get that
\[
(w_1 \ w_2 \ w_3 \ w_4) \in B_{2.3.2} \Rightarrow (z_1 + w_1 \ z_2 + w_2 \ z_3 + w_3 \ z_4 + w_4) \in B_{2.3.2}.
\]

We have the following possibilities:

(i) \((0101) \in B_{2.3.2} \Rightarrow (1000) \in B_{2.3.2}\) but \((1000), (0111) \in A_{2.3.2}\)

(ii) \((1011) \in B_{2.3.2} \Rightarrow (0110) \in B_{2.3.2}\) and \(A_{2.3.2} \cap B_{2.3.2} = \phi\).

Sub-case 2.4: If \((y_1 \ y_2 \ y_3 \ y_4) = (1 \ 0 \ 0 \ 1) \in B_2\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (0 \ 1 \ 1 \ 1) \in B_2.\)

Now to add the 7th column, \((z_1 \ z_2 \ z_3 \ z_4)\) should not belong to \(A_{2.4}\), where
\[ A_{2.4} = A_2 \cup \{(1001), (0111)\}\]
\[ = \{(1000), (0100), (0010), (0001),(1100), (0110),
(0011), (1110), (1111), (1001), (0111)\}\]

and \((z_1 \ z_2 \ z_3 \ z_4)\) should belong to \(B_{2.4}\), where
\[ B_{2.4} = B_2 \sim \{(1001), (0111)\} = \{(1010), (0101),
(1101), (1011)\}.\]

If \((z_1 \ z_2 \ z_3 \ z_4) \in B_{2.4}\), we get that
\[
(z_1 \ z_2 \ z_3 \ z_4) \in B_{2.4} \Rightarrow (y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) \in B_{2.4}.
\]

We have the following possibilities:

(i) \((1010) \in B_{2.4} \Rightarrow (0011) \in B_{2.4}\)

(ii) \((0101) \in B_{2.4} \Rightarrow (1100) \in B_{2.4}\) but \((0011), (1100), (0100), (0010) \in A_{2.4}\)

(iii) \((1101) \in B_{2.4} \Rightarrow (0100) \in B_{2.4}\)

(iv) \((1011) \in B_{2.4} \Rightarrow (0010) \in B_{2.4}\) and \(A_{2.4} \cap B_{2.4} = \phi\).
**Case III:** If \((x_1 \ x_2 \ x_3 \ x_4) = (1 \ 0 \ 1 \ 1) \in B\)

then \((x_1 \ x_2 \ x_3 \ x_4+1) = (1 \ 0 \ 1 \ 0) \in B\).

Now to add the 6th column, \((y_1 \ y_2 \ y_3 \ y_4)\) should not belong to \(A_3\), where

\[
A_3 = A \cup \{(1011), \ (1010)\} = \left\{(1000), \ (0100), \ (0010), \ (0001), \ (1100), \ (0110), \ (0011), \ (1011), \ (1010)\right\}
\]

and \((y_1 \ y_2 \ y_3 \ y_4)\) should belong to \(B_3\), where

\[
B_3 = B \sim \{(1011), \ (1010)\} = \left\{(0101), \ (1110), \ (1101)\right\} \cup \left\{(0111), \ (1001), \ (1111)\right\}
\]

If \((y_1 \ y_2 \ y_3 \ y_4) \in B_3\), we get that

\[(y_1 \ y_2 \ y_3 \ y_4) \in B_3 \Rightarrow (x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) \in B_3\]

We have the following possibilities:

(i) \((0 \ 1 \ 0 \ 1) \in B_3 \Rightarrow (1 \ 1 \ 1 \ 0) \in B_3\).

(ii) \((1 \ 1 \ 1 \ 0) \in B_3 \Rightarrow (0 \ 1 \ 0 \ 1) \in B_3\).

(iii) \((1 \ 1 \ 0 \ 1) \in B_3 \Rightarrow (0 \ 1 \ 1 \ 0) \in B_3\) but \((0110) \in A_3\) and \(A_3 \cap B_3 = \emptyset\).

