APPENDIX
Acceptance of the paper

Dear Dr. R.K. Jain and Shyam Patkar

Your paper entitled Common fixed point theorems for certain mappings has been accepted for publication in "Ultra Scientist of Physical Sciences" Vol. 8 (2) 1996 Dec 96

Faithfully yours

(Dr. A. H. Ansari)
Chief Editor

Dated 1-6-96

Place:- BHOPAL (India)
Dear Dr. Shyam Patkar,

Your paper entitled "A Common Fixed Point Theorem" has been accepted for publication in "Ultra Scientist of Physical Sciences" Vol. 8 (2) 1996.

Faithfully yours

Dated: 15/1/96

Place: BHOPAL (India)

(Dr. A. H. Ansari)
Chief Editor
REPRINT

ULTRA SCIENTIST OF PHYSICAL SCIENCES
Post Box - 93
City G.P.O. BHOPAL - 462 001 (India)

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Common fixed point theorems for certain expansion mappings

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(Acceptance Date 1 June 96)

Abstract

In this paper we obtain some common fixed point theorems for three expansion mappings which extend some results of Popa. 4

Introduction

Let $R_+$ be the set of all non-negative real numbers and let $\psi : R^3_+ \rightarrow R_+$ be a real valued function.

Definition: A real valued function $\psi : R^3_+ \rightarrow R_+$ satisfies property $(U)$ if $\psi (u,0,0) > u, \forall u \neq 0$.

In a recent paper, Popa 4 improved some fixed point theorems 1, 3, 5 for expansion mappings. His first result is as follows:

Theorem A: Let $(x,d)$ be a metric space and $f$ a self mapping of satisfying the inequality:

$$d(fx, fy) \geq \psi (d(x, y), d(x, fx), d(y, fy))$$

for all $x, y$ in $X$ with $x \neq y$, where $\psi$ satisfies property $(U)$. If $f$ has a fixed point $z$, then $z$ is unique fixed point for $f$.

Extending the results of Popa for three surjective mappings, we prove the following:

Theorem 1.: Let $(X,d)$ be a metric space and $f$, $g$ and $h$ be self mapping of $X$ satisfying the inequality:

$$(\text{ii}) \quad d(fhx, ghy) \geq \psi [d(hx, hy), d(hx, fhx), d(hy, ghy)]$$

for all $x, y \in X$ with $x \neq y$ where $\psi$ satisfies property $(U)$. If $f$, $g$ and $h$ have a common fixed point $z$ then $z$ is unique.

Proof: Suppose that $f$, $g$ and $h$ have a second fixed point $z' (\neq z)$. Then from (ii), we have

$$d(z, z') = d(fhz, ghz') \geq \psi [d(hz, hz'), d(hz, fhz), d(hz', ghz')]$$

$$= \psi [d(hz, hz'), 0, 0]$$

$$= \psi [d(z, z'), 0, 0] > d(z, z')$$

which implies $z = z'$ and this completes the proof.

Theorem 2.: Let $f$, $g$ and $h$ be surjective self-maps of a complete metric space $(X,d)$. If there exists non-negative reals $a_1$, $a_2$, $a_3$ with $2a_1 + a_2 + a_3 > 3$ such that

$$(\text{iii}) \quad d(fhx, ghy) \geq a_1 [d(hx, fhx), d(hx, hy), d(hy, ghy)]$$

$$+ a_2 d(hx, fhx) (d(hy, ghy) + a_3 d(fhx, hy)$$

$$+ d(hy, ghy) + d(hx, hy)$$
for each \( x \neq y \) in \( X \) for which \( d(hx, fhx) + d(hy, ghy) + d(hx, hy) \neq 0 \),

(iv) \( f \) and \( g \) are continuous mappings and \( h \) is orbitally continuous mapping,

(v) \( h \) commutes with \( f \) and \( g \),

then \( f, g \) and \( h \) have a common fixed point. Further if \( a_2 > 1 \), then the fixed point is unique.

**Proof:** Let \( hx_0 \in X \). Since \( f \) is surjective there exists a point \( hx_1 \in f^{-1} hx_0 \). Since \( g \) is surjective there exists a point \( hx_2 \in f^{-1} hx_1 \). Continuing in this manner, we obtain a sequence \( (hx_n) \) with \( hx_{2n+1} \in f^{-1} \) \( hx_{2n} \) and \( hx_{2n+2} \in g^{-1} hx_{2n+1} \). Assume \( hx_{2n} = hx_{2n+1} \) for some \( n \). If \( hx_{2n+1} \neq hx_{2n+2} \) then applying (iii), we have

\[
d(hx_{2n}, hx_{2n+1}) = d(fhx_{2n+1}, ghx_{2n+2}) \\
\geq a_1 [d(hx_{2n+1}, hx_{2n+1}) + d(hx_{2n+2}, hx_{2n+1})] \]

\[
+ a_2 [d(hx_{2n+1}, hx_{2n})d(hx_{2n+2}, hx_{2n+1})] \]

\[
= d(hx_{2n+1}, hx_{2n}) + d(hx_{2n+2}, hx_{2n+1}) \\
+ a_2 d^2(hx_{2n+1}, hx_{2n+2}) + d(hx_{2n+1}, hx_{2n+2})
\]

Thus, \( 0 \geq (a_1 + a_2) d(hx_{2n+2}, hx_{2n+1}) / 2 \)

which implies \( hx_{2n+1} = hx_{2n+2} \) and \( hx_{2n} \) is a common fixed point of \( f \) and \( g \). Similarly \( hx_{2n+1} = hx_{2n+2} \) for some \( n \) leads to \( hx_{2n+1} \) being a common fixed point of \( f \) and \( g \).

Suppose \( hx_n \neq hx_{n+1} \) for each \( n \). Then by (iii), we have

\[
d(hx_{2n}, hx_{2n+1}) = d(fhx_{2n+1}, ghx_{2n+2}) \\
\geq d(hx_{2n+1}, hx_{2n}) + d(hx_{2n+2}, hx_{2n+1}) \\
+ a_2 [d(hx_{2n+1}, hx_{2n})d(hx_{2n+2}, hx_{2n+1})] \\
+ a_2 d^2(hx_{2n+1}, hx_{2n+2}) \\
+ d(hx_{2n+1}, hx_{2n+2})
\]

\[
+ a_2 d^2(hx_{2n+1}, hx_{2n+2}) \\
+ d(hx_{2n+1}, hx_{2n+2})
\]

or,

\[
d^2(hx_{2n+1}, hx_{2n}) + 2d(hx_{2n+2}, hx_{2n+1}) \\
+ a_2 [d(hx_{2n+1}, hx_{2n})d(hx_{2n+2}, hx_{2n+1})] \\
+ a_2 d^2(hx_{2n+1}, hx_{2n+2}) \\
+ d(hx_{2n+1}, hx_{2n+2})
\]

or,

\[
d^2(hx_{2n+1}, hx_{2n}) + a_1 [d(hx_{2n+1}, hx_{2n+1}) + d^2(hx_{2n+2}, hx_{2n+1})] \\
+ a_2 [d(hx_{2n+1}, hx_{2n})d(hx_{2n+2}, hx_{2n+1})] \\
+ a_2 d^2(hx_{2n+1}, hx_{2n+2}) + d(hx_{2n+1}, hx_{2n+2})
\]

or,

\[
d^2(hx_{2n+1}, hx_{2n}) + 1 + 2 \frac{d(hx_{2n+2}, hx_{2n+1})}{d(hx_{2n+1}, hx_{2n})} \geq a_1 \frac{d(hx_{2n+1}, hx_{2n+1})}{d(hx_{2n+1}, hx_{2n})} + a_2 \frac{d(hx_{2n+2}, hx_{2n+1})}{d(hx_{2n+1}, hx_{2n})}
\]