(iv) \((0 \ 1 \ 1 \ 1) \in B_3 \Rightarrow (1 \ 1 \ 0 \ 0) \in B_3\) but \((1100) \in A_3\) and \(A_3 \cap B_3 = \emptyset\).

(v) \((1 \ 0 \ 0 \ 1) \in B_3 \Rightarrow (0 \ 0 \ 1 \ 0) \in B_3\) but \((0010) \in A_3\) and \(A_3 \cap B_3 = \emptyset\).

(vi) \((1 \ 1 \ 1 \ 1) \in B_3 \Rightarrow (0 \ 1 \ 0 \ 0) \in B_3\) but \((0100) \in A_3\) and \(A_3 \cap B_3 = \emptyset\).

Therefore only possibilities of 6th column are \((0101), \ (1110)\) and not \((1101), \ (0111), \ (1001), \ (1111)\).

**Sub-case 3.1:** If \((y_1 \ y_2 \ y_3 \ y_4) = (0 \ 1 \ 0 \ 1) \in B_3\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (1 \ 1 \ 1 \ 0) \in B_3\).

Now to add the 7th column, \((z_1 \ z_2 \ z_3 \ z_4)\) should not belong to \(A_{3.1}\), where

\[
A_{3.1} = A_3 \cup \{(0101), \ (1110)\}
\]

\[
= \left\{(1000), \ (0100), \ (0010), \ (0001), \ (1100), \ (0110), \ (0011), \ (1011), \ (1010), \ (0101), \ (1110)\right\}
\]
and \((z_1 z_2 z_3 z_4)\) should belong to \(B_{3,1}\), where

\[
B_{3,1} = B_3 \sim \{(0101), (1110)\} = \{(1101), (1001), (0111), (1111)\}.
\]

If \((z_1 z_2 z_3 z_4) \in B_{3,1}\), we get that

\[
(z_1 z_2 z_3 z_4) \in B_{3,1} \Rightarrow (y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) \in B_{3,1}.
\]

We have the following possibilities:

(i) \((1\ 1\ 0\ 1) \in B_{3,1} \Rightarrow (1\ 0\ 0\ 0) \in B_{3,1}\)
(ii) \((1\ 0\ 0\ 1) \in B_{3,1} \Rightarrow (1\ 1\ 0\ 0) \in B_{3,1}\)
(iii) \((0\ 1\ 1\ 1) \in B_{3,1} \Rightarrow (0\ 0\ 0\ 1) \in B_{3,1}\)
(iv) \((1\ 1\ 1\ 1) \in B_{3,1} \Rightarrow (1\ 0\ 1\ 0) \in B_{3,1}\)

Sub-case 3.2: If \((y_1 y_2 y_3 y_4) = (1\ 1\ 1\ 0) \in B_3\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (0\ 1\ 0\ 1) \in B_3\).

Now to add the 7th column, \((z_1 z_2 z_3 z_4)\) should not belong to \(A_{3,2}\),
where \(A_{3,2} = A_3 \cup \{(1110), (0101)\}\)

\[
= \{(1000), (1010), (0010), (0001), (1100), (0110), (0011), (1011), (1010), (1110), (0101)\}
\]

and \((z_1 z_2 z_3 z_4)\) should belong to \(B_{3,2}\), where

\[
B_{3,2} = B_3 \sim \{(1110), (0101)\} = \{(1101), (1001), (0111), (1111)\}.
\]

If \((z_1 z_2 z_3 z_4) \in B_{3,2}\), we get that

\[
(z_1 z_2 z_3 z_4) \in B_{3,2} \Rightarrow (y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) \in B_{3,2}.
\]
We have the following possibilities:

(i) \((1 \ 1 \ 0 \ 1) \in B_{3.2} \Rightarrow (0 \ 0 \ 1 \ 1) \in B_{3.2}\) but \((0011) \notin A_{3.2}\) and \(A_{3.2} \cap B_{3.2} = \phi\).

(ii) \((1 \ 0 \ 0 \ 1) \in B_{3.2} \Rightarrow (0 \ 1 \ 1 \ 1) \in B_{3.2}\).

(iii) \((0 \ 1 \ 1 \ 1) \in B_{3.2} \Rightarrow (1 \ 0 \ 0 \ 1) \in B_{3.2}\).