or,

\[
d^2(hx_{2n+1}, hx_{2n}) + 1 + 2 \frac{d(hx_{2n+2}, hx_{2n+1})}{d(hx_{2n+1}, hx_{2n})} \geq a_1 \frac{d(hx_{2n+1}, hx_{2n+1})}{d(hx_{2n+1}, hx_{2n})} + a_2 \frac{d(hx_{2n+2}, hx_{2n+1})}{d(hx_{2n+1}, hx_{2n})}
\]

or,

\[
(a_1 + a_2) t^2 + (a_1 + a_2 - 2) t - 1 \leq 0,
\]

where, \( t_1 = d(hx_{2n+1}, hx_{2n+2}) / d(hx_{2n+1}, hx_{2n}) \)

Now, let \( h_1 : [0, \infty) \to R \) be the function

\[
h_1(t) = (a_1 + a_2) t^2 + (a_1 + a_2 - 2) t - 1.
\]

Then \( h(0) = -1 \) and

\[
h_1(1) = 2a_1 + a_2 + a_2 - 3 > 0 \text{ from the hypothesis.}
\]
Let $K \subseteq (0.1)$ be the root of the equation $h_1(t_1) = 0$, then $h_1(t_1) \leq o$ for $t_1 \leq K$ and thus

\[(vi) \quad d(hx_{2n+2}, hx_{2n+3}) \leq Kd(hx_{2n+1}, hx_{2n})\]

Similarly, from (iii), we get

\[
d(hx_{2n+2}, hx_{2n+3}) + 2d(hx_{2n+1}, hx_{2n+2}) \geq a_1 \left[ d(hx_{2n+2}, hx_{2n+1}) + d(hx_{2n+3}, hx_{2n+2}) \right] \]

\[+ a_2 \left[ d(hx_{2n+3}, hx_{2n+1}) + d(hx_{2n+3}, hx_{2n+2}) \right] + a_3 d^2(hx_{2n+2}, hx_{2n+3}) \]

or,

\[
d(hx_{2n+2}, hx_{2n+3}) + 2d(hx_{2n+1}, hx_{2n+2}) \geq a_1 \left[ d(hx_{2n+2}, hx_{2n+1}) + d(hx_{2n+3}, hx_{2n+2}) \right] + a_2 \left[ d(hx_{2n+3}, hx_{2n+1}) + d(hx_{2n+3}, hx_{2n+2}) \right] + a_3 d^2(hx_{2n+2}, hx_{2n+3}) \]

or,

\[
(a_1 + a_2) t_2^3 + (a_1 + a_2 - 2) t_2 - 1 \leq o,
\]

where, $t_2 = \frac{d(hx_{2n+2}, hx_{2n+3})}{d(hx_{2n+2}, hx_{2n+1})}$

Again, let $h_2: [0, \infty) \rightarrow$ be the function

$h_2(t_2) = (a_1 + a_3) t_2 + (a_1 + a_2 - 2) t_2 - 1$.

Then $h_2(0) = -1$ and $h_2(1) = 2a_1 + a_2 + a_3 - 3 > o$. From the hypothesis. Let $r \in (0.1)$ be the root of equation $h_2(t_2) \leq o$ for $t_2 < r$ and thus

\[(vii) \quad d(hx_{2n+2}, hx_{2n+3}) \leq r d(hx_{2n+2}, hx_{2n+1})\]

Let $\max \{k, r\} = r \in (0.1)$. Then (vi) and (vii), we have,

\[
d(hx_{2n+2}, hx_{2n+3}) \leq \lambda d(hx_{2n}, hx_{2n+1}) \leq \lambda d(hx_{2n-1}, hx_{2n}) \leq \ldots \leq \lambda^{n+1} d(hx_0, hx_1).
\]

Then by a routine calculation it can be easily shown that $\{hx_n\}$ is a Cauchy sequence and since $X$ is complete, it converges, to a point $x \in X$. The assumption $x_n \neq x_{n+1}$, for each $n$ implies $x_n \neq x$ for almost all $n$.