(iv) \((1 \ 1 \ 1 \ 1) \in B_{3.2} \Rightarrow (0 \ 0 \ 0 \ 1) \in B_{3.2}\) but \((0001) \notin A_{3.2}\) and \(A_{3.2} \cap B_{3.2} = \phi\).

Therefore only possibilities of 7th column are \((1001), (0111)\) and not \((1101), (1111)\).

Sub-case 3.2.1: If \((z_1 \ z_2 \ z_3 \ z_4) = (1 \ 0 \ 0 \ 1) \in B_{3.2}\)

then \((y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) = (0 \ 1 \ 1 \ 1) \in B_{3.2}\).

Now to add the 8th column, \((w_1 \ w_2 \ w_3 \ w_4)\) should not belong to \(A_{3.2.1}\), where

\[
A_{3.2.1} = A_{3.2} \cup \{(1001), (0111)\}
\]

= \{(1000), (0100), (0010), (0001), (1100), (0110), (0011), (1011)\}

and \((w_1 \ w_2 \ w_3 \ w_4)\) should belong to \(B_{3.2.1}\), where

\[
B_{3.2.1} = B_{3.2} \sim \{(1001), (0111)\} = \{(1101), (1111)\}.
\]

If \((w_1 \ w_2 \ w_3 \ w_4) \in B_{3.2.1}\), we get that

\((w_1 \ w_2 \ w_3 \ w_4) \in B_{3.2.1} \Rightarrow (z_1 + w_1 \ z_2 + w_2 \ z_3 + w_3 \ z_4 + w_4) \in B_{3.2.1}\).

We have the following possibilities:

(i) \((1101) \in B_{3.2.1} \Rightarrow (0100) \in B_{3.2.1}\) but \((0100), (0110) \notin A_{3.2.1}\)

(ii) \((1111) \in B_{3.2.1} \Rightarrow (0110) \in B_{3.2.1}\) and \(A_{3.2.1} \cap B_{3.2.1} = \phi\).

Sub-case 3.2.2: If \((z_1 \ z_2 \ z_3 \ z_4) = (0 \ 1 \ 1 \ 1) \in B_{3.2}\)

then \((y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) = (1 \ 0 \ 0 \ 1) \in B_{3.2}\).
Now to add the 8\textsuperscript{th} column, \((w_1 \ w_2 \ w_3 \ w_4)\) should not belong to \(A_{3.2.2}\), where \(A_{3.2.2} = A_{3.2} \cup \{(0111), (1001)\} = A_{3.2.1}\) and \((w_1 \ w_2 \ w_3 \ w_4)\) should belong to \(B_{3.2.2}\), where \(B_{3.2.2} = B_{3.2} \sim \{(0111), (1001)\} = B_{3.2.1}\).

If \((w_1 \ w_2 \ w_3 \ w_4) \in B_{3.2.2}\) we get that
\[(w_1 \ w_2 \ w_3 \ w_4) \in B_{3.2.2} \Rightarrow (z_1 + w_1 \ z_2 + w_2 \ z_3 + w_3 \ z_4 + w_4) \in B_{3.2.2}.
\]

We have the following possibilities:

(i) \((1101) \in B_{3.2.2} \Rightarrow (1010) \in B_{3.2.2}\) but \((1010), (1000) \in A_{3.2.2}\)

(ii) \((1111) \in B_{3.2.2} \Rightarrow (1000) \in B_{3.2.2}\) and \(A_{3.2.2} \cap B_{3.2.2} = \phi\).

**Case IV:** If \((x_1 \ x_2 \ x_3 \ x_4) = (1 \ 1 \ 1 \ 1) \in B\)

then \((x_1 \ x_2 \ x_3 \ x_4 + 1) = (1 \ 1 \ 1 \ 0) \in B\).

Now to add the 6\textsuperscript{th} column, \((y_1 \ y_2 \ y_3 \ y_4)\) should not belong to \(A_4\), where

\[A_4 = A \cup \{(1111), (1110)\} = \left\{(1000), (0100), (0010), (0001),\right\}
\[(1100), (0110), (0011), (1111), (1110)\}\]

and \((y_1 \ y_2 \ y_3 \ y_4)\) should belong to \(B_4\), where

\[B_4 = B \sim \{(1111), (1110)\} = \left\{(1010), (0101), (1101),\right\}
\[(1011), (0111), (0010)\}\].