Since $f$ and $g$ are continuous mappings and $\{f h x_n\}$ and $\{g h x_n\}$ are subsequences of sequence $\{hx_n\}$, they will also converge to the same point $x \in X$, hence

$f(hx_{2n+1}) \rightarrow f x$, $g(hx_{2n+2}) \rightarrow g x$.

since $h$ commutes with $f$ and $g$, we have

\[(viii) \quad f(hx_{2n+1}) = h(fx_{2n+1}), g(hx_{2n+2}) = h(gx_{2n+2}).\]

Letting $n \rightarrow \infty$, we have from (viii)

\[fx = hx \text{ and } gx = hx \text{ i.e. } f x = g x = h x.\]

Since $h$ is orbitally continuous mapping of complete metric space $(X, d)$, we have

\[x = \lim_{n \rightarrow \infty} h^n x = \lim_{n \rightarrow \infty} h^n x = h x.\]

Hence $f x = g x = h x = x$.

i.e., $X$ is a common fixed point of $f$, $g$ and $h$.

Let, $\psi [d(hx, hy), d(hx, fhx), d(hy, ghy)]$

$= a_1 [d(hx, fhx)d(hx, hy) + d(hy, ghy)d(hx, hy)]$
\[ \begin{align*}
&\quad d(hx, fhx) + a_d d^q(hx, hy) + a_2 = \frac{d(hx, ghy) + a_2 d^2(hx, hy)}{d(hx, hy) + d(hx, fhx) + d(hy, ghy)} \\
\text{Then, if } x \neq y, \\
&\quad \psi [d(x, y), 0, 0] = a_2 d(x, y) \geq d(x, y) \\
\text{Now, from Theorem 1, } f, g \text{ and } h \text{ have a unique common fixed point.}
\end{align*} \]

**Theorem 3.** Let \( f, g \) and \( h \) be surjective self-maps of a complete metric space \((X, d)\). If there exists non-negative reals \( a_1, a_2 \) with \( a_1 < 1 \) and \( 2a_1 + a_2 - 3 > 0 \) such that

\[ \begin{align*}
&\quad (ix) \ d(fhx, ghy) \geq \frac{a_1 \{ \{ d(hx, fhx) \}^2 \}}{d(hx, fhx) + d(hy, ghy)} \\
&\quad \quad + \{ d(hy, ghy) \}^2 + a_2 \{ d(hx, hy) \} \\
&\quad \quad + \text{=} \ d(hx, hy)
\end{align*} \]

for each \( x \neq y \) in \( X \) for which \( d(hx, fhx) + d(hy, ghy) + d(hx, hy) \neq 0 \), then \( f, g \) and \( h \) have a common fixed point. Further, if \( a_4 > 1 \), then the fixed point is unique.

**Proof:** It is similar to the proof of Theorem 2.

**Theorem 4:** Let \( f, g \) and \( h \) be surjective self-maps of a complete metric space \((X, d)\). If there exists non-negative reals \( a_1, a_2, a_3, a_4 \) with \( a_1 + a_2 + a_3 + a_4 > 1 \) such that

\[ \begin{align*}
&\quad (x) \ d^k(fhx, ghy) \geq a_1 d^q(hx, fhx) d^{k-q}(hx, hy) \\
&\quad \quad + a_2 d^m(hy, ghy) d^{k-m}(hx, hy) \\
&\quad \quad + a_d d^p(hx, fhx) d^{k-p}(hy, ghy) + a_d d^k(hx, hy)
\end{align*} \]

where \( k \geq 1, q \geq 0, m \geq 0, p \geq 0 \) and \( q \leq k, m \leq k, p \leq k \) for each \( x, y \) in \( X \) with \( x = y \) and the condition (iv) and (v) hold, then \( f, g \) and \( h \) have a common fixed point. Further, if \( a_4 > 1 \), then the fixed point is unique.

**Proof:** Define a sequence \( \{hx_n\} \) as in theorem 2.

Suppose \( hx_{2n} = hx_{2n+1} \) for some \( n \). If \( hx_{2n+1} \neq hx_{2n+2} \), then applying \((x)\), we have

\[ \begin{align*}
&\quad d^k(hx_{2n}, hx_{2n+1}) = d^k(fhx_{2n+1}, ghx_{2n+2}) \\
&\quad \geq a_d d^q(hx_{2n+1}, hx_{2n}) d^{k-q}(hx_{2n+1}, hx_{2n+2}) \\
&\quad \quad + a_d d^m(hx_{2n+2}, hx_{2n+1}) d^{k-m}(hx_{2n+1}, hx_{2n+2}) \\
&\quad \quad + a_d d^p(hx_{2n+1}, hx_{2n}) d^{k-p}(hx_{2n+1}, hx_{2n+1}) \\
&\quad \quad + a_d d^k(hx_{2n+1}, hx_{2n+1})
\end{align*} \]

or,

\[ \begin{align*}
&\quad o \geq (a_4 - a_4) d^k(hx_{2n+1}, hx_{2n+2})
\end{align*} \]

which implies \( hx_{2n+1} = hx_{2n+2} \) and \( hx_{2n} \) is a common fixed point of \( f \) and \( g \). Similarly, \( hx_{n+1} = hx_{n+2} \) for some \( n \) lead to \( hx_{2n+1} \) being a common fixed point of \( f \) and \( g \).

Suppose \( hx_n \neq hx_{n+1} \) for each \( n \). then by \((x)\), we have

\[ \begin{align*}
&\quad d^k(hx_{2n}, hx_{2n+1}) \geq a_1 d^q(hx_{2n+1}, hx_{2n}) d^{k-q}(hx_{2n+1}, hx_{2n+2}) \\
&\quad \quad + a_d d^m(hx_{2n+2}, hx_{2n+1}) d^{k-m}(hx_{2n+1}, hx_{2n+2}) \\
&\quad \quad + a_d d^p(hx_{2n+1}, hx_{2n}) d^{k-p}(hx_{2n+1}, hx_{2n+1}) \\
&\quad \quad + a_4 d^k(hx_{2n+1}, hx_{2n+1})
\end{align*} \]

or,

\[ \begin{align*}
&\quad 1 \geq a_1 \frac{d^{k-q}(hx_{2n+1}, hx_{2n+2})}{d^{k-q}(hx_{2n}, hx_{2n+1})} \\
&\quad \quad \frac{d^k(hx_{2n+1}, hx_{2n+2})}{d^k(hx_{2n}, hx_{2n+1})} \\
&\quad \quad + a_4 \frac{d^k(hx_{2n+1}, hx_{2n+1})}{d^k(hx_{2n}, hx_{2n+1})}
\end{align*} \]
Common fixed ..Expansion mappings.

or,

\[ a_1 t_1^{k-q} + a_3 t_1^{k-p} + (a_2 + a_4) t_1^{k} - \leq 0 \]

where, \( t_1 = \frac{d(hx_{2n+1}, hx_{2n+2})}{d(hx_{2n}, hx_{2n+1})} \)

Now, Let \( h_1 : [a, \infty) \rightarrow R \) be the function \( h_1(t_1) = a_1 t_1^{k-q} + a_3 t_1^{k-p} + (a_2 + a_4) t_1^{k} - 1 \). Then \( h_1(a) = -1 \) and \( h_1(\lambda) = a_1 + a_3 + a_2 + a_4 - 1 \geq 0 \) from the hypothesis. Let \( k \in (0, 1) \) be the root of the equation \( h_1(t_1) = 0 \) then \( h_1(t_1) \leq 0 \) for \( t_1 < k \) and thus