If \((y_1 \ y_2 \ y_3 \ y_4) \in B_4\), we get that
\[(y_1 \ y_2 \ y_3 \ y_4) \in B_4 \Rightarrow (x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) \in B_4\.
\]

We have the following possibilities:

(i) \((1010) \in B_4 \Rightarrow (0101) \in B_4\).

(ii) \((0101) \in B_4 \Rightarrow (1010) \in B_4\).

(iii) \((1101) \in B_4 \Rightarrow (0010) \in B_4\) but \((0010) \in A_4\) and \(A_4 \cap B_4 = \phi\).
(iv) \((1011) \in B_4 \Rightarrow (0100) \in B_4\) but \((0100) \in A_4\) and \(A_4 \cap B_4 = \emptyset\).

(v) \((0111) \in B_4 \Rightarrow (1000) \in B_4\) but \((1000) \in A_4\) and \(A_4 \cap B_4 = \emptyset\).

(vi) \((1001) \in B_4 \Rightarrow (0110) \in B_4\) but \((0110) \in A_4\) and \(A_4 \cap B_4 = \emptyset\).

Therefore only possibilities of 6th column are \((1010), (0101)\) and not \((1101), (1011), (0111), (1001)\).

**Sub-case 4.1:** If \((y_1 y_2 y_3 y_4) = (1 \ 0 \ 1 \ 0) \in B_4\)

then \((x_1 + y_1 \ x_2 + y_2 \ x_3 + y_3 \ x_4 + y_4) = (0 \ 1 \ 0 \ 1) \in B_4\).

Now to add the 7th column, \((z_1 z_2 z_3 z_4)\) should not belong to \(A_{4.1}\), where
\[
A_{4.1} = A_4 \cup \{ (1010), (0101) \}
\]
= \[
\{ (1000), (0100), (0010), (0001), (1100), (0110), (0101), (1110), (1010), (0111) \}
\]
and \((z_1 z_2 z_3 z_4)\) should belong to \(B_{4.1}\), where
\[
B_{4.1} = B_4 \sim \{ (1010), (0101) \} = \{ (1101) , (0111) \}.
\]

If \((z_1 z_2 z_3 z_4) \in B_{4.1}\) we get that
\[(z_1 z_2 z_3 z_4) \in B_{4.1} \Rightarrow (y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) \in B_{4.1}\).

We have the following possibilities:

(i) \((1101) \in B_{4.1} \Rightarrow (0111) \in B_{4.1}\).

(ii) \((1011) \in B_{4.1} \Rightarrow (0001) \in B_{4.1}\) but \((0001) \in A_{4.1}\) and \(A_{4.1} \cap B_{4.1} = \emptyset\).

(iii) \((0111) \in B_{4.1} \Rightarrow (1101) \in B_{4.1}\).

(iv) \((1001) \in B_{4.1} \Rightarrow (0011) \in B_{4.1}\) but \((0011) \in A_{4.1}\) and \(A_{4.1} \cap B_{4.1} = \emptyset\).

Therefore only possibilities of 7th column are \((1101), (0111)\) and not \((1011), (1001)\).
**Sub-case 4.1.1** : If \((z_1 \ z_2 \ z_3 \ z_4) = (1 \ 1 \ 0 \ 1) \in B_{4.1}\)

then \((y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) = (0 \ 1 \ 1 \ 1) \in B_{4.1}.\)

Now to add the 8th column, \((w_1 \ w_2 \ w_3 \ w_4)\) should not belong to \(A_{4.1.1}\), where

\[
A_{4.1.1} = A_{4.1} \cup \{(1101), (0111)\} = \left\{(0000), (0001), (0010), (0011), (0100), (0101), (0110), (0111), (1000), (1001), (1100), (1101), (1110), (1111)\right\}
\]

and \((w_1 \ w_2 \ w_3 \ w_4)\) should belong to \(B_{4.1.1}\), where

\[
B_{4.1.1} = B_{4.1} - \{(0111), (1101)\} = \{
(1011), (1001)
\}.
\]