\( d(hx_{2n+2}, hx_{2n+1}) \leq kd(hx_{2n+1}, hx_{2n}) \)

Similarly, we have

\( d^k(hx_{2n+1}, hx_{2n+2}) \geq a_1 d^k(hx_{2n+2}, hx_{2n+1}) \]

\( \frac{d^k(hx_{2n+2}, hx_{2n+2})}{d^k(hx_{2n+1}, hx_{2n+2})} \geq a_1 \)

\( d(hx_{2n+3}, hx_{2n+1}) \leq d(hx_{2n+1}, hx_{2n+2}) \)

\( a_2 d(hx_{2n+3}, hx_{2n+2}) \geq a_2 \)

\( d(hx_{2n+2}, hx_{2n+2}) \leq d(hx_{2n+1}, hx_{2n+2}) \)

\( a_3 d^k(hx_{2n+2}, hx_{2n+2}) \leq a_3 \)

\( d(hx_{2n+3}, hx_{2n+2}) \leq d(hx_{2n+1}, hx_{2n+2}) \)

\( a_4 d^k(hx_{2n+2}, hx_{2n+2}) \leq a_4 \)

\( d(hx_{2n+3}, hx_{2n+2}) \leq d(hx_{2n+1}, hx_{2n+2}) \)

Again, Let \( h_2 : [a, \infty) \rightarrow R \) be the function \( h_2(t_2) = a_1 t_2^{k-q} + (a_2 + a_4) t_2^{k-p} + a_2 t_2^{k-p} + (a_3 + a_4) t_2^{k-1} \). Then \( h_2(a) = -1 \) and \( h_2(\lambda) = a_1 + a_2 + a_3 + a_4 - 1 \leq 0 \) from the hypothesis. Let \( r \in (0, 1) \) be the root of the equation \( h_2(t_2) = 0 \) then \( h_2(t_2) \leq r \) and thus

\( d(hx_{2n+3}, hx_{2n+2}) \leq r d(hx_{2n+1}, hx_{2n+1}) \)

Now, taking \( \max \{k, r\} = \lambda \in (0, 1) \), we have from (xi) and (xii)

\( d(hx_{2n+1}, hx_{2n+1}) \leq \lambda d(hx_{2n}, hx_{2n+1}) \)

\( \leq \lambda^2 d(hx_{2n-1}, hx_{2n}) \leq \cdots \leq \lambda^n d(hx, hx_1) \)

Then by a routine calculation we can show that \( \{hx_0\} \) is a couchy sequence and since \( X \) is complete, it converges to a point \( x \in X \).

Now, using (iv) and (v) and following the proof of Theorem 2, we observe that \( X \) is a common fixed point of \( f, g \) and \( h \).

Let, \( \psi [d(hx, hy), d(hx, fhx), d(hy, ghx)] \)

\( = [a_1 d^p(hx, fhx) d^{k-p}(hx, hy) + a_4 d^m(hy, ghx) d^{k-m}(hx, hy) + a_3 d^p(hx, fhx) \]

\( d^{k-p}(hx, hy) + a_4 d^k(hx, hy) \]^{1/k}

Then, if \( x \neq y \), we have

\( \psi (d(x, x), 0, 0) = a_4^{1/k} d(x, y) > d(x, y) \)

Therefore, from theorem 1, \( x \) is unique.

This completes the proof.

Remarks:

1. On taking \( f = g, h = 1 \) (identity map) in Theorem 1, it reduces to theorem A.
2) On some setting $f = g$, $h = I$ is Theorem 2, 3 and 4, we obtain some other results of Popa\textsuperscript{3,4,5}.

References


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A common fixed point theorem for continuous densifying mappings

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(Acceptance Date 16 September 96)

Abstract

In this paper we shall prove a common fixed point theorem for
continuous densifying mappings which extend some well known results due
to Furi & Vignoli¹, Iseki² and Jain and Dixit³.

Extending the results of Furi & Vignoli¹ & Iseki², Jain and Dixit³ obtained the follow-
ing.

**Theorem A**: Let S be a continuous, densifying mapping of a bounded complete
dense metric space (X,d) into itself.

If for every x, y ∈ X, x ≠ y, x ≠ Sx, y ≠ Sy,

\[ d(Sx, Sy) < a_1 d(x, y) + a_2 d(x, Sx) + d(y, Sy) \]

+ \[ a_3 d(x, Sy) + d(y, Sx) \]

+ \[ a_4 d(x, Sx) d(y, Sy) \]

\[ d(x, y) \]

Where \( a_1, a_2, a_3, a_4 \) are non-negative reals
satisfying \( a_1 + 2a_2 + 2a_3 + a_4 = 1 \), then S has a fixed point.

We further extend Theorem A for a pair of continuous densifying mappings of a complete metric space and prove:

**Theorem**: Let S and T be two continuous densifying mappings of a bounded complete metric space (X, d) satisfying

\[ \text{(1.1) } d(Sx, STy) < h \left( d(x, Ty), d(x, Sx), d(Ty, STy), d(x, STy), d(Ty, Sx), d(x, Sx), d(Ty, Sy) \right) \]

\[ \frac{d(x, Sx) d(Ty, STy)}{d(x, STy)} \]

\[ a(x, Ty) d(x, STy) d(Ty, Sx), \]

b(x, Ty) (d(Ty, Sx) d(x, STy))^{1/2} \]

For every \( x, y \in X \) and \( Sx \neq STy \), where \( a(x, y) \)
and \( b(x, y) \) are non-negative real functions satisfying

\[ a(x, y) \leq \{ d(x, Ty) \}^{-1}, \quad b(x, y) \leq 1 \]

(1.2) \( ST = TS \)

(1.3) \( h : (R+)^9 \rightarrow R_+ = [0, \infty) \) is non-decreasing in each co-ordinate variable and \( g(t) = h(t, t, t, C_1 t, C_2 t, t, t, t, t) \leq t \) for each t ≥ 0,
where \( C_1 + C_2 \leq 2 \). Then S and T have a
unique common fixed point in X.

**Proof**: Let \( x_0 \in X \) and the sequence
\{\( x_n \)\} be defined by \( Sx_{2n} = x_{2n+1}, Tx_{2n+1} = x_{2n+2} \) for \( n = 0, 1, 2, 3, \ldots \), if

\[ A = \{x_{2n+1} : n = 0, 1, 2 \ldots \}, \]

then

\[ ST(A) = \{x_{2n+1} : n = 1, 2, 3, \ldots \}. \] therefore

\[ ST(A) \subseteq A \] and by the continuity of S and T,

\[ ST(A) \subseteq \overline{ST(A)} \subseteq \overline{A}. \]

Hence A is invariant under ST and is bounded. Now let \( \alpha (A) > 0 \), Then

\[ \alpha (A) = \alpha (ST(A) \cup \{x_1\}) \]

\[ = \max \{ \alpha (ST(A)), \alpha \{x_1\} \} \]
= a (ST(A) < a (A).

giving a contradiction which implies that
a (A) = 0, and therefore A is precompact
since x is complete metric space, A is com-

pact. Define a real valued function f on x
by f(x) = d(Tx, STx ) by the hypothesis,
d(Tx, STx) is continuous on the compact
subset A. Hence d(Tx, STx) has a minimum
point u in A. To prove that u is a fixed
point of S, Suppose u ≠ Su, Tu ≠ TSu and
STu ≠ STu. Then by (1.1) and (1.2), We have