If \((w_1 \ w_2 \ w_3 \ w_4) \in B_{4.1.1}\), we get that

\[
(w_1 \ w_2 \ w_3 \ w_4) \in B_{4.1.1} \Rightarrow (z_1 + w_1 \ z_2 + w_2 \ z_3 + w_3 \ z_4 + w_4) \in B_{4.1.1}.
\]

We have the following possibilities :

(i) \((1 \ 0 \ 1 \ 1) \in B_{4.1.1} \Rightarrow (0 \ 1 \ 1 \ 0) \in B_{4.1.1}\) but \((0110), (0100) \in A_{4.1.1}\) and \(A_{4.1.1} \cap B_{4.1.1} = \emptyset\).

(ii) \((1 \ 0 \ 0 \ 1) \in B_{4.1.1} \Rightarrow (0 \ 1 \ 0 \ 0) \in B_{4.1.1}\)

**Sub-case 4.1.2** : If \((z_1 \ z_2 \ z_3 \ z_4) = (0 \ 1 \ 1 \ 1) \in B_{4.1}\)

then \((y_1 + z_1 \ y_2 + z_2 \ y_3 + z_3 \ y_4 + z_4) = (1 \ 1 \ 0 \ 1) \in B_{4.1}.\)

Now to add the 8th column, \((w_1 \ w_2 \ w_3 \ w_4)\) should not belong to \(A_{4.1.2}\), where

\[
A_{4.1.2} = A_{4.1} \cup \{(0111), (1101)\} = A_{4.1.1} \text{ and (}w_1 \ w_2 \ w_3 \ w_4\) should belong to \(B_{4.1.2}\), where \(B_{4.1.2} = B_{4.1} - \{(0111), (1101)\} = B_{4.1.1}\).
\]

If \((w_1 \ w_2 \ w_3 \ w_4) \in B_{4.1.2}\) we get that

\[
(w_1 \ w_2 \ w_3 \ w_4) \in B_{4.1.2} \Rightarrow (z_1 + w_1 \ z_2 + w_2 \ z_3 + w_3 \ z_4 + w_4) \in B_{4.1.2}.
\]

We have the following possibilities :
Sub-case 4.2: If \((y_1, y_2, y_3, y_4) = (0 1 0 1) \in B_4\)

then \((x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) = (1 0 1 0) \in B_4\).

Now to add the 7th column, \((z_1, z_2, z_3, z_4)\) should not belong to \(A_{4.2}\), where

\[
A_{4.2} = A_4 \cup \{(0101), (1010)\} \subseteq \{(0000), (0100), (0010), (1100), (0110), (1001), (1011), (0111), (1110), (1111)\}
\]

and \((z_1, z_2, z_3, z_4)\) should belong to \(B_{4.2}\),

where \(B_{4.2} = B_4 \sim \{(0101), (1010)\} = \{(1101), (1011), (0111), (1001)\}\).

If \((z_1, z_2, z_3, z_4) \in B_{4.2}\), we get that

\[(z_1, z_2, z_3, z_4) \in B_{4.2} \Rightarrow (y_1 + z_1, y_2 + z_2, y_3 + z_3, y_4 + z_4) \in B_{4.2}\].

We have the following possibilities:

(i) \((1 1 0 1) \in B_{4.2} \Rightarrow (1 0 0 0) \in B_{4.2}\)

(ii) \((1 0 1 1) \in B_{4.2} \Rightarrow (1 1 1 0) \in B_{4.2}\)

(iii) \((0 1 1 1) \in B_{4.2} \Rightarrow (0 0 1 0) \in B_{4.2}\)

(iv) \((1 0 0 1) \in B_{4.2} \Rightarrow (1 1 0 0) \in B_{4.2}\)

Therefore for all the possible values of \((x_1, x_2, x_3, x_4)\) in \(B\) and corresponding eligible values of \((y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4), (w_1, w_2, w_3, w_4)\) in \(B\), we have a contradiction in each case and hence it is not possible to construct parity check matrix \(H\) for the desired code.

Hence the result.