\[ f(Su) = d(TSu, STSu) \]
\[ = d(STu, STSu) \]
\[ < h\{d(Tu,TSu), d(Tu,STu), d(TSu, STSu), d(Tu,STu), d(Tu,STu)\} \]
\[ \frac{d(Tu,STu)}{d(STu,STSu)} \]
\[ \frac{d(Tu,STu)}{d(Tu,STu)} \]
\[ a(Tu,Tsu) d(Tu,STsu) d(Tsu,STsu), \]
\[ b(Tu,Tsu) (d(TSu,STu) d(Tu,STSu))^{1/2} \]
\[ \leq h\{d(STu,STSu), d(TSu,STSu), d(TSu,STSu), 2d(TSu,STSu), 0, d(TSu,STSu), d(TSu,STSu), 0, 0\} \]
\[ \leq g(d(TSu,STSu)) \]
\[ \leq d(TSu,STSu). \]

a contradiction, Hence u ∈ X is a fixed
point of S, i.e. Su = u and STu = TSu = Tu,
Now we shall prove that Tu = u.

If possible, let Tu ≠ u. Then (1.1) gives
\[ d(u, Tu) = d(Su, STu) \]
\[ < d\{d(u, Tu), 0, 0 d(u, Tu), 0, 0, d(u, Tu)\} \]
\[ \leq g(d(u, Tu)) \]
\[ \leq d(u, Tu) \]
a contradiction, which proves that Tu = u.

Finally, to show the uniqueness of u, let
v(≠ u) be another common fixed point of S
and T. Then from (1.1) we have
\[ d(u,v) = d(Su, STv) \]
\[ < h\{d(u,v), 0, 0, d(u,v), 0, 0, d(u,v), d(u,v)\} \]
\[ \leq g(d(u,v)) \]
\[ \leq d(u,v) \]

which yields u = v. This completes the proof
of the theorem.

Corollary 1: Let S and T be two continuous
densifying mappings of a bounded
complete metric space (X, d) satisfying

\[ (1.4) d(Sx,STy) < \max \{d(d(x,Ty), \frac{1}{2} \{d(x,Sx) + d(Ty,STy)\}, \frac{1}{2} \{d(x,STy) + d(Ty,Sx)\}, \]
\[ d(x,Sx) d(Ty,STy), d(x,Sx) d(Ty,STy), d(x,Ty) \]
\[ d(x,Sx) d(x,Ty) \}
\[ a(x, Ty) d(x, STy) d(Ty, Sx), b(x, Ty) \]
\[ (d(Ty, Sx) d(x, STy))^{1/2} \]

For all x,y ∈ X and Sx ≠ STy, where a(x,y)
and b(x,y) are non-negative real functions
satisfying
\[ a(x, y) \leq d(x, Ty)^{-1}, b(x, y) \leq 1 \]
and (1.2) holds. Then S and T have a unique
common fixed point in X.

Taking T = I (identity mapping) in Corollary
1, We have,

Corollary 2: Let S be a continuous
densifying mapping of a bounded complete
metric space (X, d) satisfying

\[ d(Sx, Sy) < \max \{d(x,y), \frac{1}{2} \{d(x,Sx) + d(y, Sy)\}, \]
\[ \frac{1}{2} \{d(x, Sy) + d(y, Sx)\}, \frac{d(x,Sx) d(y, Sy)}{d(x,y)} \]
\[
\frac{d(x,Sx) \ d(y,Sy)}{d(Sx,Sy)}, \quad \frac{a(x,y) \ d(x,Sy) \ d(y,Sx)}{d(Sx,Sy)} \quad b(x,y) \left\{d(y,Sx) \ d(x,Sy)\right\}^{1/2}
\]

for all \( x, y \in X \) with \( Sx \neq Sy \), where \( a(x,y) \) and \( b(x, y) \) are non-negative reals functions, satisfying

\[ a(x, y) \leq d(x, Ty)^{-1}, \quad b(x, y) \leq 1 \text{ and (1.2) holds.} \]

Then \( S \) has a unique fixed point in \( X \). It is to be remarked that Corollary 2, includes Theorem A.

I am thankful to Dr. R.K. Jain for his valuable suggestions during the preparation of this paper.

References